ON ASYMPTOTIC-NUMERICAL SOLVERS FOR DIFFERENTIAL EQUATIONS WITH HIGHLY OSCILLATORY FORCING TERMS

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Abstract. We present a method to compute efficiently solutions of systems of ordinary differential equations that possess highly oscillatory forcing terms. This approach is based on asymptotic expansions in inverse powers of the oscillatory parameter, and features two fundamental advantages with respect to standard ODE solvers: firstly, the construction of the numerical solution is more efficient when the system is highly oscillatory, and secondly, the cost of the computation is essentially independent of the oscillatory parameter. Numerical examples are provided, motivated by problems in electronic engineering.

1. Introduction. In this paper we are concerned with an initial-value problem involving an ODE system of the form

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + g_\omega(t)\boldsymbol{f}(\boldsymbol{y}(t)), \quad t \geq 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{1.1}$$

where g_ω is a rapidly oscillating scalar function of a frequency related to the oscillatory parameter $\omega \gg 1$, while $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^d$ is an analytic function. We further asume that the spectral abcissa $\alpha[A]$ is nonpositive or that the logarithmic norm $\mu[A]$ is nonpositive, so that the solution of the linear problem $\boldsymbol{y}'(t) = A\boldsymbol{y}(t)$ remains bounded.

Differential equations of this type abound in a wide variety of different contexts, notably in the modelling of circuits in Radio Frequency communication systems, see for instance [5], [9], [10]. In a broader context, this setting corresponds to the situation where a physical system modelled by the differential equation is subject to a high frequency forcing term (which could for instance be an electromagnetic wave or a mechanical excitation) and we wish to analyse the behaviour of the system on a time scale which is much larger than the period of the forcing function.

Oscillatory systems of differential equations have been widely studied in the literature, but in most cases it is the linear part of the equation which is responsible for the oscillatory character, for example when A has eigenvalues with large imaginary part, see [7], [8]. In our case the matrix A will be independent of ω, therefore the highly oscillatory behaviour of the solution originates solely in the forcing term.

Typical examples of highly oscillatory forcing terms in this context are

$$g_\omega(t) = \mathrm{e}^{\mathrm{i}\omega t}, \quad g_\omega(t) = \mathrm{e}^{\eta \cos\omega t}, \quad g_\omega(t) = \mathrm{e}^{\eta(t)\cos\omega t}. \tag{1.2}$$

The first example involves just a Fourier oscillator, whereas the second and third appear in the modelling of nonlinear circuits involving diodes and subject to various amplitude and digital modulation formats, respectively.

In terms of the structure of the forcing function, a general framework is given by *modulated Fourier expansion* (MFE), that is:

$$g_\omega(t) = \sum_{j=-\infty}^{\infty} \alpha_j(t)\mathrm{e}^{\mathrm{i}j\omega t}, \qquad t \geq 0. \tag{1.3}$$

Modulated Fourier expansions have already been used in the context of geometric integration of Hamiltonian systems, see [2] and [6, Ch. XIII], but it turns out that

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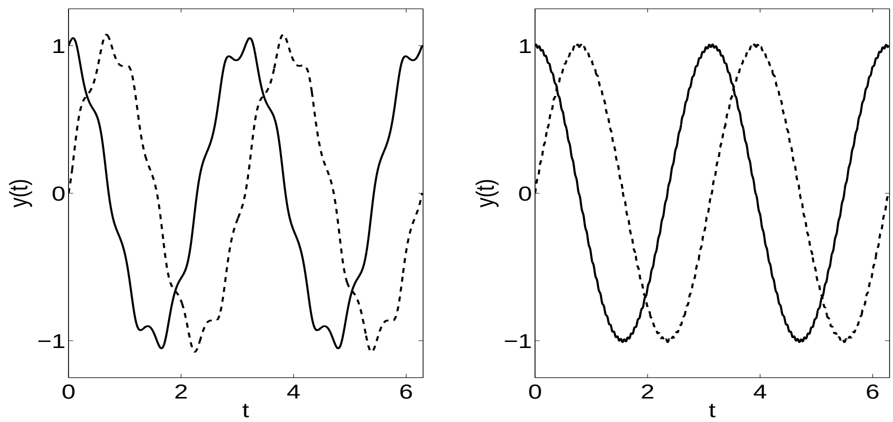

FIG. 1.1. *Real (solid line) and imaginary part (dashed line) of the solution of (1.4) for* $\omega = 10$ *(left) and* $\omega = 100$ *(right).*

they provide the natural framework for the study of this type of differential equations as well. They allow us to treat not only periodic forcing terms (in which case the coefficients $\alpha_j(t)$ will be independent of t), but more general cases, since the forcing term is periodic in the variable ωt but not necessarily in t.

In order to gain intuition about the expected behaviour of the solutions of such type of differential equations, let us consider the scalar example

$$y' = 2\mathrm{i}y + \mathrm{e}^{\mathrm{i}\omega t}y^2, \quad t \geq 0, \qquad y(0) = 1, \tag{1.4}$$

whose exact solution is

$$y(t) = \frac{\left(1 + \frac{2}{\omega}\right)\mathrm{e}^{2\mathrm{i}t}}{1 + \frac{2-\mathrm{i}}{\omega} + \frac{\mathrm{i}}{\omega}\mathrm{e}^{\mathrm{i}(2+\omega)t}}.$$

Note that $y(t)$ is an $\mathcal{O}(\omega^{-1})$ perturbation of a periodic function. We have sketched in Fig.1.1 the real and imaginary parts of y.

Another example is given by the equation

$$y' = \mathrm{i}y - 5\mathrm{e}^{\mathrm{i}\omega t}y^2, \quad t \geq 0, \qquad y(0) = 1. \tag{1.5}$$

This example may look similar to the previous one, except that now the vector field is dominated by the nonlinear, high frequency term. This is reflected by the plots in Fig.1.2.

These examples and similar ones suggest the intuitive idea that the solution of a system of ODEs with a highly oscillatory forcing term consists of a non-oscillatory base function superimposed with small and very fast oscillations, and that the amplitude of these oscillations decreases when ω grows. The presence of these very small oscillations makes the problem notoriously difficult to solve numerically using standard ODE routines like the ones available in MATLAB or MAPLE, because one has to use an exceedingly small stepsize in order to control the error.

The reason for the poor performance of classical numerical methods for solving ODEs in the presence of high oscillation lies at the very heart of the standard numerical theory, which is essentially based on Taylor expansion of the solution. In any numerical method of order p with step h, the error scales roughly like

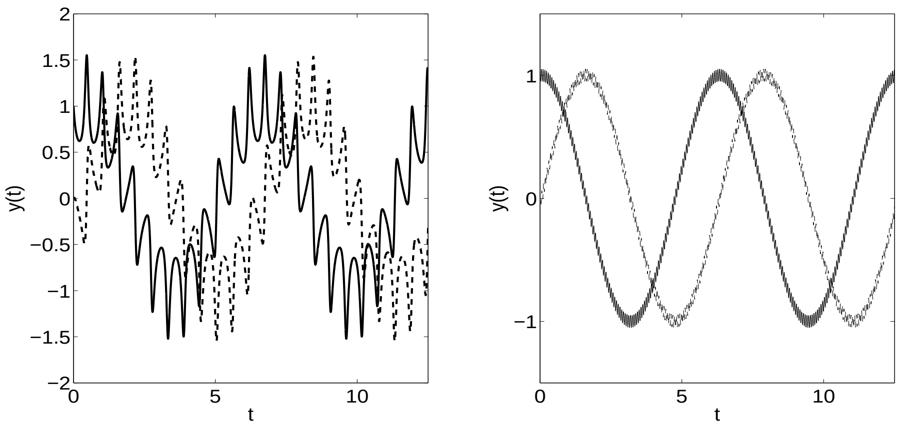

FIG. 1.2. *Real (solid line) and imaginary part (dashed line) of the solution of (1.5) for* $\omega = 10$ *(left) and* $\omega = 100$ *(right).*

$h^{p+1}\boldsymbol{y}^{(p+1)}(t)$. Since the derivatives of highly oscillatory functions grow very fast, typically $\boldsymbol{y}^{(p+1)}(t) = \mathcal{O}(\omega^{p+1})$, we require h to be extremely small in order to keep the error down to an acceptable size.

In this paper we use ideas recently developed in the theory of highly oscillatory problems to devise an alternative and efficient method to compute the solutions of this type of equations. More precisely, we construct combined asymptotic-numerical solvers, based on asymptotic expansions for large values of ω rather than Taylor expansions. The procedure presents two remarkable properties compared to the standard discretization method of ODEs: firstly, the algorithm becomes more efficient for large values of the oscillatory parameter, and secondly, the computational effort is essentially independent of ω.

2. General setting. In order to justify the general form of the solution as a base function superimposed with oscillations of decreasing amplitude when ω grows, our first step is to write the solution of (1.1) using nonlinear variation of constants, see [7]. More precisely, considering

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + g_\omega(t)\boldsymbol{f}(\boldsymbol{y}(t)), \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0$$

as a perturbation of the linear equation with the same initial value,

$$\boldsymbol{z}'(t) = A\boldsymbol{z}(t), \qquad \boldsymbol{z}(0) = \boldsymbol{y}_0,$$

we have

$$\begin{aligned}\boldsymbol{y}(t) &= \boldsymbol{z}(t) + \mathrm{e}^{tA}\int_0^t g_\omega(x)\mathrm{e}^{-xA}\boldsymbol{f}(\boldsymbol{y}(x))\,\mathrm{d}x \qquad (2.1)\\ &= \mathrm{e}^{tA}\boldsymbol{y}_0 + \mathrm{e}^{tA}\int_0^t g_\omega(x)\mathrm{e}^{-xA}\boldsymbol{f}(\boldsymbol{y}(x))\,\mathrm{d}x.\end{aligned}$$

Our essential *ansatz* in constructing asymptotic-numerical solvers is that the solution $\boldsymbol{y}(t)$ admits an asymptotic expansion of the form:

$$\boldsymbol{y}(t) \sim \sum_{s=0}^{\infty} \frac{1}{\omega^s}\boldsymbol{\psi}_s(t) \qquad \omega \gg 1, \qquad (2.2)$$

where $\boldsymbol{\psi}_s(t) \sim \mathcal{O}(1)$, $\omega \gg 1$, for $s \in \mathbb{Z}_+$.[1]

Taking into account formula (2.1) and the structure of the forcing term in (1.3), it makes sense to look for a solution which is periodic in the variable ωt. Actually, a reasonable assumption in these conditions is that each $\boldsymbol{\psi}_s(t)$ in (2.2), except when $s = 0$, has itself the form of a modulated Fourier expansion:

$$\boldsymbol{\psi}_s(t) = \sum_{j=-\infty}^{\infty} \boldsymbol{a}_{s,j}(t) \mathrm{e}^{\mathrm{i} j\omega t}, \qquad s \geq 1. \tag{2.3}$$

In order to satisfy the initial condition, we impose $\boldsymbol{\psi}_0(0) = \boldsymbol{y}(0) = \boldsymbol{y}_0$, which means that $\boldsymbol{\psi}_s(0) = \boldsymbol{0}$ for $s \geq 1$, or equivalently

$$\sum_{j=-\infty}^{\infty} \boldsymbol{a}_{0,j}(0) = \boldsymbol{y}_0, \qquad \sum_{j=-\infty}^{\infty} \boldsymbol{a}_{s,j}(0) = \boldsymbol{0}, \qquad s \geq 1.$$

Instead of discretizing the highly oscillatory integral in (2.2) using standard quadrature, a procedure which is very expensive and numerically not too reliable, we will expand that integral in inverse powers of ω and then identify the functions $\boldsymbol{\psi}_n(t)$ as the terms that multiply the same inverse power of the oscillatory parameter. Equivalently, we will try to compute the coefficients of the MFE of the different terms $\boldsymbol{\psi}_n(t)$. To this end, we note that since $\boldsymbol{f}$ is analytic, we can expand it into Taylor series around $\boldsymbol{\psi}_0$, assuming that this term will give the main contribution:

$$\boldsymbol{f}(\boldsymbol{\psi}_0 + \boldsymbol{\theta}) = \sum_{m=0}^{\infty} \frac{1}{m!} \boldsymbol{\varphi}_m(\boldsymbol{\psi}_0, \overbrace{\boldsymbol{\theta}, \boldsymbol{\theta}, \ldots, \boldsymbol{\theta}}^{m \text{ times}}).$$

Here $\boldsymbol{\varphi}_m$ is an m-tensor related to the mth derivative of $\boldsymbol{f}$ at $\boldsymbol{\psi}_0$,

$$\begin{aligned}
\boldsymbol{\varphi}_0(\boldsymbol{\psi}_0) &= \boldsymbol{f}(\boldsymbol{\psi}_0), \\
\boldsymbol{\varphi}_1(\boldsymbol{\psi}_0, \boldsymbol{\theta}) &= \frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0)}{\partial \boldsymbol{y}} \boldsymbol{\theta}, \\
(\boldsymbol{\varphi}_2(\boldsymbol{\psi}_0, \boldsymbol{\theta}, \boldsymbol{\theta}))_k &= \sum_{i=1}^{d} \sum_{j=1}^{d} \theta_i \frac{\partial^2 f_k(\boldsymbol{\psi}_0)}{\partial y_i \partial y_j} \theta_j, \qquad k = 1, 2, \ldots, d,
\end{aligned}$$

etc. In general we can write

$$(\boldsymbol{\varphi}_m(\boldsymbol{\psi}_0, \boldsymbol{\theta}, \ldots, \boldsymbol{\theta}))_k = \sum_{i_1=1}^{d} \cdots \sum_{i_m=1}^{d} \frac{\partial^m f_k(\boldsymbol{\psi}_0)}{\partial y_{i_1} \cdots \partial y_{i_m}} \theta_{i_1} \theta_{i_2} \cdots \theta_{i_m}, \qquad k = 1, 2, \ldots, d.$$

Note that each $\boldsymbol{\varphi}_m(\boldsymbol{\psi}_0, \boldsymbol{\theta}, \ldots, \boldsymbol{\theta})$ is linear in each of the $\boldsymbol{\theta}_k$s. Substituting the Taylor

[1] This ansatz can be generalised to a more general expansion, e.g. $\sum_{s=0}^{\infty} \omega^{-\kappa_s} \boldsymbol{\psi}_s(t)$, where $\{\kappa_s\}_{s\geq 0}$ is a strictly increasing nonnegative sequence such that $\lim_{s\to\infty} \kappa_s = +\infty$, but for the sake of simplicity we will restrict ourselves to this model.

expansion of $\boldsymbol{f}$ into (2.1), we have

$$
\begin{aligned}
\boldsymbol{y}(t) &\sim \sum_{n=0}^{\infty} \frac{1}{\omega^n} \boldsymbol{\psi}_n(t,\omega) && (2.4)\\
&\sim \mathrm{e}^{tA}\boldsymbol{y}_0 + \mathrm{e}^{tA} \int_0^t g_\omega(x) \mathrm{e}^{-xA} \boldsymbol{f}\left(\sum_{n=0}^{\infty} \frac{1}{\omega^n} \boldsymbol{\psi}_n(x,\omega) \right) \mathrm{d}x \\
&= \mathrm{e}^{tA}\boldsymbol{y}_0 \\
&\quad + \mathrm{e}^{tA} \sum_{m=0}^{\infty} \frac{1}{m!} \int_0^t g_\omega(x) \mathrm{e}^{-xA} \boldsymbol{\varphi}_m\left(\boldsymbol{\psi}_0(x,\omega), \sum_{n_1=1}^{\infty} \frac{\boldsymbol{\psi}_{n_1}(x,\omega)}{\omega^{n_1}}, \ldots, \sum_{n_m=1}^{\infty} \frac{\boldsymbol{\psi}_{n_m}(x,\omega)}{\omega^{n_m}} \right) \mathrm{d}x \\
&= \mathrm{e}^{tA}\boldsymbol{y}_0 + \mathrm{e}^{tA} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{\omega^{n_1+\cdots+n_m}} \int_0^t \boldsymbol{\gamma}_{m,\boldsymbol{n}}(x) g_\omega(x)\, \mathrm{d}x,
\end{aligned}
$$

where the term

$$
\boldsymbol{\gamma}_{m,\boldsymbol{n}}(x) = \mathrm{e}^{-xA} \boldsymbol{\varphi}_m(\boldsymbol{\psi}_0(x), \boldsymbol{\psi}_{n_1}(x), \ldots, \boldsymbol{\psi}_{n_m}(x)).
$$

is nonoscillatory. Once we express the functions $\boldsymbol{\psi}_n(t)$ as modulated Fourier series, as in (2.3), we obtain:

$$
\begin{aligned}
\boldsymbol{y}(t) &\sim \mathrm{e}^{tA}\boldsymbol{y}_0 && (2.5)\\
&\quad + \mathrm{e}^{tA} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{\omega^{|\boldsymbol{n}|}} \sum_{j_1,\ldots,j_m=-\infty}^{\infty} \int_0^t \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(x) \mathrm{e}^{\mathrm{i}|j|\omega x} g_\omega(x)\, \mathrm{d}x,
\end{aligned}
$$

where the nonoscillatory term is now

$$
\boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(x) = \mathrm{e}^{-xA} \boldsymbol{\varphi}_m(\boldsymbol{\psi}_0(x), \boldsymbol{a}_{n_1,j_1}(x), \ldots, \boldsymbol{a}_{n_m,j_m}(x)) \tag{2.6}
$$

and we have used the notation $|\boldsymbol{j}| = j_1 + j_2 + \ldots j_m$.

This is the general form of the solution, and naturally the structure (and complexity) of the $\boldsymbol{\psi}_n(t)$ functions will depend both on the particular oscillator $g_\omega(t)$ that we are considering and on the nonlinear term $\boldsymbol{f}$, through the $\boldsymbol{\varphi}_m$ functions.

Now we need to expand the integrand in (2.5) in inverse powers of ω, and for each $s \geq 1$ identify the functions $\boldsymbol{\psi}_s(t)$ as the terms that multiply the ω^{-s} elements. For this it is useful, given a multi-index $\boldsymbol{n} = (n_1, n_2, \ldots, n_m)$, to define the following sets:

$$
\mathbb{I}_s = \{(m;\boldsymbol{n}) : m \in \mathbb{Z}_+, \boldsymbol{n} \in \mathbb{N}^m, |\boldsymbol{n}| = s\}, \tag{2.7}
$$

where again $|\boldsymbol{n}| := n_1 + n_2 + \ldots + n_m$. It is understood that $\mathbb{I}_0 = \{(0)\}$, and the first few sets are

$$
\mathbb{I}_1 = \{(1;1)\}, \qquad \mathbb{I}_2 = \{(1;2),(2;1,1)\}, \qquad \mathbb{I}_3 = \{(1;3),(2;2,1),(2;1,2),(3;1,1,1)\}.
$$

Each value of s corresponds to terms that multiply the same inverse power of ω. However, in order to determine how many terms we should add in each step in the Taylor expansion of $\boldsymbol{f}$ and in the modulated Fourier expansion of the forcing term, we must take into account the characteristics of the oscillator $g_\omega(x)$, since integration by parts introduces additional inverse powers of ω. In this sense, we present the following definition:

Definition 2.1. *We say that the oscillator $g_\omega(t)$ is* soft *if, given any* C^∞ *function* $h(t)$*, it is true that*

$$\int_0^t h(x) g_\omega(x)\,\mathrm{d}x \sim \sum_{k=1}^{\infty} \frac{1}{\omega^k}[h_k(t)\sigma_k(t,\omega) - h_k(0)\sigma_k(0,\omega)], \qquad \omega \gg 1. \tag{2.8}$$

We say that $g_\omega(t)$ is hard *if*

$$\int_0^t h(x) g_\omega(x)\,\mathrm{d}x \sim \mathcal{F}[h] + \sum_{k=1}^{\infty} \frac{1}{\omega^k}[h_k(t)\sigma_k(t,\omega) - h_k(0)\sigma_k(0,\omega)]. \tag{2.9}$$

Here the functions $\sigma_k(t,\omega)$ are independent of h, while $h_k(x)$ is a linear combination of $h(x), h'(x), \ldots, h^{(k-1)}(x)$, and we assume that $\mathcal{F}$ is a linear operator.

Equivalently, and recalling formula (1.3), an oscillator is soft if $\boldsymbol{\alpha}_0 = 0$ and hard otherwise.

In the case of a soft oscillator, we remark that the expansion (2.8) begins with terms of order $\mathcal{O}(\omega^{-1})$. Therefore, in order to find the $\mathcal{O}(\omega^{-s})$ terms in $\boldsymbol{\psi}_s(t)$, we only need to consider an expansion up to the set $\mathbb{I}_{s-1}$.

In the case of hard oscillators, however, the oscillator itself will introduce a $\mathcal{O}(1)$ term, which means both that the solution of the base equation $\boldsymbol{\psi}_0(t)$ may not be explicit and also that we need all the terms up to (and including) $\mathbb{I}_s$ in order to obtain $\boldsymbol{\psi}_s(t)$. That will necessarily include the function $\boldsymbol{\psi}_s(t)$ inside the integral, giving an implicit integral (or differential) equation.

In any case, we will show in the examples below that this strategy will allow us to construct increasingly accurate approximations to the solution $\boldsymbol{y}(t)$ by adding more terms in (2.2). A very important point to bear in mind is that we do *not* solve any oscillatory ODE to obtain the solution $\boldsymbol{y}(t)$. Instead, we construct $\boldsymbol{\psi}_0(t)$, which is the solution of a nonoscillatory equation, and then add subsequent terms $\boldsymbol{\psi}_s(t)$, which are computed by finding the coefficients of the modulated Fourier expansion $\boldsymbol{a}_{s,j}(t)$ and then assembling the Fourier expansion itself.

In striking contrast to what happens in the classical setting, note that our procedure is likely to be more efficient the larger ω is, since we will need fewer terms in the asymptotic approximation to obtain similar accuracy. Moreover, observe that the only time-stepping needed is when solving the ODE for the zeroth term $\boldsymbol{\psi}_0(t)$, which is nonoscillatory and hence affordable with standard algorithms.

3. The linear oscillator. Let us first consider the case of a Fourier oscillator that involves a single frequency

$$g_\omega(x) = \mathrm{e}^{\mathrm{i}\omega x}.$$

Upon repeated integration by parts, it is straightforward to show that for any smooth function $\boldsymbol{h}(t)$

$$\int_0^t \boldsymbol{h}(x)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x \sim -\sum_{k=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{k+1}}[\boldsymbol{h}^{(k)}(t)\mathrm{e}^{\mathrm{i}\omega t} - \boldsymbol{h}^{(k)}(0)]. \tag{3.1}$$

This is the simplest example of a soft oscillator, and the zeroth term of the expansion is available explicitly:

$$\boldsymbol{\psi}_0(t) = \mathrm{e}^{tA}\boldsymbol{y}_0. \tag{3.2}$$

In order to find higher order terms, we substitute the asymptotic expansion (3.1) in (2.5). Note that

$$\begin{aligned}\int_0^t \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(x)\mathrm{e}^{\mathrm{i}|\boldsymbol{j}|\omega x} g_\omega(x)\,\mathrm{d}x &= \int_0^t \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(x)\mathrm{e}^{\mathrm{i}(|\boldsymbol{j}|+1)\omega x}\,\mathrm{d}x \\ &\sim -\sum_{k=0}^{\infty}\frac{\mathrm{i}^{k+1}}{((|\boldsymbol{j}|+1)\omega)^{k+1}}[\boldsymbol{\gamma}^{(k)}_{m,\boldsymbol{n},\boldsymbol{j}}(t)\mathrm{e}^{\mathrm{i}(|\boldsymbol{j}|+1)\omega t} - \boldsymbol{\gamma}^{(k)}_{m,\boldsymbol{n},\boldsymbol{j}}(0)]\end{aligned}$$

if $|\boldsymbol{j}| \neq -1$, otherwise the integral is nonoscillatory. This is a simple example of resonance (understood as the multiplication of oscillatory terms that results in a nonoscillatory integral). However, as we will see later on, when the forcing term is a Fourier oscillator that includes only the frequency ωt, this phenomenon does not occur.

Grouping equal powers of ω, we obtain

$$\begin{aligned}\boldsymbol{\psi}_{s+1}(t) = -\mathrm{e}^{tA}\sum_{k=0}^{s}\sum_{(m,\boldsymbol{n})\in\mathbb{I}_{s-k}}\frac{\mathrm{i}^{k+1}}{m!}\times \qquad\qquad\qquad\qquad\qquad\qquad (3.3)\\ \left[\sum_{|\boldsymbol{j}|\neq -1}\frac{1}{(|\boldsymbol{j}|+1)^{k+1}}\left[\boldsymbol{\gamma}^{(k)}_{m,\boldsymbol{n},\boldsymbol{j}}(t)\mathrm{e}^{\mathrm{i}(|\boldsymbol{j}|+1)\omega t} - \boldsymbol{\gamma}^{(k)}_{m,\boldsymbol{n},\boldsymbol{j}}(0)\right] + \sum_{|\boldsymbol{j}|=-1}\int_0^t \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(x)\,\mathrm{d}x\right]\end{aligned}$$

for $s \geq 0$. Observe that in order to construct the function $\boldsymbol{\psi}_{s+1}(t)$ we need to add contributions from all the spaces from $\mathbb{I}_0$ to $\mathbb{I}_s$.

For instance, to compute $\boldsymbol{\psi}_1(t)$ we substitute $p = 0$ above to get

$$\begin{aligned}\boldsymbol{\psi}_1(t) &= -\mathrm{e}^{tA}\sum_{(m,\boldsymbol{n})\in\mathbb{I}_0}\frac{\mathrm{i}}{m!}\sum_{j_1,\ldots,j_m=-\infty}^{\infty}\left[\boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(t)\mathrm{e}^{\mathrm{i}(|\boldsymbol{j}|+1)\omega t} - \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(0)\right] \\ &= -\mathrm{i}\mathrm{e}^{tA}\left[\boldsymbol{\gamma}_{0,(0),0}(t)\mathrm{e}^{\mathrm{i}(|\boldsymbol{j}|+1)\omega t} - \boldsymbol{\gamma}_{0,(0),0}(0)\right].\end{aligned}$$

Now, since $\boldsymbol{\gamma}_{0,(0),0}(x) = \mathrm{e}^{-xA}\boldsymbol{f}(\boldsymbol{\psi}_0(x))$, we obtain

$$\boldsymbol{\psi}_1(t) = \mathrm{i}\left[\mathrm{e}^{tA}\boldsymbol{f}(\boldsymbol{y}_0) - \mathrm{e}^{\mathrm{i}\omega t}\boldsymbol{f}(\mathrm{e}^{tA}\boldsymbol{y}_0)\right]. \qquad (3.4)$$

Similarly,

$$\begin{aligned}\boldsymbol{\psi}_2(t) = &-\mathrm{i}\mathrm{e}^{tA}\sum_{\substack{j=-\infty\\ j\neq -1}}^{\infty}\frac{1}{j+1}\left[\boldsymbol{\gamma}_{1,(1),j}(t)\mathrm{e}^{\mathrm{i}(|\boldsymbol{j}|+1)\omega t} - \boldsymbol{\gamma}_{1,(1),j}(0)\right] \\ &- \mathrm{i}\mathrm{e}^{tA}\int_0^t \boldsymbol{\gamma}_{1,(1),-1}(x)\,\mathrm{d}x + \mathrm{e}^{tA}\left[\boldsymbol{\gamma}'_{0,(0),0}(t)\mathrm{e}^{\mathrm{i}(|\boldsymbol{j}|+1)\omega t} - \boldsymbol{\gamma}'_{0,(0),0}(0)\right].\end{aligned}$$

Now observe that $\boldsymbol{\gamma}_{1,(1),-1}(x) = 0$, since $\boldsymbol{a}_{1,-1}(x) \equiv 0$ from the previous formula for $\boldsymbol{\psi}_1(t)$. Moreover,

$$\boldsymbol{\gamma}_{1,(1),j}(x) = \mathrm{e}^{-xA}\boldsymbol{\varphi}_1(\boldsymbol{\psi}_0(x), \boldsymbol{a}_{1,j}(x)) = \mathrm{e}^{-xA}\frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(x))}{\partial \boldsymbol{y}}\boldsymbol{a}_{1,j}(x)$$

and

$$\boldsymbol{\gamma}'_{0,(0),0}(x) = \mathrm{e}^{-xA}\left[-A\boldsymbol{f}(\boldsymbol{\psi}_0(x)) + \frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(x))}{\partial \boldsymbol{y}}\boldsymbol{\psi}'_0(x)\right].$$

Substituting the known values, it turns out that we have the following structure, emphasising only highly oscillatory terms:

$$\boldsymbol{\psi}_2(t,\omega) = \boldsymbol{a}_{2,0}(t) + \mathrm{e}^{\mathrm{i}\omega t}\boldsymbol{a}_{2,1}(t) + \mathrm{e}^{2\mathrm{i}\omega t}\boldsymbol{a}_{2,2}(t),$$

where

$$\begin{aligned}
\boldsymbol{a}_{2,0}(t) &= \mathrm{e}^{tA}\left\{A\boldsymbol{f}(\boldsymbol{y}_0) - \frac{\partial \boldsymbol{f}(\boldsymbol{y}_0)}{\partial \boldsymbol{y}}\left[A\boldsymbol{y}_0 + \tfrac{1}{2}\boldsymbol{f}(\boldsymbol{y}_0)\right]\right\},\\
\boldsymbol{a}_{2,1}(t) &= -A\boldsymbol{f}(\mathrm{e}^{tA}\boldsymbol{y}_0) + \mathrm{e}^{tA}\frac{\partial \boldsymbol{f}(\mathrm{e}^{tA}\boldsymbol{y}_0)}{\partial \boldsymbol{y}}\left[A\boldsymbol{y}_0 + \boldsymbol{f}(\boldsymbol{y_0})\right],\\
\boldsymbol{a}_{2,2}(t) &= -\tfrac{1}{2}\frac{\partial \boldsymbol{f}(\mathrm{e}^{tA}\boldsymbol{y}_0)}{\partial \boldsymbol{y}}\boldsymbol{f}(\mathrm{e}^{tA}\boldsymbol{y}_0).
\end{aligned}$$

Observe that the number of frequencies present in each of the $\boldsymbol{\psi}_s(t)$ functions grows as we increase s, and actually we have a straightforward pattern in accordance with our ansatz (2.3), in a simpler setting:

THEOREM 3.1. *For $s \ge 0$ the functions $\boldsymbol{\psi}_s(t)$ have the form of (finite) modulated Fourier expansions:*

$$\boldsymbol{\psi}_s(t) = \sum_{j=0}^{s}\boldsymbol{a}_{s,j}(t)\mathrm{e}^{\mathrm{i}j\omega t},$$

where each $\boldsymbol{a}_{s,j}(t)$ is non-oscillatory, and indeed independent of ω.

Proof. The result is clear for $s = 0$, suppose that it holds for s. It follows from the general formula (3.3) that

$$\boldsymbol{\psi}_{s+1}(t) = -\mathrm{e}^{tA}\sum_{k=0}^{s}\sum_{(m,\boldsymbol{n})\in\mathbb{I}_{s-k}}\frac{\mathrm{i}^{k+1}}{m!}\sum_{\substack{0\le j_i\le n_i\\ i=1,\ldots,m}}\frac{1}{(|\boldsymbol{j}|+1)^{k+1}}\left[\boldsymbol{\gamma}^{(k)}_{m,\boldsymbol{n},\boldsymbol{j}}(t)\mathrm{e}^{\mathrm{i}(|\boldsymbol{j}|+1)\omega t} - \boldsymbol{\gamma}^{(k)}_{m,\boldsymbol{n},\boldsymbol{j}}(0)\right].$$

Note that

$$|\boldsymbol{j}| + 1 \le |\boldsymbol{n}| + 1 \le s - k + 1 \le s + 1,$$

and therefore $\boldsymbol{\psi}_{s+1}(t)$ is indeed of the stipulated form and our *ansatz* is true. Moreover, the coefficients of the modulated Fourier expansion of $\boldsymbol{\psi}_s(t)$ can be written explicitly in the form

$$\begin{aligned}
\boldsymbol{a}_{s,0}(t) &= -\mathrm{e}^{tA}\sum_{k=1}^{s}\sum_{(m,\boldsymbol{n})\in\mathbb{I}_{s-k}}\frac{1}{m!}\sum_{\substack{0\le j_i\le n_i\\ i=1,\ldots,m}}\frac{\mathrm{i}^k}{(|\boldsymbol{j}|+1)^k}\,\boldsymbol{\gamma}^{(k-1)}_{m,\boldsymbol{n},\boldsymbol{j}}(0),\\
\boldsymbol{a}_{s,j}(t) &= \mathrm{e}^{tA}\sum_{k=1}^{s}\sum_{(m,\boldsymbol{n})\in\mathbb{I}_{s-k}}\frac{1}{m!}\sum_{\substack{0\le j_i\le n_i\\ i=1,\ldots,m}}\frac{\mathrm{i}^k}{(|\boldsymbol{j}|+1)^k}\,\boldsymbol{\gamma}^{(k-1)}_{m,\boldsymbol{n},\boldsymbol{j}}(t), \qquad j = 1,\ldots,s.
\end{aligned}$$

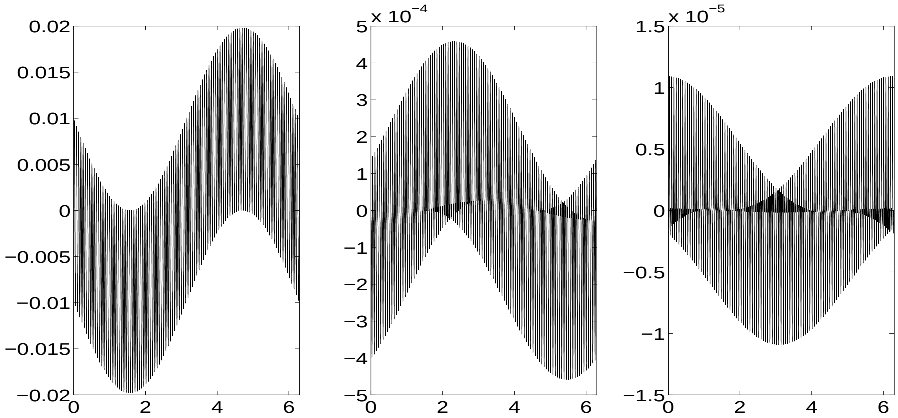

FIG. 3.1. *Real part of the error in the approximation of the solution of (3.5) for $\omega = 100$. Left to right, $e_s(t,\omega)$ for $s = 0, 1, 2$.*

□

We observe that this result rules out the possibility of resonance, as mentioned before, since $\boldsymbol{a}_{s,-1}(t) = \mathbf{0}$ for all $s \geq 0$.

To illustrate the performance of the approximation, let us go back to the examples from the introduction:

$$y' = 2\mathrm{i}y + \mathrm{e}^{\mathrm{i}\omega t}y^2, \quad t \geq 0, \qquad y(0) = 1, \tag{3.5}$$

$$y' = \mathrm{i}y - 5\mathrm{e}^{\mathrm{i}\omega t}y^2, \quad t \geq 0, \qquad y(0) = 1. \tag{3.6}$$

We can compute the first terms of the asymptotic approximation and compare with the exact solution. We define the absolute error

$$e_s(t) = y(t) - \sum_{m=0}^{s} \frac{\psi_m(t)}{\omega^m},$$

for $s \geq 0$. Figures (3.1) and (3.2) display the real part of the error for $s = 0, 1, 2$ for $\omega = 100$ and $\omega = 500$ (the imaginary part follows a similar pattern).

We note that in place of $g_\omega(t) = \mathrm{e}^{\mathrm{i}\omega t}$, we can take a more general exponential oscillator:

$$g_\omega(t) = \mathrm{e}^{\mathrm{i}\omega g(t)},$$

where $g' \neq 0$ in $[0, t]$: in that case

$$\int_0^t \boldsymbol{h}(x) g_\omega(x)\,\mathrm{d}x \sim -\sum_{k=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{k+1}} \left[\boldsymbol{h}_k(t) \frac{\mathrm{e}^{\mathrm{i}\omega g(t)}}{g'(t)} - \boldsymbol{h}_k(0) \frac{\mathrm{e}^{\mathrm{i}\omega g(0)}}{g'(0)} \right],$$

where

$$\boldsymbol{h}_0(x) = \boldsymbol{h}(x), \qquad \boldsymbol{h}_{k+1}(x) = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\boldsymbol{h}_k(x)}{g'(x)}, \quad k \in \mathbb{Z}_+.$$

The results can be extended to this case without any other conceptual difficulty, at the price of more complicated algebra.

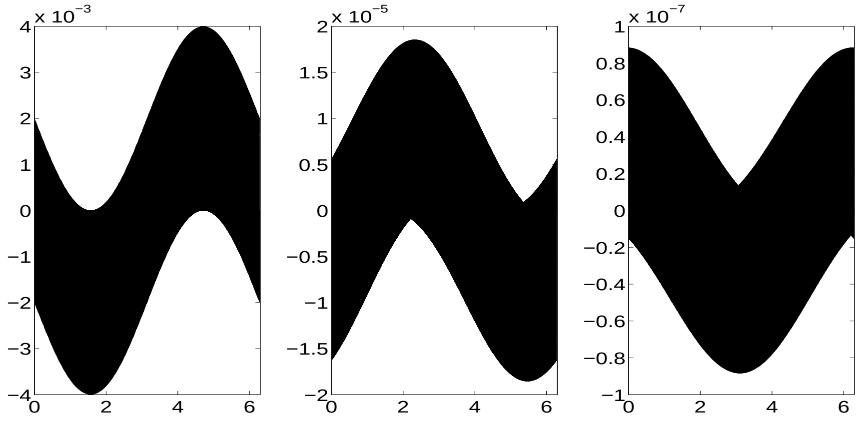

FIG. 3.2. *Same as figure 3.1 but with* $\omega = 500$.

4. The Expcos oscillator and digital modulation. Another generalisation of the previous model that one can consider is given by a forcing term that has a proper modulated Fourier expansion:

$$g_\omega(t) = \sum_{j=-\infty}^{\infty} \alpha_j(t) \mathrm{e}^{\mathrm{i}j\omega t}.$$

In this case similar analysis is possible, although the general structure of the $\boldsymbol{\psi}_s(t)$ functions is bound to be more complicated. Note in particular that if $\alpha_0(t) \neq 0$ then the oscillator is of hard type, according to our previous definition.

4.1. General setting. An important case, significantly more complicated than that of the Fourier oscillator, is given by

$$g_\omega(t) = \mathrm{e}^{\eta \cos \omega t},$$

already considered in [3], [4]. The Fourier expansion of the oscillator is given in terms of modified Bessel functions:

$$\mathrm{e}^{\eta \cos \omega t} = I_0(\eta) + 2 \sum_{m=1}^{\infty} I_m(\eta) \cos m\omega t$$

see [1, Eq. 9.6.34]. The series converges very fast for fixed values of η due to the asymptotic behaviour of the modified Bessel functions for large orders, see for instance [1, Eq. 9.7.7], and the convergence is also uniform for fixed values of t.

This expansion implies that this oscillator $g_\omega(t)$ introduces a non-oscillatory contribution, since integration term by term gives

$$\int_0^t \boldsymbol{h}(x) g_\omega(x)\, \mathrm{d}x = I_0(\eta) \int_0^t \boldsymbol{h}(x)\, \mathrm{d}x + 2 \sum_{m=1}^{\infty} I_m(\eta) \int_0^t \boldsymbol{h}(x) \cos m\omega x\, \mathrm{d}x,$$

for any smooth function $\boldsymbol{h}(t)$. Therefore in this case $g_\omega(t)$ is a hard oscillator.

A more general case of this oscillator can be obtained by allowing η to vary

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + \mathrm{e}^{\eta(t) \cos \omega t} \boldsymbol{f}(\boldsymbol{y}(t)), \quad t \geq 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0. \tag{4.1}$$

In a typical example in applications, the function $\eta(t)$ will be piecewise constant

$$\eta(t) \equiv \eta_i, \quad t \in [\tau_i, \tau_{i+1}) \qquad m \in \mathbb{Z}_+,$$

where

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots \qquad \text{and} \qquad \theta_0, \theta_1, \ldots \in \mathbb{R}$$

are given. We suppose that

$$\min \omega(\tau_{i+1} - \tau_i) \gg 1,$$

so the oscillator $\cos \omega t$ undergoes many oscillations in each interval $[\tau_i, \tau_{i+1}]$. Thus we want to solve the ODE

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + \mathrm{e}^{\eta(t)\cos\omega t}\boldsymbol{f}(\boldsymbol{y}(t)), \qquad \boldsymbol{y}(\tau_i) = \boldsymbol{y}_i,$$

where $t \in [\tau_i, \tau_{i+1})$ and the initial values $\boldsymbol{y}_i$ in each subinterval are given. Naturally, if $\eta(t)$ is constant we will set $i = 0$ and $\tau_i = 0$. Also, in order to simplify notation, we will omit the subscript i in the sequel, in the understanding that η or τ stand for η_i or τ_i if needed.

As before, we assume that $\boldsymbol{y}(t)$ admits an asymptotic representation in inverse powers of ω, where each term $\boldsymbol{\psi}_n(t)$ can be expressed as a modulated Fourier series:

$$\boldsymbol{\psi}_n(t) = \sum_{j=-\infty}^{\infty} \boldsymbol{a}_{n,j}(t)\mathrm{e}^{\mathrm{i}j\omega t}, \qquad n \in \mathbb{N}. \tag{4.2}$$

The first term, $\boldsymbol{\psi}_0(t)$, obeys the base nonoscillatory ODE:

$$\boldsymbol{\psi}_0'(t) = A\boldsymbol{\psi}_0(t) + I_0(\eta)\boldsymbol{f}(\boldsymbol{\psi}_0(t)), \quad t \geq 0, \qquad \boldsymbol{\psi}_0(\tau) = \boldsymbol{y}(\tau). \tag{4.3}$$

Note than, unlike what happened in the case of the Fourier oscillator, this base equation is in general nonlinear, and the solution may be not be available analytically. Yet, being nonoscillatory, it can be computed easily with conventional numerical software.

Using the linearity of $\boldsymbol{\varphi}_m$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{\omega^n}\boldsymbol{\psi}_n(t) \\
&= \mathrm{e}^{tA}\boldsymbol{y}_0 \\
&\quad + \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{\omega^{|\boldsymbol{n}|}} \mathrm{e}^{tA} \int_\tau^t \mathrm{e}^{-xA}\boldsymbol{\varphi}_m(\boldsymbol{\psi}_0(x), \boldsymbol{\psi}_{n_1}(x), \ldots, \boldsymbol{\psi}_{n_m}(x))\mathrm{e}^{\eta\cos\omega x}\,\mathrm{d}x \\
&= \mathrm{e}^{tA}\boldsymbol{y}_0 + \mathrm{e}^{tA}\int_\tau^t \mathrm{e}^{-xA}\boldsymbol{f}(\boldsymbol{\psi}_0(x))\mathrm{e}^{\eta\cos\omega(x+\tau)}\,\mathrm{d}x \\
&\quad + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{\omega^{|\boldsymbol{n}|}} \sum_{j_1,\ldots,j_m=-\infty}^{\infty} \mathrm{e}^{tA} \int_\tau^t \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(x)\mathrm{e}^{\mathrm{i}|\boldsymbol{j}|\omega x}\mathrm{e}^{\eta\cos\omega x}\,\mathrm{d}x
\end{aligned}$$

where as usual

$$\boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(x) = \mathrm{e}^{-xA}\boldsymbol{\varphi}_m(\boldsymbol{\psi}_0(x), \boldsymbol{a}_{n_1,j_1}(x), \ldots, \boldsymbol{a}_{n_m,j_m}(x)).$$

Now we need an asymptotic expansion of the last integral in inverse powers of ω. Given $q \in \mathbb{N}$ and a smooth function $\boldsymbol{h}(t)$, we observe that

$$\begin{aligned}
&\int_\tau^t \boldsymbol{h}(x)\mathrm{e}^{\mathrm{i}q\omega x}\mathrm{e}^{\eta\cos\omega x}\,\mathrm{d}x \\
&\sim I_q(\eta)\int_\tau^t \boldsymbol{h}(x)\,\mathrm{d}x + \sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}[\boldsymbol{h}^{(2k)}(t)\alpha_{q,k}(t) - \boldsymbol{h}^{(2k)}(\tau)\alpha_{q,k}(\tau)] \\
&+ \sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}[\boldsymbol{h}^{(2k+1)}(t)\beta_{q,k}(t) - \boldsymbol{h}^{(2k+1)}(\tau)\beta_{q,k}(\tau)],
\end{aligned}$$

where, for every $q \in \mathbb{N}$, $k \in \mathbb{Z}_+$,

$$\begin{aligned}
\alpha_{q,k}(x) &= -\mathrm{i}\sum_{j=1}^{\infty}\frac{I_{q-j}(\eta)\mathrm{e}^{\mathrm{i}j\omega x} - I_{q+j}(\eta)\mathrm{e}^{-\mathrm{i}j\omega x}}{j^{2k+1}}, \\
\beta_{q,k}(x) &= \sum_{j=1}^{\infty}\frac{I_{q-j}(\eta)\mathrm{e}^{\mathrm{i}j\omega x} + I_{q+j}(\eta)\mathrm{e}^{-\mathrm{i}j\omega x}}{j^{2k+2}}.
\end{aligned}$$

Since for all $x \in \mathbb{R}$

$$\overline{\alpha_{q,k}(x)} = \alpha_{-q,k}(x), \qquad \overline{\beta_{q,k}(x)} = \beta_{-q,k}(x),$$

we can extend the definition of $\alpha_{q,k}$ and $\beta_{q,k}$ to all $q \in \mathbb{Z}$ and obtain an asymptotic expansion valid for all $q \in \mathbb{Z}$, $q \neq 0$. Moreover, for $q = 0$ we have

$$\alpha_{0,k}(x) = S_k(x), \qquad \beta_{0,k}(x) = C_k(x),$$

where

$$S_k(x) = 2\sum_{m=1}^{\infty}\frac{I_m(\eta)}{m^{2k+1}}\sin m\omega x, \qquad C_k(x) = 2\sum_{m=1}^{\infty}\frac{I_m(\eta)}{m^{2k+2}}\cos m\omega x. \tag{4.4}$$

In the sequel we will omit the parameter η if no confusion arises, in order to simplify notation. We also observe that given the asymptotic behaviour of the modified Bessel functions for large order m, the previous series converge absolutely and uniformly for fixed values of η and k and for $x \in \mathbb{R}$.

Using this expansion, we can write

$$\begin{aligned}
\boldsymbol{y}(t) &\sim \sum_{n=0}^{\infty}\frac{1}{\omega^n}\boldsymbol{\psi}_n(t) \\
&= \mathrm{e}^{tA}\boldsymbol{y}_0 + \sum_{s=0}^{\infty}\frac{1}{\omega^s}\sum_{(m,\boldsymbol{n})\in\mathbb{I}_s}\frac{1}{m!}\sum_{j_1,\ldots,j_m=-\infty}^{\infty}\left\{I_{|\boldsymbol{j}|}(\eta)\mathrm{e}^{tA}\int_\tau^t \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(x)\,\mathrm{d}x\right. \\
&\quad + \sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}\mathrm{e}^{tA}\left[\boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}^{(2k)}(t)\alpha_{|\boldsymbol{j}|,k}(t) - \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}^{(2k)}(\tau)\alpha_{|\boldsymbol{j}|,k}(\tau)\right] \\
&\quad \left. + \sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}\mathrm{e}^{tA}\left[\boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}^{(2k+1)}(t)\beta_{|\boldsymbol{j}|,k}(t) - \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}^{(2k+1)}(\tau)\beta_{|\boldsymbol{j}|,k}(\tau)\right]\right\}
\end{aligned}$$

The procedure now consists of collecting all the $\mathcal{O}(\omega^{-p})$ terms on the right and equating them to $\omega^{-p}\boldsymbol{\psi}_p(t)$. We do so separately for odd and even p. Thus, for any $p \in \mathbb{Z}_+$,

$$
\begin{aligned}
\boldsymbol{\psi}_{2p+1}(t) = & \sum_{(m,\boldsymbol{n})\in\mathbb{I}_{2p+1}} \frac{1}{m!} \sum_{j_1,\ldots,j_m=-\infty}^{\infty} I_{|\boldsymbol{j}|}(\eta)\mathrm{e}^{tA} \int_\tau^t \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(x)\,\mathrm{d}x \\
+ \sum_{k=0}^{p} & \sum_{(m,\boldsymbol{n})\in\mathbb{I}_{2(p-k)}} \frac{(-1)^k}{m!} \sum_{j_1,\ldots,j_m=-\infty}^{\infty} \mathrm{e}^{tA}\left[\boldsymbol{\gamma}^{(2k)}_{m,\boldsymbol{n},\boldsymbol{j}}(t)\alpha_{|\boldsymbol{j}|,k}(t) - \boldsymbol{\gamma}^{(2k)}_{m,\boldsymbol{n},\boldsymbol{j}}(\tau)\alpha_{|\boldsymbol{j}|,k}(\tau)\right] \\
+ \sum_{k=0}^{p-1} & \sum_{(m,\boldsymbol{n})\in\mathbb{I}_{2(p-k)-1}} \frac{(-1)^k}{m!} \sum_{j_1,\ldots,j_m=-\infty}^{\infty} \mathrm{e}^{tA}\left[\boldsymbol{\gamma}^{(2k+1)}_{m,\boldsymbol{n},\boldsymbol{j}}(t)\beta_{|\boldsymbol{j}|,k}(t) - \boldsymbol{\gamma}^{(2k+1)}_{m,\boldsymbol{n},\boldsymbol{j}}(\tau)\beta_{|\boldsymbol{j}|,k}(\tau)\right].
\end{aligned}
\tag{4.5}
$$

Likewise, for all $p \in \mathbb{Z}_+$,

$$
\begin{aligned}
\boldsymbol{\psi}_{2p+2}(t) = & \sum_{(m,\boldsymbol{n})\in\mathbb{I}_{2p+2}} \frac{1}{m!} \sum_{j_1,\ldots,j_m=-\infty}^{\infty} I_{|\boldsymbol{j}|}(\eta)\mathrm{e}^{tA} \int_\tau^t \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(x)\,\mathrm{d}x \\
+ \sum_{k=0}^{p} & \sum_{(m,\boldsymbol{n})\in\mathbb{I}_{2(p-k)+1}} \frac{(-1)^k}{m!} \sum_{j_1,\ldots,j_m=-\infty}^{\infty} \mathrm{e}^{tA}\left[\boldsymbol{\gamma}^{(2k)}_{m,\boldsymbol{n},\boldsymbol{j}}(t)\alpha_{|\boldsymbol{j}|,k}(t) - \boldsymbol{\gamma}^{(2k)}_{m,\boldsymbol{n},\boldsymbol{j}}(\tau)\alpha_{|\boldsymbol{j}|,k}(\tau)\right] \\
+ \sum_{k=0}^{p} & \sum_{(m,\boldsymbol{n})\in\mathbb{I}_{2(p-k)}} \frac{(-1)^k}{m!} \sum_{j_1,\ldots,j_m=-\infty}^{\infty} \mathrm{e}^{tA}\left[\boldsymbol{\gamma}^{(2k+1)}_{m,\boldsymbol{n},\boldsymbol{j}}(t)\beta_{|\boldsymbol{j}|,k}(t) - \boldsymbol{\gamma}^{(2k+1)}_{m,\boldsymbol{n},\boldsymbol{j}}(\tau)\beta_{|\boldsymbol{j}|,k}(\tau)\right].
\end{aligned}
\tag{4.6}
$$

We remark that when we try to solve for the modulated Fourier coefficients of, say, $\boldsymbol{\psi}_{2s+1}(t)$, those coefficients appear on the right hand side under the integral sign as well, due to the presence of the index $(2p+1,(1,1,\ldots,1)) \in \mathbb{I}_{2p+1}$ in the first sum. The standard procedure in this case is first to identify the coefficients $\boldsymbol{a}_{2s+1,j}(t)$ for $j \neq 0$ and then solve a differential (or integral) equation for the non-oscillatory term $\boldsymbol{a}_{2s+1,0}(t)$. Similar reasoning, with obvious changes, applies to the computation of $\boldsymbol{\psi}_{2s+2}(t)$.

We illustrate this idea in the next subsections by computing $\boldsymbol{\psi}_1(t)$ and $\boldsymbol{\psi}_2(t)$.

4.2. The first term. Using (4.5) with $p=0$ we obtain:

$$
\begin{aligned}
\boldsymbol{\psi}_1(t) = & \sum_{(m,\boldsymbol{n})\in\mathbb{I}_1} \frac{1}{m!} \sum_{j_1,\ldots,j_m=-\infty}^{\infty} I_{|\boldsymbol{j}|}(\eta)\mathrm{e}^{tA} \int_\tau^t \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(x)\,\mathrm{d}x \\
+ & \sum_{(m,\boldsymbol{n})\in\mathbb{I}_0} \frac{1}{m!} \sum_{j_1,\ldots,j_m=-\infty}^{\infty} \mathrm{e}^{tA}\left[\boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(t)\alpha_{|\boldsymbol{j}|,0}(t) - \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(\tau)\alpha_{|\boldsymbol{j}|,0}(\tau)\right] \\
= & \sum_{j=-\infty}^{\infty} I_j(\eta)\mathrm{e}^{tA}\int_\tau^t \boldsymbol{\gamma}_{1,(1),j}(x)\,\mathrm{d}x + \mathrm{e}^{tA}\left[\boldsymbol{\gamma}_{0,(0),0}(t)\alpha_{0,0}(t) - \boldsymbol{\gamma}_{0,(0),0}(\tau)\alpha_{0,0}(\tau)\right].
\end{aligned}
$$

Since for all $x \in \mathbb{R}$

$$
\boldsymbol{\gamma}_{0,(0),0}(x) = \mathrm{e}^{-xA}\boldsymbol{\varphi}_0(\boldsymbol{\psi}_0(x)) = \mathrm{e}^{-xA}\boldsymbol{f}(\boldsymbol{\psi}_0(x)),
$$

$$\boldsymbol{\gamma}_{1,(1),j}(x) = \mathrm{e}^{-xA}\boldsymbol{\varphi}_1(\boldsymbol{\psi}_0(x), \boldsymbol{a}_{1,j}(x)) = \mathrm{e}^{-xA}\frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(x))}{\partial \boldsymbol{y}}\boldsymbol{a}_{1,j}(x),$$

and $\alpha_{0,0}(x) = S_0(x)$, it follows that if we write $\boldsymbol{\psi}_1(t)$ in modulated Fourier series, then

$$\begin{aligned}\boldsymbol{\psi}_1(t) = \sum_{j=-\infty}^{\infty} \boldsymbol{a}_{1,j}(t)\mathrm{e}^{\mathrm{i}j\omega t} = \mathrm{e}^{tA}\sum_{j=-\infty}^{\infty} I_j(\eta)\int_\tau^t \mathrm{e}^{-xA}\frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(x))}{\partial \boldsymbol{y}}\boldsymbol{a}_{1,j}(x)\,\mathrm{d}x \\ + \boldsymbol{f}(\boldsymbol{\psi}_0(t))S_0(t) - \mathrm{e}^{(t-\tau)A}\boldsymbol{f}(\boldsymbol{y}(\tau))S_0(\tau).\end{aligned}$$

Note the presence of the coefficients $\boldsymbol{a}_{1,j}(t)$ on both sides of the equation due to the first index $(1,(1))$, as we mentioned before. Identifying coefficients we get

$$\boldsymbol{a}_{1,j}(t) = \frac{-\mathrm{i}I_j(\eta)}{j}\boldsymbol{f}(\boldsymbol{\psi}_0(t)), \qquad j \neq 0,$$

and $\boldsymbol{a}_{1,j}(t) = -\boldsymbol{a}_{1,-j}(t)$ for $j \geq 1$, using the fact that for integer orders $I_{-n}(\eta) = I_n(\eta)$, see for instance [1, Eq. 9.6.6]. The nonoscillatory term $\boldsymbol{a}_{1,0}(t)$ is implicitly defined by

$$\boldsymbol{a}_{1,0}(t) = \mathrm{e}^{tA}I_0(\eta)\int_\tau^t \mathrm{e}^{-xA}\frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(x))}{\partial \boldsymbol{y}}\boldsymbol{a}_{1,0}(x)\,\mathrm{d}x - \mathrm{e}^{(t-\tau)A}\boldsymbol{f}(\boldsymbol{y}(\tau))S_0(\tau).$$

Observe that because of the symmetry of the coefficients $\boldsymbol{a}_{1,j}(t)$ for $j \neq 0$, the only remaining term in the sum is the one corresponding to $j = 0$. Multiplying by e^{-tA}, differentiating, multiplying by e^{tA} and rearranging term results in the ODE

$$\boldsymbol{a}'_{1,0}(t) = \left[A + I_0(\eta)\frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(t))}{\partial \boldsymbol{y}}\right]\boldsymbol{a}_{1,0}(t), \qquad \boldsymbol{a}_{1,0}(\tau) = -\boldsymbol{f}(\boldsymbol{y}(\tau))S_0(\tau). \tag{4.7}$$

In general, we cannot write the solution of (4.7) (with the given initial values) explicitly, but it can be approximated very well with standard numerical methods for ODEs, since there are no oscillatory components present. In fact, we can write:

$$\boldsymbol{a}_{1,0}(t) = -\boldsymbol{\Omega}(t)\boldsymbol{f}(\boldsymbol{y}(\tau))S_0(\tau), \tag{4.8}$$

where $\boldsymbol{\Omega}(t)$ is the solution of the matrix ODE

$$\boldsymbol{\Omega}'(t) = \left[A + I_0(\eta)\frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(t))}{\partial \boldsymbol{y}}\right]\boldsymbol{\Omega}(t), \qquad \boldsymbol{\Omega}(\tau) = I. \tag{4.9}$$

Incidentally, note that this is the variational equation corresponding to the ODE (4.3).

Finally, assembling everything we obtain

$$\boldsymbol{\psi}_1(t) = \boldsymbol{f}(\boldsymbol{\psi}_0(t))S_0(t) - \boldsymbol{\Omega}(t)\boldsymbol{f}(\boldsymbol{y}(\tau))S_0(\tau), \tag{4.10}$$

Observe that $\boldsymbol{\psi}_1(\tau) = \boldsymbol{y}_i$, which is consistent with the way we have imposed the initial conditions in (4.3).

4.3. The second term. Once we substitute $p=0$ in (4.6), we obtain

$$\boldsymbol{\psi}_2(t) = \sum_{(m,\boldsymbol{n})\in\mathbb{I}_2} \frac{1}{m!} \sum_{j_1,\ldots,j_m=-\infty}^{\infty} I_{|\boldsymbol{j}|}(\eta)\mathrm{e}^{tA} \int_\tau^t \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(x)\,\mathrm{d}x \tag{4.11}$$

$$+ \sum_{(m,\boldsymbol{n})\in\mathbb{I}_1} \frac{1}{m!} \sum_{j_1,\ldots,j_m=-\infty}^{\infty} \mathrm{e}^{tA}\left[\boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(t)\alpha_{|\boldsymbol{j}|,0}(t) - \boldsymbol{\gamma}_{m,\boldsymbol{n},\boldsymbol{j}}(\tau)\alpha_{|\boldsymbol{j}|,0}(\tau)\right]$$

$$+ \sum_{(m,\boldsymbol{n})\in\mathbb{I}_0} \frac{1}{m!} \sum_{j_1,\ldots,j_m=-\infty}^{\infty} \mathrm{e}^{tA}\left[\boldsymbol{\gamma}'_{m,\boldsymbol{n},\boldsymbol{j}}(t)\beta_{|\boldsymbol{j}|,0}(t) - \boldsymbol{\gamma}'_{m,\boldsymbol{n},\boldsymbol{j}}(\tau)\beta_{|\boldsymbol{j}|,0}(\tau)\right].$$

The first term comprises two different sums, originating in the two indices in $\mathbb{I}_2$, which are $(m,\boldsymbol{n})=(1;2)$ and $(m,\boldsymbol{n})=(2;1,1)$. Specifically:

$$\sigma_1(t) = \sum_{j=-\infty}^{\infty} I_j(\eta)\mathrm{e}^{tA}\int_\tau^t \boldsymbol{\gamma}_{1,(2),j}(x)\,\mathrm{d}x = \sum_{j=-\infty}^{\infty} I_j(\eta)\mathrm{e}^{tA}\int_\tau^t \mathrm{e}^{-xA}\boldsymbol{\varphi}_1(\boldsymbol{\psi}_0(x),\boldsymbol{a}_{2,j}(x))\,\mathrm{d}x,$$

and

$$\begin{aligned}\sigma_2(t) &= \tfrac{1}{2}\sum_{j_1,j_2=-\infty}^{\infty} I_{j_1+j_2}(\eta)\mathrm{e}^{tA}\int_\tau^t \boldsymbol{\gamma}_{2,(1,1),(j_1,j_2)}(x)\,\mathrm{d}x\\ &= \tfrac{1}{2}\sum_{j_1,j_2=-\infty}^{\infty} I_{j_1+j_2}(\eta)\mathrm{e}^{tA}\int_\tau^t \mathrm{e}^{-xA}\boldsymbol{\varphi}_2(\boldsymbol{\psi}_0(x),\boldsymbol{a}_{1,j_1}(x),\boldsymbol{a}_{1,j_2}(x))\,\mathrm{d}x,\end{aligned}$$

With regard this last sum $\sigma_2(t)$, we can prove the following

PROPOSITION 4.1. *The term $\sigma_2(t)$ satisfies:*

$$\begin{aligned}\sigma_2(t) &= \mathrm{e}^{tA}\rho\int_\tau^t \mathrm{e}^{-xA}\boldsymbol{\varphi}_2(\boldsymbol{\psi}_0(x),\boldsymbol{f}(\boldsymbol{\psi}_0(x)),\boldsymbol{f}(\boldsymbol{\psi}_0(x)))\,\mathrm{d}x\\ &\quad+ \tfrac{1}{2}I_0(\eta)S_0^2(\tau)\mathrm{e}^{tA}\int_\tau^t \mathrm{e}^{-xA}\boldsymbol{\varphi}_2(\boldsymbol{\psi}_0(x),\boldsymbol{\Omega}(x)\boldsymbol{f}(\boldsymbol{y}(\tau)),\boldsymbol{\Omega}(x)\boldsymbol{f}(\boldsymbol{y}(\tau)))\,\mathrm{d}x + \mathcal{O}(\omega^{-1}),\end{aligned}$$

where

$$\rho = I_0(\eta)\sum_{j=1}^{\infty}\frac{I_j^2(\eta)}{j^2}.$$

Proof. We note that if we substitute the explicit value of the coefficients $\boldsymbol{a}_{1,j}(t)$, then we obtain

$$\begin{aligned}&\tfrac{1}{2}\sum_{j_1,j_2=-\infty}^{\infty} I_{j_1+j_2}(\eta)\mathrm{e}^{tA}\int_\tau^t \mathrm{e}^{-xA}\boldsymbol{\varphi}_2(\boldsymbol{\psi}_0(x),\boldsymbol{a}_{1,j_1}(x),\boldsymbol{a}_{1,j_2}(x))\,\mathrm{d}x,\\ &= -\tfrac{1}{2}\sum_{j_1,j_2\neq 0} I_{j_1+j_2}(\eta)\frac{I_{j_1}(\eta)}{j_1}\frac{I_{j_2}(\eta)}{j_2}\mathrm{e}^{tA}\int_\tau^t \mathrm{e}^{-xA}\boldsymbol{\varphi}_2(\boldsymbol{\psi}_0(x),\boldsymbol{f}(\boldsymbol{\psi}_0(x)),\boldsymbol{f}(\boldsymbol{\psi}_0(x))\,\mathrm{d}x,\\ &+ \tfrac{1}{2}I_0(\eta)S_0^2(\tau)\mathrm{e}^{tA}\int_\tau^t \mathrm{e}^{-xA}\boldsymbol{\varphi}_2(\boldsymbol{\psi}_0(x),\boldsymbol{\Omega}(x)\boldsymbol{f}(\boldsymbol{y}(\tau)),\boldsymbol{\Omega}(x)\boldsymbol{f}(\boldsymbol{y}(\tau)))\,\mathrm{d}x + \mathcal{O}(\omega^{-1}).\end{aligned}$$

Observe that when only one of the indices j_1 or j_2 is equal to 0 then all the terms in the sum cancel because of the symmetry of the coefficients $\boldsymbol{a}_{1,j}(t)$ with respect to j, except when $j_1 = j_2 = 0$, which corresponds to the last integral above.

Now, using again the property that $I_{-n}(\eta) = I_n(\eta)$ for integer n and the fact that we can reorder the double sum due to absolute convergence, we have

$$
\begin{aligned}
-\tfrac{1}{2} &\sum_{j_1,j_2\neq 0} I_{j_1+j_2}(\eta)\frac{I_{j_1}(\eta)}{j_1}\frac{I_{j_2}(\eta)}{j_2} \\
&= -\sum_{j_1,j_2=1}^{\infty} \frac{I_{j_1}(\eta)I_{j_2}(\eta)}{j_1 j_2}\left[I_{j_1+j_2}(\eta) - I_{j_1-j_2}(\eta)\right] \\
&= -\sum_{j_1,j_2=1}^{\infty} \frac{I_{j_1}(\eta)I_{j_2}(\eta)I_{j_1+j_2}(\eta)}{j_1 j_2 (j_1+j_2)}(j_1 - j_2) + I_0(\eta)\sum_{j=1}^{\infty}\frac{I_j^2(\eta)}{j^2} \\
&= I_0(\eta)\sum_{j=1}^{\infty}\frac{I_j^2(\eta)}{j^2},
\end{aligned}
$$

since the first sum vanishes due to symmetry. ☐

The second term in (4.11) yields

$$
\sigma_3(t) = \sum_{j=-\infty}^{\infty}\left[\boldsymbol{\varphi}_1(\boldsymbol{\psi}_0(t), \boldsymbol{a}_{1,j}(t))\alpha_{j,0}(t) - \mathrm{e}^{tA}\boldsymbol{\varphi}_1(\boldsymbol{\psi}_0(\tau), \boldsymbol{a}_{1,j}(\tau))\alpha_{j,0}(\tau)\right].
$$

Finally, the last one is

$$
\begin{aligned}
\sigma_4(t) &= \mathrm{e}^{tA}\left[\boldsymbol{\gamma}'_{0,(0),0}(t)\beta_{0,0}(t) - \boldsymbol{\gamma}'_{0,(0),0}(\tau)\beta_{0,0}(\tau)\right] \\
&= \mathrm{e}^{tA}\left[\boldsymbol{\gamma}'_{0,(0),0}(t)C_0(t) - \boldsymbol{\gamma}'_{0,(0),0}(\tau)C_0(\tau)\right].
\end{aligned}
$$

Note that $\boldsymbol{\gamma}_{0,(0),0}(x) = \mathrm{e}^{-xA}\boldsymbol{f}(\boldsymbol{\psi}_0(x))$, and therefore

$$
\boldsymbol{\gamma}'_{0,(0),0}(x) = \mathrm{e}^{-xA}\left[-A\boldsymbol{f}(\boldsymbol{\psi}_0(x)) + \frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(x))}{\partial \boldsymbol{y}}\boldsymbol{\psi}'_0(x)\right],
$$

where we can substitute the base equation $\boldsymbol{\psi}'_0(x) = A\boldsymbol{\psi}_0(x) + I_0(\eta)\boldsymbol{f}(\boldsymbol{\psi}_0(x))$. Assembling everything together, we get:

$$
\begin{aligned}
\sum_{j=-\infty}^{\infty}\boldsymbol{a}_{2,j}(t)\mathrm{e}^{\mathrm{i}j\omega t} &= \sum_{j=-\infty}^{\infty} I_j(\eta)\mathrm{e}^{tA}\int_\tau^t \mathrm{e}^{-xA}\boldsymbol{\varphi}_1(\boldsymbol{\psi}_0(x), \boldsymbol{a}_{2,j}(x))\,\mathrm{d}x \qquad (4.12) \\
&+ \mathrm{e}^{tA}\rho\int_\tau^t \mathrm{e}^{-xA}\boldsymbol{\varphi}_2(\boldsymbol{\psi}_0(x), \boldsymbol{f}(\boldsymbol{\psi}_0(x)), \boldsymbol{f}(\boldsymbol{\psi}_0(x)))\,\mathrm{d}x \\
&+ \tfrac{1}{2}I_0(\eta)S_0^2(\tau)\mathrm{e}^{tA}\int_\tau^t \mathrm{e}^{-xA}\boldsymbol{\varphi}_2(\boldsymbol{\psi}_0(x), \boldsymbol{\Omega}(x)\boldsymbol{f}(\boldsymbol{y}(\tau)), \boldsymbol{\Omega}(x)\boldsymbol{f}(\boldsymbol{y}(\tau)))\,\mathrm{d}x \\
&+ \sum_{j=-\infty}^{\infty}\left[\boldsymbol{\varphi}_1(\boldsymbol{\psi}_0(t), \boldsymbol{a}_{1,j}(t))\alpha_{j,0}(t) - \mathrm{e}^{tA}\boldsymbol{\varphi}_1(\boldsymbol{y}(\tau), \boldsymbol{a}_{1,j}(\tau))\alpha_{j,0}(\tau)\right] \\
&+ \mathrm{e}^{tA}\left[\boldsymbol{\gamma}'_{0,(0),0}(t)C_0(t) - \boldsymbol{\gamma}'_{0,(0),0}(\tau)C_0(\tau)\right].
\end{aligned}
$$

Identifying oscillatory terms, we deduce that for $j \neq 0$

$$\boldsymbol{a}_{2,j}(t) = -\frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(t))}{\partial \boldsymbol{y}} \boldsymbol{f}(\boldsymbol{\psi}_0(t)) \sum_{s\neq 0} \frac{I_s(\eta) I_{j-s}(\eta)}{js} + \mathrm{e}^{tA} \boldsymbol{\gamma}'_{0,(0),0}(t) \frac{I_j(\eta)}{j^2} \tag{4.13}$$
$$+ \mathrm{i} \frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(t))}{\partial \boldsymbol{y}} \boldsymbol{\Omega}(t) \boldsymbol{f}(\boldsymbol{y}(\tau)) S_0(\tau) \frac{I_j(\eta)}{j}$$

In order to construct the term $\boldsymbol{a}_{2,0}(t)$, we observe first that

$$\sum_{j\neq 0} I_j(\eta) \mathrm{e}^{tA} \int_\tau^t \mathrm{e}^{-xA} \boldsymbol{\varphi}_1(\boldsymbol{\psi}_0(x), \boldsymbol{a}_{2,j}(x))\,\mathrm{d}x$$
$$= -\sum_{j\neq 0} I_j(\eta) \sum_{s\neq 0} \frac{I_s(\eta) I_{j-s}(\eta)}{js} \mathrm{e}^{tA} \int_\tau^t \mathrm{e}^{-xA} \left[\frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(x))}{\partial \boldsymbol{y}}\right]^2 \boldsymbol{f}(\boldsymbol{\psi}_0(x))\,\mathrm{d}x$$
$$+ 2\sum_{j=1}^\infty \frac{I_j^2(\eta)}{j^2} \mathrm{e}^{tA} \int_\tau^t \frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(x))}{\partial \boldsymbol{y}} \boldsymbol{\gamma}'_{0,(0),0}(x)\,\mathrm{d}x$$
$$= -2\rho \mathrm{e}^{tA} \int_\tau^t \mathrm{e}^{-xA} \left[\frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(x))}{\partial \boldsymbol{y}}\right]^2 \boldsymbol{f}(\boldsymbol{\psi}_0(x))\,\mathrm{d}x$$
$$+ 2\sum_{j=1}^\infty \frac{I_j^2(\eta)}{j^2} \mathrm{e}^{tA} \int_\tau^t \frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(x))}{\partial \boldsymbol{y}} \boldsymbol{\gamma}'_{0,(0),0}(x)\,\mathrm{d}x.$$

Multiplying (4.12) by e^{-tA}, differentiating and multiplying by e^{tA}, we get

$$\boldsymbol{a}'_{2,0}(t) = \left[A + I_0(\eta) \frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(t))}{\partial \boldsymbol{y}}\right] \boldsymbol{a}_{2,0}(t) + \boldsymbol{B}(t), \tag{4.14}$$

where

$$\boldsymbol{B}(t) = \rho \left(\boldsymbol{\varphi}_2(\boldsymbol{\psi}_0(t), \boldsymbol{f}(\boldsymbol{\psi}_0(t)), \boldsymbol{f}(\boldsymbol{\psi}_0(t)) - 2\left[\frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(t))}{\partial \boldsymbol{y}}\right]^2 \boldsymbol{f}(\boldsymbol{\psi}_0(t))\right) \tag{4.15}$$
$$+ 2\sum_{j=1}^\infty \frac{I_j^2(\eta)}{j^2} \mathrm{e}^{tA} \frac{\partial \boldsymbol{f}(\boldsymbol{\psi}_0(t))}{\partial \boldsymbol{y}} \boldsymbol{\gamma}'_{0,(0),0}(t)$$
$$+ \tfrac{1}{2} I_0(\eta) S_0^2(\tau) \boldsymbol{\varphi}_2(\boldsymbol{\psi}_0(t), \boldsymbol{\Omega}(t)\boldsymbol{f}(\boldsymbol{y}(\tau)), \boldsymbol{\Omega}(t)\boldsymbol{f}(\boldsymbol{y}(\tau))),$$

with the initial condition given by $\boldsymbol{\psi}_2(\tau) = \boldsymbol{0}$, that is

$$\boldsymbol{a}_{2,0}(\tau) = -\sum_{j\neq 0} \boldsymbol{a}_{2,j}(\tau) \mathrm{e}^{\mathrm{i}j\omega\tau}$$
$$= \frac{\partial \boldsymbol{f}(\boldsymbol{y}(\tau))}{\partial \boldsymbol{y}} \boldsymbol{f}(\boldsymbol{y}(\tau)) \sum_{j\neq 0} \mathrm{e}^{\mathrm{i}j\omega\tau} \sum_{s\neq 0} \frac{I_s(\eta) I_{j-s}(\eta)}{js} - \boldsymbol{\gamma}'_{0,(0),0}(\tau) C_0(\tau)$$
$$+ \frac{\partial \boldsymbol{f}(\boldsymbol{y}(\tau))}{\partial \boldsymbol{y}} \boldsymbol{f}(\boldsymbol{y}(\tau)) S_0^2(\tau)$$
$$= \frac{\partial \boldsymbol{f}(\boldsymbol{y}(\tau))}{\partial \boldsymbol{y}} \boldsymbol{f}(\boldsymbol{y}(\tau)) \sum_{s=1}^\infty \frac{I_s(\eta)}{s} \sum_{j=1}^\infty \frac{I_{j-s}(\eta) - I_{j+s}(\eta)}{j} \cos j\omega\tau - \boldsymbol{\gamma}'_{0,(0),0}(\tau) C_0(\tau)$$
$$+ \frac{\partial \boldsymbol{f}(\boldsymbol{y}(\tau))}{\partial \boldsymbol{y}} \boldsymbol{f}(\boldsymbol{y}(\tau)) S_0^2(\tau).$$

We note that for implementation purposes all these series should be truncated once a certain prescribed accuracy is reached. Due to the very rapid decay of the modified Bessel functions when the order increases, in most cases we need only to consider a few terms.

4.4. Examples. In this section we present several examples that illustrate the construction that we have explained in the previous sections. In all cases we will compare the approximation given by the first few terms of the asymptotic-numerical solver with the exact solution (which may be analytically available or either computed numerically with standard MATLAB routines up to prescribed accuracy).

We stress that the values of ω that we use are much smaller than the ones normally present in applications. This restriction is essentially imposed by the fact that the comparison with the exact solution should be reliable and affordable. Increasing ω will benefit the asymptotic-numerical solver, since the approximation with a fixed number of terms will be more accurate, and the computational cost will be roughly similar.

In our first example we will take $\eta(t)$ to be constant, and for simplicity we deal with a scalar equation:

$$y' = \mathrm{i}y + \mathrm{e}^{\cos\omega t}y^2, \quad t \geq 0, \qquad y(0) = 1. \tag{4.16}$$

The exact solution

$$y(t) = \frac{\mathrm{e}^{\mathrm{i}t}}{1 - \int_0^t \mathrm{e}^{\mathrm{i}x+\cos\omega x}\,\mathrm{d}x}$$

cannot be expressed in a finite manner by elementary functions.

The solution of (4.16) is periodic with period 2π. This is not easily visible from the explicit form of $y(t)$ but can be seen from our asymptotic expansion. It is enough to show that the integral in $y(t)$ is periodic of the given period. However, it follows from the Expcos asymptotics that

$$\int_0^t \mathrm{e}^{\mathrm{i}x+\cos\omega x}\,\mathrm{d}x \sim -\mathrm{i}I_0(1)(\mathrm{e}^{\mathrm{i}t}-1) + \mathrm{e}^{\mathrm{i}t}\sum_{k=0}^{\infty}\frac{S_k(t)}{\omega^{2k+1}} + \mathrm{i}\sum_{k=0}^{\infty}\frac{1}{\omega^{2k+2}}[C_k(t)\mathrm{e}^{\mathrm{i}t} - C_k(0)],$$

where $S_k(t)$ and $C_k(t)$ are given by (4.4) and are clearly periodic with period 2π, as of course is $\mathrm{e}^{\mathrm{i}t}$. Therefore, it follows at once that

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}x+\cos\omega x}\,\mathrm{d}x = 0$$

and y is indeed periodic of period 2π. The base function satisfies the ODE

$$\psi_0'(t) = \mathrm{i}\psi_0(t) + I_0(1)\psi_0^2(t), \qquad \psi_0(0) = 1,$$

which can be solved analytically:

$$\psi_0(t) = \frac{\mathrm{e}^{\mathrm{i}t}}{1 + \mathrm{i}I_0(1)(\mathrm{e}^{\mathrm{i}t}-1)}.$$

Higher order functions $\psi_s(t)$ can be obtained from the results of previous sections. Note that we can take $i = 0$ and $\tau_0 = 0$ throughout, since $\eta(t)$ is constant, and also that $\psi_s(t) \equiv 0$ for $s \geq 3$, which greatly simplifies the results. For instance, from (4.10) we deduce that

$$\psi_1(t) = S_0(t)\psi_0^2(t),$$

since $S_0(0) = 0$. Actually the variational equation (4.9) can be solved explicitly as well, since

$$\Omega(t) = \exp\left(tA + I_0(1)\int_0^t \frac{\partial f(\psi_0(s))}{\partial y}\,\mathrm{d}s\right),$$

and in this case

$$I_0(1)\int_0^t \frac{\partial f(\psi_0(s))}{\partial y}\,\mathrm{d}s = 2I_0(1)\int_0^t \psi_0(s)\,\mathrm{d}s = -2\log(1 + \mathrm{i}I_0(1)(\mathrm{e}^{\mathrm{i}t} - 1)),$$

and thus $\Omega(t) = \mathrm{e}^{-\mathrm{i}t}\psi_0^2(t)$.

The solution of the equation and the errors in the approximation when considering the first few terms are sketched in Fig. 4.1.

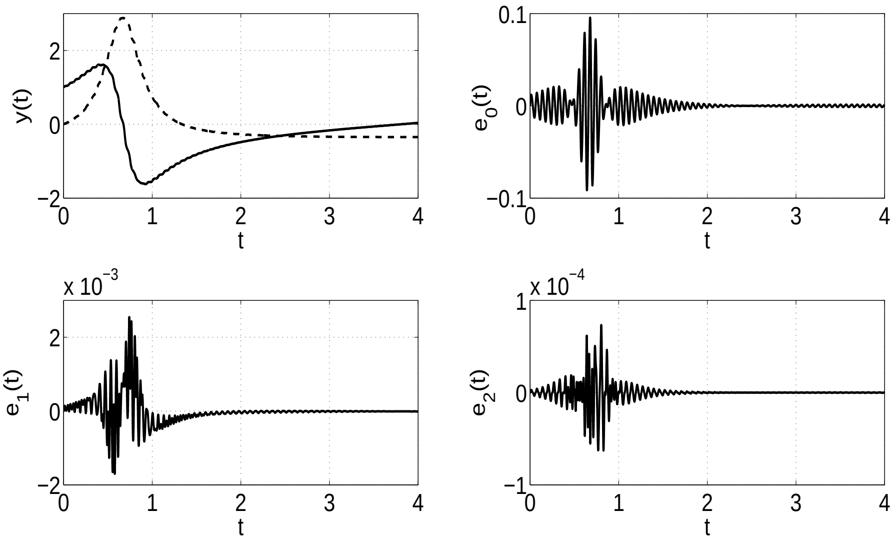

FIG. 4.1. *The exact solution of (4.16) for $\omega = 100$ (top left, real part in solid line and imaginary part in dashed line) and the real part of the errors $e_0(t)$ (top right), $e_1(t)$ (bottom left) and $e_2(t)$ (bottom right).*

As a matter of fact, we can obtain more ψ_ss by direct expansion. Thus, expanding the integral in the denominator of $y(t)$, we have

$$\begin{aligned} y(t) &\sim \frac{\psi_0(t)}{1 - \frac{1}{\omega}S_0(t)\psi_0(t) - \frac{\mathrm{i}}{\omega^2}[C_0(t) - \mathrm{e}^{-\mathrm{i}t}C_0(0)]\psi_0(t) - \frac{1}{\omega^3}S_1(t)\psi_0(t) + \ldots} \\ &\approx \psi_0(t) + \frac{1}{\omega}S_0(t)\psi_0^2(t) + \frac{1}{\omega^2}\{\mathrm{i}[C_0(t) - \mathrm{e}^{-\mathrm{i}t}C_0(0)]\psi_0^2(t) + S_0^2(t)\psi_0^3(t)\} \\ &\quad + \frac{1}{\omega^3}\{S_1(t)\psi_0^2(t) + 2\mathrm{i}S_0(t)[C_0(t) - \mathrm{e}^{-\mathrm{i}t}C_0(0)]\psi_0^3(t) + S_0^3(t)\psi_0^4(t)\}. \end{aligned}$$

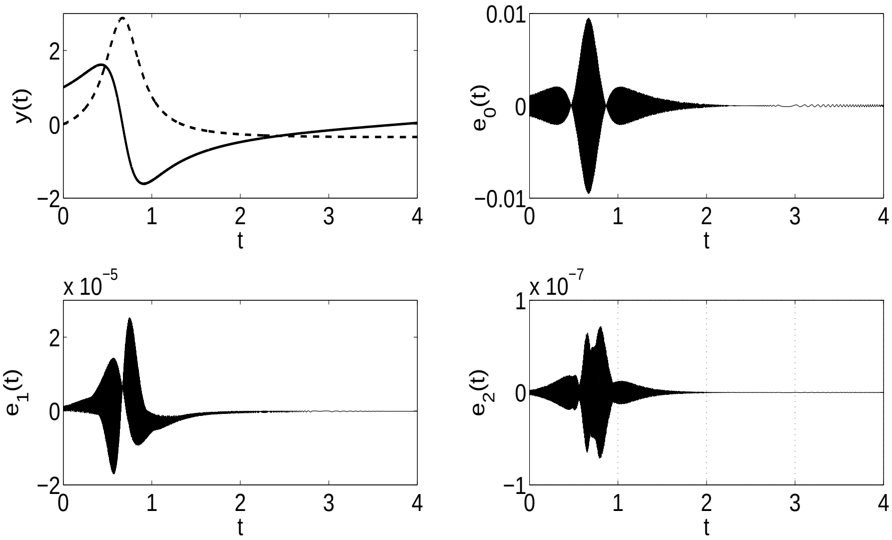

FIG. 4.2. *The exact solution of (4.16) for* $\omega = 1000$ *(top left) and the real part of the errors* $e_0(t)$ *(top right),* $e_1(t)$ *(bottom left) and* $e_2(t)$ *(bottom right).*

Note that ψ_1 is exactly as predicted by our theory.

We make a remark on the wide discrepancy of the error – up to three degrees of magnitude. The reason is in the relative size of $|y|$ in different parts of the period (cf. Fig. 4.1 and 4.2). It is easy to check that

$$\psi_m(t) = S_0^m(t)\psi_0^{m+1}(t) + \text{lower-order terms in } \psi_0.$$

Since $\psi_0(t) \approx y(t)$, this means that, when $|\psi_0(t)S_0(t)| > 1$, the mth component $\psi_m(t)/\omega^m$ scales roughly like $|\psi_0(t)S_0(t)/\omega|^m$. On the other hand, when $|\psi_0(t)|$ is small,

$$\psi_m(t) = \begin{cases} \mathrm{i}[C_{n-1}(t) - \mathrm{e}^{-\mathrm{i}t}C_{n-1}(0)]\psi_0^2(t), & m = 2n, \\ S_n(t)\psi_0^2(t), & m = 2n+1 \end{cases} + \text{higher-order terms in } \psi_0.$$

Hence, in this regime $\psi_m(t)/\omega^m$ scales roughly like $|\psi_0(t)|^2/\omega^m$. This explains the different scaling of the error.

As a second example, we present an equation whose solution is not explicitly available:

$$y'(t) = 2\mathrm{i}y(t) + \mathrm{e}^{\cos\omega t}\tanh(y(t)), \qquad y(0) = 1. \tag{4.17}$$

In this case, all the computations have to be carried out numerically in MATLAB, using a grid of points on the interval of interest. As a comparison, we have solved the equation with the standard `ode45` solver, using `reltol` and `abstol` equal to 10^{-12}.

In order to construct the asymptotic-numerical solver, we need to solve the fol-

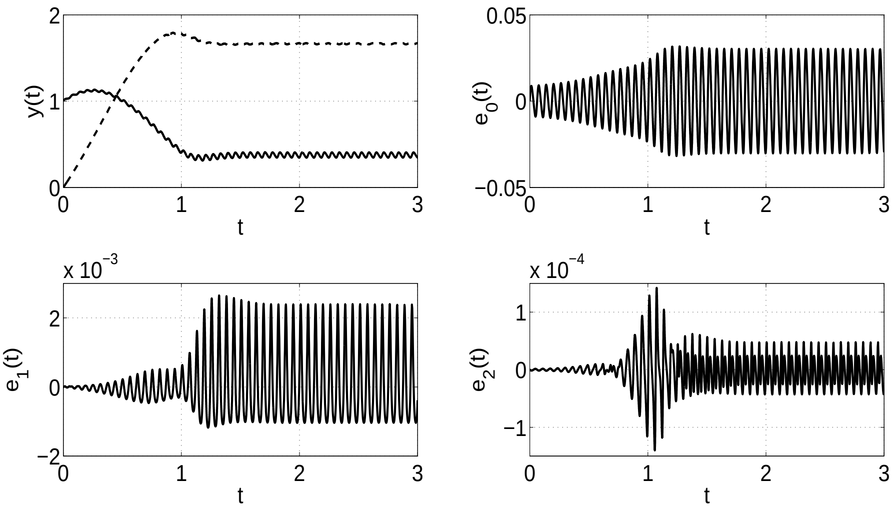

FIG. 4.3. *The exact solution of (4.17) for $\omega = 100$ (top left, real part in solid line and imaginary part in dashed line) and the real part of the errors $e_0(t)$ (top right), $e_1(t)$ (bottom left) and $e_2(t)$ (bottom right).*

lowing 3×3 (nonoscillatory) system:

$$\begin{bmatrix} \psi_0(t) \\ \Omega(t) \\ a_{2,0}(t) \end{bmatrix}' = \begin{bmatrix} A & 0 & 0 \\ 0 & A + I_0(1)\frac{\partial f(\psi_0(t))}{\partial y} & 0 \\ 0 & 0 & A + I_0(1)\frac{\partial f(\psi_0(t))}{\partial y} \end{bmatrix} \begin{bmatrix} \psi_0(t) \\ \Omega(t) \\ a_{2,0}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} I_0(1) + f(\psi_0(t)) \\ 0 \\ B(t) \end{bmatrix},$$

where in this example $A = 2\mathrm{i}$, $f(y(t)) = \tanh(y(t))$ and $B(t)$ is given in (4.15). Note that this system is significantly cheaper to solve than the original equation using standard ODE routines, since it is nonoscillatory. Of course, it is possible to add more terms if needed, for example $a_{3,0}(t)$, at the price of more complicated computations.

Once $\psi_0(t)$, $\Omega(t)$ and $a_{2,0}(t)$ are constructed (on a certain grid given by the tolerance we impose in MATLAB), we can compute $\psi_1(t)$ using (4.10). Note that in this case $\tau = 0$, so

$$\psi_1(t) = S_0(t) f(\psi_0(t)) = S_0(t) \tanh(\psi_0(t)),$$

which can be easily computed on the same grid as $\psi_0(t)$. Observe that the series $S_0(t)$ converges fast for fixed t due to the rapid decay of the modified Bessel functions. The same is true for the different series that we have to compute for the coefficients $a_{2,j}(t)$ given in (4.13).

In Figures 4.3 and 4.4 we illustrate the approximations using $\psi_s(t)$ for $s \leq 2$. For the sake of clarity and space we only plot the real part of the errors, since the

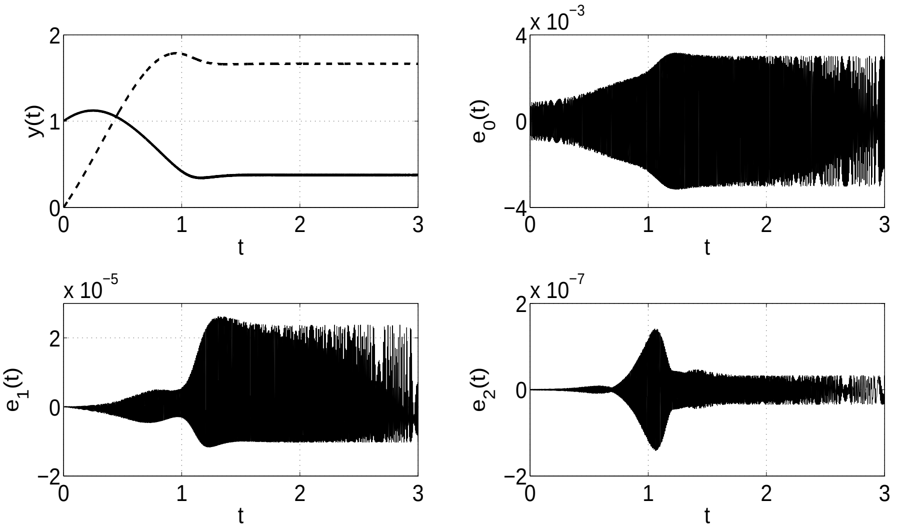

FIG. 4.4. *Same as Figure 4.3 but with $\omega = 1000$.*

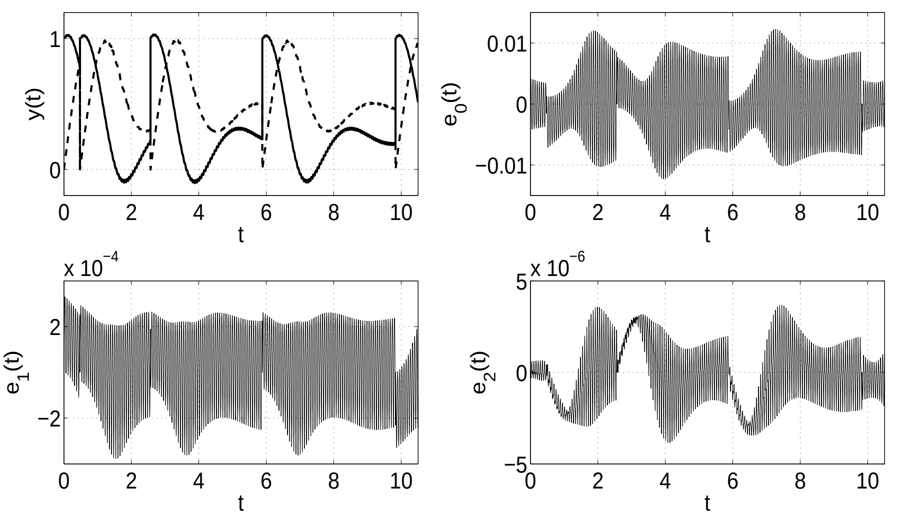

FIG. 4.5. *The exact solution of (4.18) for $\omega = 100$ (top left) and the real part of the errors $e_0(t)$ (top right), $e_1(t)$ (bottom left) and $e_2(t)$ (bottom right).*

imaginary part follows an analogous pattern. As before, we use the notation

$$e_s(t) = y(t) - \sum_{m=0}^{s} \frac{\psi_m(t)}{\omega^m}, \qquad s \geq 0.$$

The final example that we present includes a variable η. For simplicity we restrict ourselves to the case where $\eta(t)$ is piecewise constant, and if fact we take $\eta(t)$

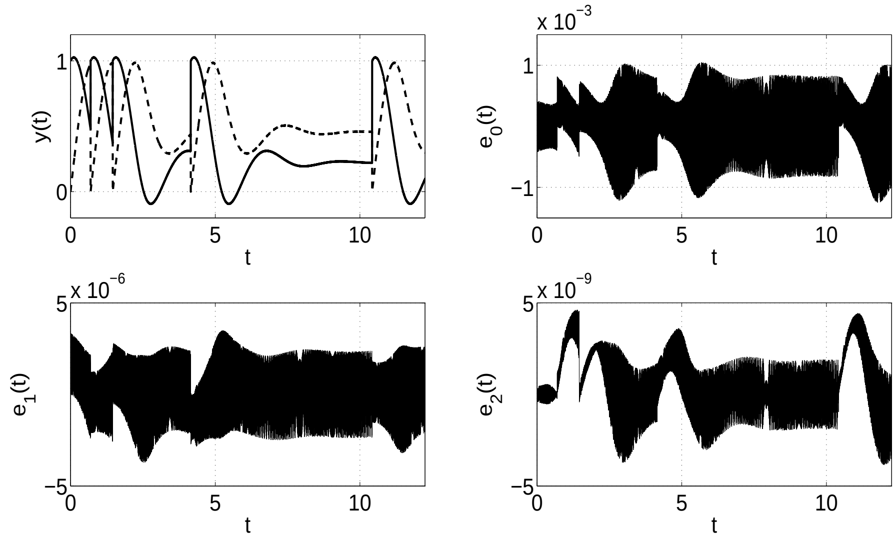

FIG. 4.6. *Similar to Figure 4.5, but with a different random grid and* $\omega = 1000$.

alternating between 1 and -1 on a grid that we generate randomly in MATLAB. The ODE that we take as an example is

$$y'(t) = 2\mathrm{i}y(t) + \mathrm{e}^{\eta(t)\cos\omega t}\mathrm{e}^{-y(t)}, \tag{4.18}$$

with $y(\tau_i) = 1$ fixed for all i. Figures 4.5 and 4.6 depict the solution and absolute errors for $\omega = 100$ and $\omega = 1000$.

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REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, vol. 55 of National Bureau of Standards Applied Mathematics Series, U.S. Government Printing Office, Washington, 1964.

[2] D. COHEN, E. HAIRER, AND C. LUBICH, *Modulated Fourier expansions of highly oscillatory differential equations*, Found. Comput. Maths., 3 (2005), pp. 327–450.

[3] M. CONDON, A. DEAÑO, AND A. ISERLES, *On highly oscillatory problems arising in electronic engineering*, ESAIM: Mathematical Modeling and Numerical Analysis, 43 (2009), pp. 785–804.

[4] M. CONDON, A. DEAÑO, A. ISERLES, K. MACZYNSKI, AND T. XU, *On numerical methods for highly oscillatory problems in circuit simulation*, To appear in COMPEL, (2009).

[5] E. DAUTBEGOVIC, M. CONDON, AND C. BRENNAN, *An efficient nonlinear circuit simulation technique*, IEEE Trans. Microwave Theory and Applications, 53 (2005), pp. 548–555.

[6] E. HAIRER, C. LUBICH, AND G. WANNER, *Geometric Numerical Integration, 2nd edition*, Springer Verlag, 2006.

[7] E. Hairer, S. Nørsett, and G. Wanner, *Solving Ordinary Differential Equations I. Nonstiff Problems*, Springer Verlag, 1993.

[8] M. Khanamirian, *Quadrature methods for highly oscillatory linear and nonlinear systems of ordinary differential equations: Part I*, BIT, 48 (2008), pp. 743–761.

[9] J. C. Pedro and N. Carvalho, *Simulation of RF circuits driven by modulated signals without bandwidth constraints*, Proc IEEE International Symp. on Microwave Theory and Tech., 3 (2002), pp. 2173–2176.

[10] J. Roychowdhury, *Analyzing circuits with widely separated time scales using numerical PDE methods*, IEEE Transactions on circuits and systems - I, 48 (2001), pp. 578–594.