On the singular values of the Fox–Li operator

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Abstract

The Fox–Li integral operator

$$\mathcal{F}_\omega[f] = \int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}\omega(x-y)^2}\,\mathrm{d}x, \qquad y \in [-1,1],$$

plays an important role in laser engineering. However, the mathematical analysis of its spectrum is still rather incomplete. In the present paper we obtain significant new insight into the spectral properties, by giving a complete description of the set of singular values of $\mathcal{F}_\omega$. In addition, numerical experiments appear to point to a deep connection between our analysis and the spectrum of the Fox–Li operator.

1 Introduction

The Fox–Li operator is

$$\mathcal{F}_\omega[f](y) = \int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}\omega(x-y)^2}\,\mathrm{d}x, \qquad y \in [-1,1], \tag{1.1}$$

where $f \in \mathrm{L}_2[-1,1]$ (Cochran & Hinds 1974). Its spectrum has important applications in laser and maser engineering (Fox & Li 1961); unfortunately little is rigorously known about it. $\mathcal{F}_\omega$ is compact, hence $\sigma(\mathcal{F}_\omega)$ consists of a countable number of eigenvalues, accumulating at the origin. Computation (cf. Fig. 1.1) seems to indicate that they lie on a spiral, commencing at $\approx \pi^{\frac{1}{2}}/(-\mathrm{i}\omega)^{\frac{1}{2}}$ and rotating clockwise to the origin, except that, strenuous efforts notwithstanding, the precise shape of this spiral is yet unknown.

Indeed, rigorous results on the Fox–Li spectrum are fairly sparse. Henry Landau (1977/78) determined the pseudo-spectrum $\sigma_\varepsilon(\mathcal{F}_\omega)$, Michael Berry and his collaborators have written a number of papers on physical aspects of the spectrum and its applications in laser theory (Berry 2001, Berry, Strom & van Saarlos 2001, Berry 2003) and, in a recent paper, the current authors have analysed a number of efficient numerical methods for the determination of $\sigma(\mathcal{F}_\omega)$ and, with greater generality, of spectra of integral operators with high oscillation (Brunner, Iserles & Nørsett 2010).

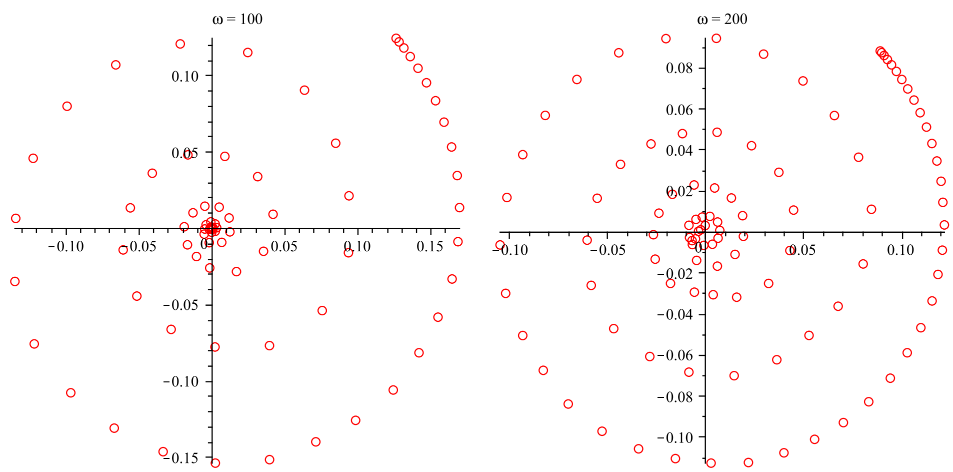

Figure 1.1: The spectra of $\mathcal{F}_{100}$ and $\mathcal{F}_{200}$.

The purpose of this brief paper is to characterize completely $s(\mathcal{F}_\omega)$, the set of *singular values* of the operator (1.1). We recall that, given a Hilbert-space operator $T : \mathcal{H} \to \mathcal{H}$, the singular values of T are square roots of the eigenvalues of the self-adjoint, positive semidefinite operator TT^*.

Unlike $\sigma(\mathcal{F}_\omega)$, the set $s(\mathcal{F}_\omega)$ consists of real, nonnegative points. The spiral goes away! However, computational evidence indicates that it is replaced by a different, arguably just as striking, feature: Asymptotically in ω, $s(\mathcal{F}_\omega)$ consists of just two distinct points!

Fig. 1.2 displays the approximate singular values obtained from nine $(2N+1)\times(2N+1)$ matrices approximating $\mathcal{F}_\omega\mathcal{F}_\omega^*$ for three different values of ω. (We defer the description of how these matrices were constructed to the next section.) The following features should be easily ascertained by the observant reader:

1. The set $s(\mathcal{F}_\omega)$ is mostly composed from just two values, one at the origin and the other strictly positive. There are few additional values at the interface but it would not be

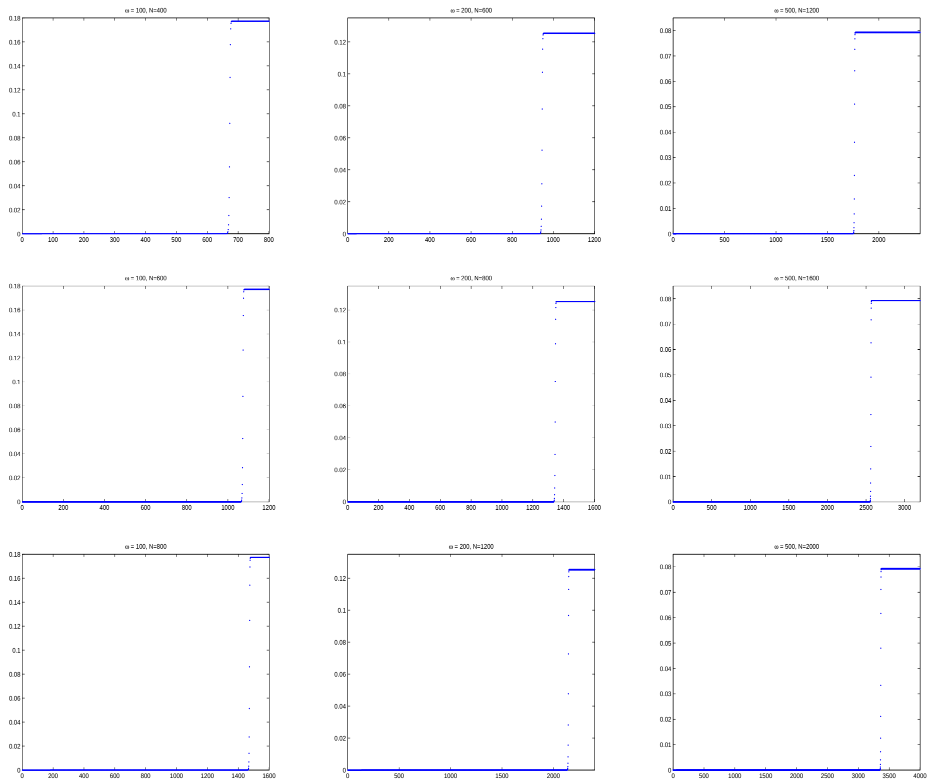

Figure 1.2: Singular values originating in $(2N+1)\times(2N+1)$ matrices approximating $\mathcal{F}_\omega\mathcal{F}_\omega^*$ for different values of ω and N.

surprising altogether were they to go away as $N\to\infty$.

2. The above positive singular value depends upon ω but is independent of N, hence it is likely to stay put as $N\to\infty$.

3. The multiplicity of the positive singular value seems to be independent of N. In other words, and disregarding the few intermediate singular values, it appears that the *rank* of the matrix is asymptotically independent of N.

We prove in the sequel that all these observations are true. We do so by discretizing the spectral problem $\mathcal{F}_\omega\mathcal{F}_\omega^*[f]=\mu f$. The outcome is an algebraic eigenvalue problem with a self-adjoint Toeplitz matrix, and this allows us to use the Grenander–Szegő theory explicating the asymptotic behaviour of finite sections of self-adjoint Toeplitz–Laurent operators. Combining this with basic asymptotic results in a comprehensive description of $\sigma(\mathcal{F}_\omega\mathcal{F}_\omega^*)$ and leads to our complete description of the set of singular values of $\mathcal{F}_\omega$.

We conclude the paper with a number of observations on the spectrum of $\mathcal{F}_\omega$. Firstly, we explain why our method of analysis of singular values falls short in explicating the spectrum of the Fox–Li operator. Secondly, we exhibit initial observations linking $\sigma(\mathcal{F}_\omega)$ with the Jacobi theta function θ_3.

2 The singular values of $\mathcal{F}_\omega$

2.1 The operator $\mathcal{F}_\omega\mathcal{F}_\omega^*$

The set $s(\mathcal{F}_\omega)$ of the singular values of $\mathcal{F}_\omega$ coincides with the square roots of the eigenvalues of the positive semidefinite operator $\mathcal{M}_\omega = \mathcal{F}_\omega\mathcal{F}_\omega^*$. But, for every $f \in \mathrm{L}_2[-1,1]$,

$$\begin{aligned}\mathcal{M}_\omega[f](y) &= \int_{-1}^{1}\int_{-1}^{1} f(x_1)\mathrm{e}^{\mathrm{i}\omega(x_1-x_2)^2}\,\mathrm{d}x_1\mathrm{e}^{-\mathrm{i}\omega(x_2-y)^2}\,\mathrm{d}x_2\\ &= \mathrm{e}^{-\mathrm{i}\omega y^2}\int_{-1}^{1} f(x_1)\mathrm{e}^{\mathrm{i}\omega x_1^2}\int_{-1}^{1}\mathrm{e}^{2\mathrm{i}\omega(y-x_1)x_2}\,\mathrm{d}x_2\,\mathrm{d}x_1\\ &= \mathrm{e}^{-\mathrm{i}\omega y^2}\int_{-1}^{1}\mathrm{e}^{\mathrm{i}\omega x^2} f(x)\frac{\sin 2\omega(y-x)}{\omega(y-x)}\,\mathrm{d}x.\end{aligned}$$

Lemma 1 *The set $s(\mathcal{F}_\omega)$ coincides with the square roots of the eigenvalues of the operator*

$$\mathcal{L}_\omega[f](y) = \int_{-1}^{1} f(x)\frac{\sin(2\omega(y-x))}{\omega(y-x)}\,\mathrm{d}x, \qquad y \in [-1,1]. \tag{2.1}$$

Proof Since $\mathcal{M}_\omega$ is compact and has solely point spectrum, we have proved above that $\mu \in s(\mathcal{F}_\omega)$ iff there exists $g \in \mathrm{L}_2[-1,1]$ such that

$$\mathrm{e}^{-\mathrm{i}\omega y^2}\int_{-1}^{1}\mathrm{e}^{\mathrm{i}\omega x^2} g(x)\frac{\sin 2\omega(y-x)}{\omega(y-x)}\,\mathrm{d}x = \mu^2 g(y), \qquad y \in [-1,1].$$

The theorem follows by letting $f(y) = \mathrm{e}^{\mathrm{i}\omega y^2} g(y)$. □

2.2 A general highly oscillatory convolution-type problem

The operator $\mathcal{L}_\omega$ is a special case of a general self-adjoint integral operator of convolution type,

$$\mathcal{K}_\omega[f](y) = \int_{-1}^{1} f(x)K_\omega(y-x)\,\mathrm{d}x, \qquad y \in [-1,1], \tag{2.2}$$

where the *kernel* K_ω is analytic and even in $[-2,2]$ and oscillates rapidly for $\omega \gg 1$. It is a compact operator and it follows from standard Sturm–Liouville theory that $\sigma(\mathcal{K}_\infty)$ is a point spectrum with a single accumulation point at the origin. The spectral problem (2.1) is a special case of

$$\mathcal{K}_\omega[f](y) = \mu f(y), \qquad y \in [-1,1], \tag{2.3}$$

which we consider next.

We discretize $\mathcal{K}_\omega$ at equidistant points: this, we hasten to say, is emphatically *not* an effective numerical method for the solution of (2.3), certainly falling well short of the *finite section techniques* of (Brunner et al. 2010). However, 'naive' discretization at equidistant points has a crucial feature which we exploit in our subsequent analysis.

Equidistant quadrature

$$\mathcal{K}_\omega[f](y) \approx \frac{1}{N}\sum_{m=-N}^{N} f(\tfrac{m}{N})K_\omega(y-\tfrac{m}{N}),$$

where $N\in\mathbb{N}$ is sufficiently large, is applied for $2N+1$ equidistant values of y, whereby (2.3) is approximated by the algebraic eigenvalue problem

$$K^{[N]}\boldsymbol{f}^{[N]} = \mu^{[N]}\boldsymbol{f}^{[N]},$$

where

$$K^{[N]}_{m,n} = \tau^{[N]}_{n-m}, \quad m,n=-N,\ldots,N, \qquad \text{where} \qquad \tau^{[N]}_n = \frac{1}{N}K_\omega(\tfrac{n}{N}), \quad n=-2N,\ldots,2N,$$

while $f^{[N]}_m \approx f(\frac{m}{N})$. Our main observation, enabling subsequent analysis of this paper, is that $K^{[N]}$ is a self-adjoint *Toeplitz* matrix with the symbol

$$\tau^{[N]}(z) = \sum_{n=-N}^{N} \tau^{[N]}_n z^n,$$

which is analytic on $|z|=1$. This is the time to recall a few pertinent facts from the theory of Toeplitz matrices and operators (Böttcher & Silberman 2006, Grenander & Szegő 1984). Thus, given the symbol $\tau(z)=\sum_{n=-\infty}^{\infty}\tau_n z^n$, analytic on $|z|=1$, the bi-infinite *Toeplitz–Laurent operator* T, where $T_{m,n}=\tau_{m-n}$, $m,n\in\mathbb{Z}$, has the continuous spectrum

$$\sigma(T) = \{\tau(\mathrm{e}^{\mathrm{i}\theta}) \,:\, \theta\in[-\pi,\pi]\}.$$

Let $T^{[N]}$ be an $(2N+1)\times(2N+1)$ section of T, i.e. an $(2N+1)\times(2N+1)$ matrix such that $T^{[N]}_{m,n}=\tau_{m-n}$, $m,n=-N,\ldots,N$. Provided that T is Hermitian (or, with greater generality, normal) the Grenander–Szegő theorem (Grenander & Szegő 1984) claims that eigenvalues $\sigma(T^{[N]})$ are equidistributed along the curve $\sigma(T)$ as $N\to\infty$. In other words, letting $\mathrm{d}\nu^{[N]}$ be the *enumeration measure* of the eigenvalues of $T^{[N]}$,

$$\mathrm{d}\nu^{[N]}(t) = \frac{1}{2N+1}\sum_{k=1}^{2N+1}\delta(t-\lambda^{[N]}_k),$$

where $\lambda^{[N]}_k$ are the eigenvalues of $T^{[N]}$ and δ is the Dirac delta function, then, as $N\to\infty$, $\mathrm{d}\nu^{[N]} \stackrel{*}{\to} \mathrm{d}\nu$, where $\mathrm{d}\nu$ is supported on the curve $\sigma(T)$. Moreover, the proportion of eigenvalues tending to any measurable subset of $\sigma(T)$ equals the proportion of the measure of this set to the measure of the entire spectral curve.

Although $K^{[N]}$ tends (in a well-defined sense) to the integral operator $\mathcal{K}_\omega$ as $N\to\infty$, we cannot use the Grenander–Szegő theorem in this setting because $K^{[N]}$ is not a section of $\mathcal{K}_\omega$.

Instead, for every $N \gg 1$ we consider the Toeplitz–Laurent operator $\tilde{K}^{[N]}$ with the symbol $\tau^{[N]}$. It thus follows that $\sigma(K^{[N]})$ approaches

$$\sigma(\tilde{K}^{[N]}) = \left\{\tau^{[N]}(\mathrm{e}^{\mathrm{i}\theta}) \,:\, -\pi \le \theta \le \pi\right\} = \left\{ \frac{1}{N} \sum_{n=-N}^{N} K_\omega(\tfrac{n}{N}) \mathrm{e}^{\mathrm{i}n\theta} \,:\, -\pi \le \theta \le \pi \right\}.$$

However,

$$\frac{1}{N} \sum_{n=-N}^{N} K_\omega(\tfrac{n}{N}) \mathrm{e}^{\mathrm{i}n\theta} = \int_{-1}^{1} K_\omega(t) \mathrm{e}^{\mathrm{i}N\theta t}\,\mathrm{d}t + \mathcal{O}\big(N^{-2}\big) = \hat{K}_\omega(N\theta) + \mathcal{O}\big(N^{-2}\big)\,,$$

where $\hat{K}_\omega$ is the *truncated Fourier transform* of K_ω,

$$\hat{K}_\omega(s) = \int_{-1}^{1} K_\omega(t) \mathrm{e}^{\mathrm{i}st}\,\mathrm{d}t.$$

This is perhaps the moment to impose an extra condition on K_ω, namely that its truncated Fourier transform exists for $\omega \gg 1$. We refrain from doing so because, in the general case, it is sufficient for it to exist in the sense of distributions while, for the application we have in mind, the existence is trivially assured.

Theorem 2 *It is true that*

$$\sigma(\mathcal{K}_\omega) \subset \{\hat{K}_\omega(t) \,:\, t \in \mathbb{R}\}. \tag{2.4}$$

Proof Follows at once from our analysis, because in any reasonable metric (e.g. the Hausdorff distance) compactness of $\mathcal{K}_\omega$ implies that $\sigma(K^{[N]}) \stackrel{N\to\infty}{\longrightarrow} \sigma(\mathcal{K}_\omega)$. □

2.3 The spectrum of $\mathcal{M}_\omega$ explained

Setting

$$K_\omega(y) = \frac{\sin(2\omega y)}{\omega y}, \qquad y \in [-2, 2],$$

we recover $\mathcal{M}_\omega = \mathcal{F}_\omega \mathcal{F}_\omega^*$. It is clearly an even and analytic function, hence within the scope of our analysis. Its truncated Fourier transform can be calculated directly, e.g. with MAPLE:

$$\begin{aligned}\hat{K}_\omega(t) = -\frac{\mathrm{i}}{2\omega}[&\mathrm{Ei}_1(-\mathrm{i}t + 2\mathrm{i}\omega) - \mathrm{Ei}_1(-\mathrm{i}t - 2\mathrm{i}\omega) - \mathrm{Ei}_1(\mathrm{i}t - 2\mathrm{i}\omega) + \mathrm{Ei}_1(\mathrm{i}t + 2\mathrm{i}\omega)\\ &+ \log(-\mathrm{i}t + 2\mathrm{i}\omega) - \log(-\mathrm{i}t - 2\mathrm{i}\omega) - \log(\mathrm{i}t - 2\mathrm{i}\omega) + \log(\mathrm{i}t + 2\mathrm{i}\omega)].\end{aligned}$$

However, a simple calculation confirms that

$$\log(-\mathrm{i}t + 2\mathrm{i}\omega) - \log(-\mathrm{i}t - 2\mathrm{i}\omega) - \log(\mathrm{i}t - 2\mathrm{i}\omega) + \log(\mathrm{i}t + 2\mathrm{i}\omega) = \begin{cases} 2\pi\mathrm{i}, & |t| < 2\omega, \\ 0, & |t| > 2\omega. \end{cases}$$

Moreover, employing the standard asymptotic expansion of the exponential integral,

$$\mathrm{Ei}_1(z) \sim \mathrm{e}^{-z} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{z^{n+1}}, \qquad |\arg z| < \frac{3\pi}{2}, \quad |z| \gg 1$$

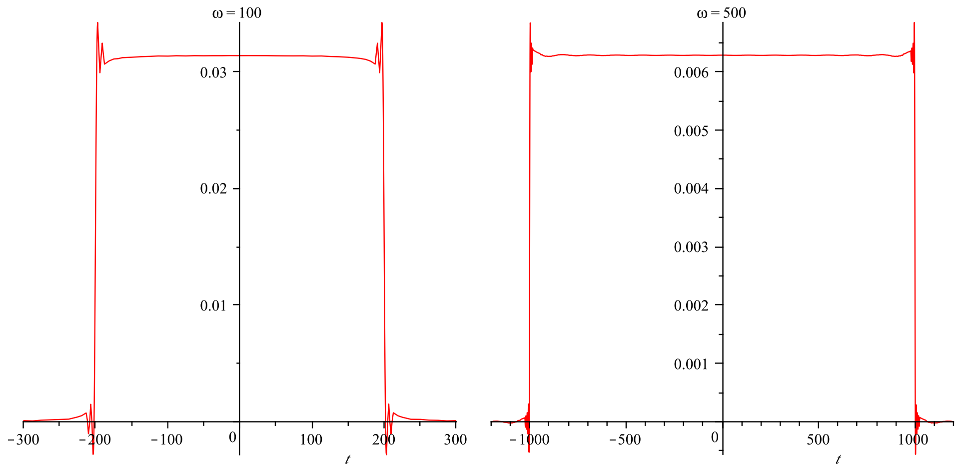

Figure 2.1: Truncated Fourier transforms $\hat{K}_{100}$ and $\hat{K}_{500}$.

(Abramowitz & Stegun 1964, p. 231), we deduce that

$$\begin{aligned} &-\frac{\mathrm{i}}{4\omega}[\mathrm{Ei}_1(-\mathrm{i}t+2\mathrm{i}\omega)-\mathrm{Ei}_1(-\mathrm{i}t-2\mathrm{i}\omega)-\mathrm{Ei}_1(\mathrm{i}t-2\mathrm{i}\omega)+\mathrm{Ei}_1(\mathrm{i}t+2\mathrm{i}\omega)] \\ &\sim \frac{2\mathrm{e}^{\pi\mathrm{i}t}\cos 2\omega}{t^2-4\omega^2}+\mathcal{O}\big(\omega^{-4}\big)\,, \end{aligned}$$

valid for $t\in\mathbb{R}$ except in a neighbourhood of $t=\pm 2\omega$. However, for $t=\pm 2\omega$ we have

$$\hat{K}_\omega(\pm 2\omega)=\frac{1}{2\mathrm{i}\omega}[\mathrm{Ei}_1(4\mathrm{i}\omega)-\mathrm{Ei}_1(-4\mathrm{i}\omega)]$$

and using the asymptotic formula again, we have

$$\hat{K}_\omega(\pm 2\omega)\pm-\frac{\cos 4\omega}{4\omega^2}+\mathcal{O}\big(\omega^{-4}\big)\,.$$

We deduce that

$$\hat{K}_\omega(t)\sim\begin{cases}\frac{\pi}{\omega}, & |t|<2\omega,\\ 0, & |t|\ge 2\omega\end{cases}+\mathcal{O}\big(\omega^{-2}\big)\,. \tag{2.5}$$

Fig. 2.1 displays truncated Fourier transforms for $\omega=100$ and $\omega=500$ and it is vividly clear that they approximate a scaled indicator function of the interval $[-2\omega,2\omega]$.

Note that we could have proved (2.5) by a simpler and most plausible argument, yet one which does not provide us with an easy means of estimating the $\mathcal{O}\big(\omega^{-2}\big)$ terms. Thus,

$$\hat{K}_\omega(t)=\int_{-1}^{1}\frac{\sin(2\omega x)}{\omega x}\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}x=\frac{1}{\omega}\int_{-2\omega}^{2\omega}\frac{\sin x}{x}\mathrm{e}^{\mathrm{i}xt/(2\omega)}\,\mathrm{d}x=\frac{1}{\omega}\hat{K}^\infty\!\left(\frac{t}{2\omega}\right)+\mathcal{O}\big(\omega^{-2}\big)\,,$$

where $\hat{K}^{\infty}(t) = \int_{-\infty}^{\infty} K(x)\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}x$ is the standard (i.e., not truncated) Fourier transform of the kernel K. Since

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}x = \begin{cases} \pi, & |t| < 1, \\ 0, & |t| \ge 1, \end{cases}$$

we deduce (2.5).

Theorem 3 *The spectrum of* $\mathcal{M}_\omega$ *consists of* $4\omega/\pi + \mathcal{O}(\omega^{-1})$ *eigenvalues* $\pi/\omega + \mathcal{O}(\omega^{-2})$, *the remaining eigenvalues being* $\mathcal{O}(\omega^{-2})$.

Proof We have already proved that, up to $\mathcal{O}(\omega^{-2})$, $\sigma(\mathcal{M}_\omega)$ consists just of two distinct points, namely 0 and π/ω. It remains to determine κ_ω, the number of the nonzero eigenvalues. To this end we calculate $\mathrm{tr}\,K^{[N]}$ in two different ways. Firstly,

$$\mathrm{tr}\,K^{[N]} = \sum_{\mu\in\sigma(K^{[N]})} \mu = \frac{\pi}{\omega}\kappa_\omega^{[N]} + \mathcal{O}(\omega^{-2}, N^{-2}),$$

where $\kappa_\omega^{[N]}$ is the number of eigenvalues $\pi/\omega + \mathcal{O}(\omega^{-2})$ of the $(2N+1)\times(2N+1)$ matrix $K^{[N]}$. Secondly,

$$\mathrm{tr}\,K^{[N]} = \sum_{m=-N}^{N} K_{n,n}^{[N]} = \frac{2N+1}{N} K_\omega(0) = 4 + \mathcal{O}(N^{-1}).$$

Therefore $\kappa_\omega^{[N]} = 4\omega/\pi + \mathcal{O}(\omega^{-1}, N^{-2})$ and the theorem follows by letting $N \to \infty$. □

Corollary 1 *The Fox–Li operator* $\mathcal{F}_\omega$ *has* $4\omega/\pi + \mathcal{O}(\omega^{-1})$ *singular values* $(\pi/\omega)^{\frac{1}{2}} + \mathcal{O}(\omega^{-2})$ *and an infinity of singular values which are* $\mathcal{O}(\omega^{-2})$.

This result completely confirms the numerical evidence of Fig. 1.2.

3 Back to the Fox–Li spectrum

Inasmuch as singular values of $\mathcal{F}_\omega$ are interesting, the real prize is the spectrum of the operator. It is therefore of interest to apply the method of analysis of the last section to the Fox–Li operator directly. Thus, we discretize $\mathcal{F}_\omega$ at $2N+1$ equidistant points, whereby $\mathcal{F}_\omega[f] = \lambda f$ is approximated by the algebraic eigenvalue problem

$$B^{[N]}\boldsymbol{f}^{[N]} = \lambda^{[N]}\boldsymbol{f}^{[N]}, \tag{3.1}$$

where

$$B_{m,n}^{[N]} = \upsilon_{n-m}^{[N]}, \quad m,n = -N,\ldots,N, \quad \text{with} \quad \upsilon_n^{[N]} = \frac{1}{N}\mathrm{e}^{\mathrm{i}\omega n^2/N^2}, \ \ n = -2N,\ldots,2N.$$

The matrix $B^{[N]}$ is symmetric and Toeplitz, but it is not normal. We denote its symbol by

$$\upsilon^{[N]}(z) = \sum_{n=-N}^{N} \upsilon_n^{[N]} z^n.$$

Lemma 4 *It is true that*

$$\upsilon^{[N]}(\mathrm{e}^{\mathrm{i}\theta}) \sim \frac{1}{2}\frac{\pi^{\frac{1}{2}}\mathrm{e}^{-\frac{N^2\theta^2}{4\omega}}}{(-\mathrm{i}\omega)^{\frac{1}{2}}}\left[\mathrm{erf}\left(\frac{1}{2}\frac{\mathrm{i}N\theta}{(-\mathrm{i}\omega)^{\frac{1}{2}}}+(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\mathrm{erf}\left(\frac{1}{2}\frac{\mathrm{i}N\theta}{(-\mathrm{i}\omega)^{\frac{1}{2}}}-(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right] \tag{3.2}$$

for all $-\pi \le \theta \le \pi$ *and* $N \gg 1$.

Proof Similarly to Subsection 2.2, we approximate the finite sum

$$\upsilon^{[N]}(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{N}\sum_{n=-N}^{N}\mathrm{e}^{\mathrm{i}\omega(\frac{n}{N})^2+\mathrm{i}\theta n}$$

by an integral, which can be computed explicitly, and let $N \gg 1$. □

Lemma 4 implies that

$$\left\{\lim_{N\to\infty}\upsilon^{[N]}(\mathrm{e}^{\mathrm{i}\theta}) \,:\, |\theta|\le\pi\right\} \subseteq \{\hat{F}_\omega(t) \,:\, t\in\mathbb{R}\},$$

where

$$\hat{F}_\omega(t) = \frac{1}{2}\frac{\pi^{\frac{1}{2}}\mathrm{e}^{-t^2/(4\omega)}}{(-\mathrm{i}\omega)^{\frac{1}{2}}}\left[\mathrm{erf}\left(\frac{1}{2}\frac{\mathrm{i}t}{(-\mathrm{i}\omega)^{\frac{1}{2}}}+(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\mathrm{erf}\left(\frac{1}{2}\frac{\mathrm{i}t}{(-\mathrm{i}\omega)^{\frac{1}{2}}}-(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right], \quad t\in\mathbb{R}.$$

Fig. 3.1 displays real and imaginary parts of $\hat{F}_\omega$, as well as its absolute value, all scaled by $(-\mathrm{i}\omega/\pi)^{\frac{1}{2}}$, for two different values of ω. It is clear that real and imaginary parts oscillate wildly, decaying for $|t| > 2\omega$, while the scaled absolute value seems to approximate the indicator function of the interval $[-2\omega, 2\omega]$. Much of this intuition is confirmed by basic asymptotics. Since

$$\mathrm{erf} z \sim 1 - \frac{\mathrm{e}^{-z^2}}{\pi^{\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2})_n}{z^{2n+1}}, \qquad |\arg z| < \frac{3\pi}{4}, \quad |z| \gg 1$$

(Abramowitz & Stegun 1964, p. 298), we have

$$\hat{F}_\omega(t) \sim 2\mathrm{e}^{\mathrm{i}\omega}\frac{\sin t}{t} + \mathcal{O}\left(t^{-2}\right), \qquad |t| \gg \omega^{\frac{1}{2}},$$

hence the gentle decay for large $|t|$. Moreover, applying the asymptotic formula to

$$\hat{F}_\omega(t) = \frac{1}{2}\frac{\pi^{\frac{1}{2}}\mathrm{e}^{-t^2/(4\omega)}}{(-\mathrm{i}\omega)^{\frac{1}{2}}}\left[\mathrm{erf}\left((-\mathrm{i}\omega)^{\frac{1}{2}}+\frac{1}{2}\frac{\mathrm{i}t}{(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)+\mathrm{erf}\left((-\mathrm{i}\omega)^{\frac{1}{2}}-\frac{1}{2}\frac{\mathrm{i}t}{(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)\right]$$

we obtain

$$\hat{F}_\omega(t) \sim \frac{\pi^{\frac{1}{2}}}{(-\mathrm{i}\omega)^{\frac{1}{2}}}\mathrm{e}^{-\frac{1}{4}\mathrm{i}t^2/\omega} + \mathcal{O}\left(\omega^{-1}\right), \qquad \omega^{\frac{1}{2}} \gg |t|.$$

This is consistent with Fig. 3.1.

Unfortunately, since $\upsilon^{[N]}$ is not a normal operator, there is no immediate connection between the mapping of $|z| = 1$ under its symbol and the spectrum of $\mathcal{F}_\omega$. Having said so,

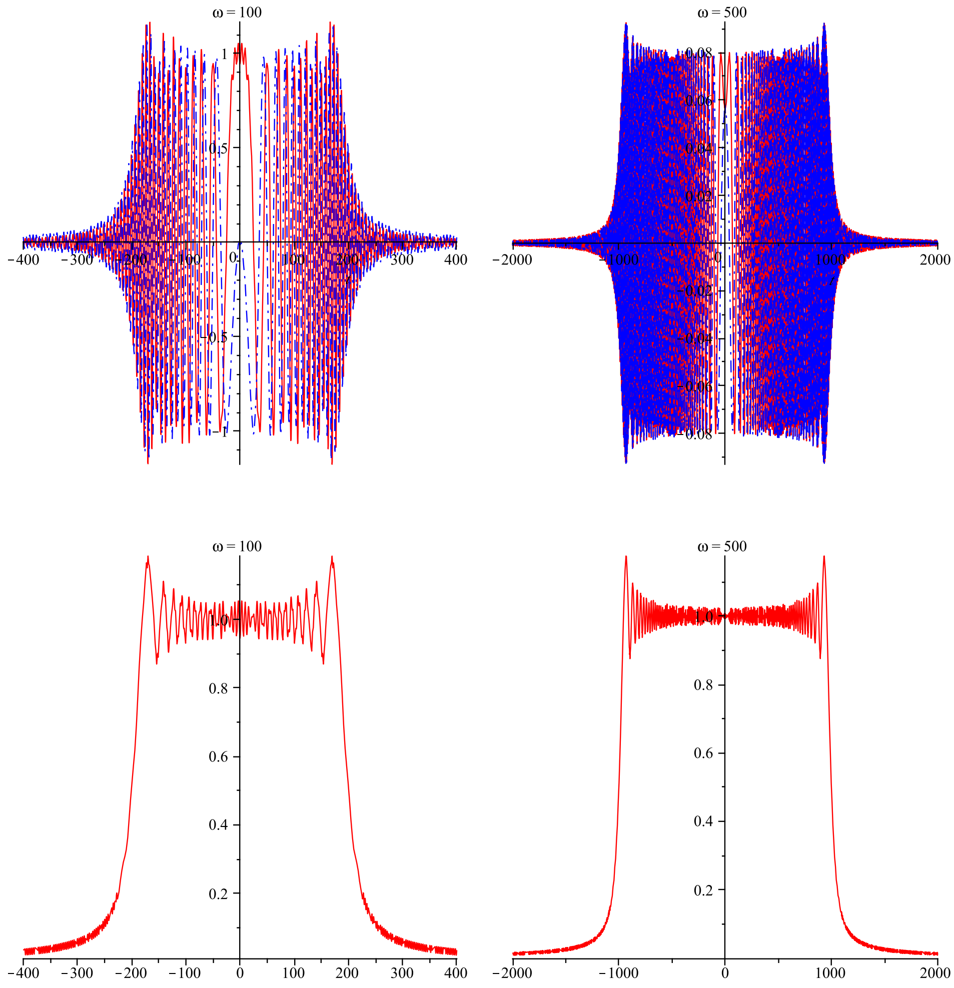

Figure 3.1: In the top row: real (solid) and imaginary (dash–dotted) parts of $(-\mathrm{i}\omega/\pi)^{\frac{1}{2}}\hat{F}_\omega(t)$ for $\omega = 100$ (left) and $\omega = 500$. In the bottom row: $(\omega/\pi)^{\frac{1}{2}}|\hat{F}_\omega(t)$ for the same values of ω

we note that computation indicates that $\hat{F}_\omega(0) = \pi^{\frac{1}{2}}\mathrm{erf}((-\mathrm{i}\omega)^{\frac{1}{2}})/(-\mathrm{i}\omega)^{\frac{1}{2}}$ is either the spectral radius $\rho(\mathcal{F}_\omega)$ or at least an exceedingly good approximation thereof. We do not have an explanation for this observation.

Another interesting observation, so far without any obvious implications for the spectrum of the Fox–Li operator, is the close connection of the symbol $\upsilon^{[N]}$ with the *Jacobi theta function* θ_3 (Rainville 1960, p. 314). Recalling the definition of $\upsilon^{[N]}$, we have

$$\upsilon^{[N]}(\mathrm{e}^{2\mathrm{i}\alpha}) = \frac{1}{N}\left[1 + 2\sum_{k=1}^{N} q_N^{k^2}\cos(2\alpha k)\right],$$

where $q_n = \mathrm{e}^{\mathrm{i}\omega/N^2}$. Compare this with the standard definition

$$\theta_3(\alpha, q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2}\cos(2\alpha k).$$

The snag is that absolute convergence of the above expansion requires $|q| < 1$, while $|q_N| = 1$: what makes $\upsilon^{[N]}$ stay 'nice' when $N \to \infty$ is the normalizing factor $1/N$.

Yet, there appears to be a deep connection between the theta function and $\sigma(\mathcal{F}_\omega)$, and this is confirmed by our numerical experimentation. Thus, we consider sequences $\boldsymbol{q} = \{q_{N,\omega}\}_{N\in\mathbb{N}}$ such that $|q_{N,\omega}| < 1$ for all $N \in \mathbb{N}$, $\lim_{N\to\infty} q_{N,\omega}q_N^{-1} = 1$, and examine

$$\kappa^{[N]}(\alpha) = \frac{\theta_3(\alpha, q_{N,\omega})}{N}, \qquad N \gg 1, \quad -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$$

(it is enough, by symmetry, to restrict α to $[0, \frac{\pi}{2}]$). Everything now depends on the specific choice of the sequence $\boldsymbol{q}$: in our experience, we need to attenuate $|q_N|$ by exactly the right amount, to obtain a good fit with the Fox–Li spiral. After a large number of trials, we have used

$$q_{N,\omega} = 1 - \frac{\omega^{\frac{1}{2}}}{2^{\frac{1}{2}}N^2},$$

and this results in Fig. 3.2, where we have superimposed the theta function curve on the eigenvalues of $\mathcal{F}_\omega$ for $\omega = 100$ and $\omega = 200$. Although the match is far from perfect, in particular in the 'intermediate' regime along the spiral, and we can provide neither rigorous proof nor intuitive explanation, there is enough in the figure to indicate that, at the very least, we are on the right track in seeking the explicit form for the 'spectral spiral' of $\sigma(\mathcal{F}_\omega)$.

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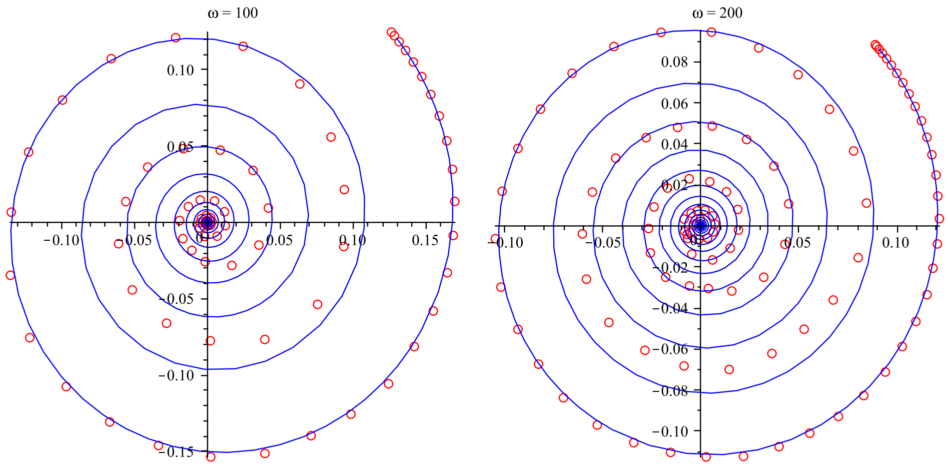

Figure 3.2: Attenuated theta spirals for $\omega = 100$ and $\omega = 200$.

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