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Asymptotic solvers for oscillatory systems of differential equations

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Abstract

We describe an asymptotic method for approximating solutions of systems of ODEs with oscillatory forcing terms. The approach is based on asymptotic expansions in inverse powers of the oscillatory parameter ω and on modulated Fourier expansions. We revise some relevant examples, including problems that appear in the modelling of mechanical and electronic systems, and for which the method is superior to standard methods.

Key words : *Initial value problems, oscillatory ordinary differential equations, modulated Fourier expansions, asymptotic expansions*
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1. Introduction

The efficient and accurate solution of systems of ordinary differential equations (ODEs) subject to oscillatory forcing terms is highly relevant in several aspects of the modelling of electronic circuits subject to amplitude and frequency modulation, see for instance [3, 12, 21] and also [10] and the references therein. More precisely, in this review we are concerned with systems of ODEs of the form

$$\boldsymbol{y}'(t) = \boldsymbol{h}(\boldsymbol{y}(t)) + g_\omega(t)\boldsymbol{f}(\boldsymbol{y}(t)), \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{1}$$

where $\boldsymbol{y}(t) : \mathbb{R} \to \mathbb{R}^d$, $\boldsymbol{f}(\boldsymbol{y}), \boldsymbol{h}(\boldsymbol{y}) : \mathbb{R}^d \to \mathbb{R}^d$ are analytic functions (generally nonlinear), and the scalar term $g_\omega(t)$ can be written in the form of a modulated Fourier

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expansion (MFE), that is

$$g_\omega(t) = \sum_{m=-\infty}^{\infty} a_m(t) e^{im\omega t}. \tag{2}$$

Observe that we allow the coefficients $a_m(t)$ to depend on the variable t, and also that within this setting we cover the case of ODEs with oscillatory forcing terms (when the function $\boldsymbol{f}(\boldsymbol{y}(t))$ is constant). In general, we make the important assumption that the functions $\boldsymbol{h}$ and $\boldsymbol{f}$ are independent of ω. Under certain conditions it is possible to allow dependence on ω in the differential equation with only minor changes in the general setting, see the details below.

Typical examples of forcing terms are

$$g_\omega(t) = e^{i\omega t}, \quad g_\omega(t) = e^{\eta \cos \omega t}. \tag{3}$$

In this latter case, the forcing term has full spectrum, as follows from the Fourier expansion

$$e^{\eta \cos \omega t} = I_0(\eta) + 2 \sum_{m=1}^{\infty} I_m(\eta) \cos m\omega t, \tag{4}$$

in terms of the modified Bessel functions $I_m(\eta)$, see [1, Eq. 9.6.34]. Other examples include two different large frequencies $g_{\omega_1,\omega_2}(t) = \sin \omega_1 t \, \sin \omega_2 t$, and combinations of the above. See for instance [11], and also [23, 22], where this type of problem is formulated in terms of PDEs.

From a mathematical point of view, the use of standard methods of numerical ODEs (such as Runge–Kutta) is problematic in this context, since the oscillatory behaviour of the solutions of the differential equation imposes exceedingly small stepsize which is both too expensive for implementation and leads to an accumulation of round-off error due to the large number of steps needed to integrate the ODE in a given interval.

As a toy example, consider the following second order linear equation with forcing term:

$$y''(t) + y(t) = 2 \sin \omega t, \quad y(0) = 1, \quad y'(0) = 0.$$

or in matrix form

$$\boldsymbol{y}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{y}(t) + \sin \omega t \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \tag{5}$$

If we set $\omega = 10^4$ (a very modest value, given the high frequencies in which we are interested), and solve it with the MATLAB standard `ode45` routine in the interval $t \in [0, 10]$, with initial values $y(0) = 1$, $y'(0) = 0$, the number of steps needed for a fixed relative tolerance is

RelTol	Steps
10^{-4}	42.233
10^{-6}	127.329
10^{-8}	345.189

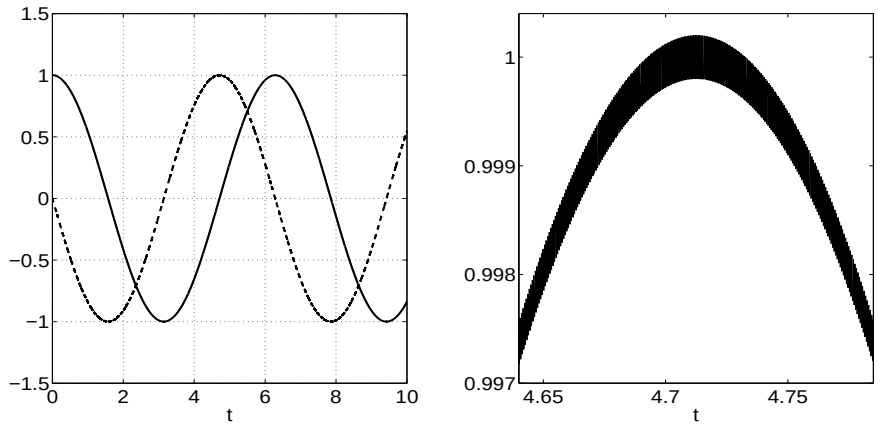

Figure 1: On the left, the function $y(t)$ (solid) and its derivative $y'(t)$ (dashed), solutions of the system (5), with $\omega = 10^4$. On the right, detailed plot of the derivative $y'(t)$, showing the rapid oscillations that are superimposed on the smooth curve on the left.

The plots shown in Figure 1 are quite revealing of the behaviour of the solutions of this kind of systems of ODEs, and they give a crucial insight that leads to the *ansatz* that we propose later on. Observe that the (apparently) slowly varying function $y'(t)$ is superimposed with tiny oscillations (of amplitude $\mathcal{O}(\omega^{-1})$), which impose a very small stepsize on a standard time-stepping routine. See a similar example in [8]. This is consistent with the behaviour observed in the example.

The reason for the poor performance of standard methods in this setting is a direct consequence of their underlying mathematical construction. Methods such as Runge–Kutta are essentially based on Taylor expansion and matching the exact and the numerical solution up to a certain order. In any numerical method of order p with step h, the error scales roughly like $h^{p+1}\boldsymbol{y}^{(p+1)}(t)$, and since the amplitudes of the derivatives of highly oscillatory functions grow very fast, typically $\boldsymbol{y}^{(p+1)}(t) = \mathcal{O}(\omega^{p+1})$ (consider the standard example $\boldsymbol{y}(t) = e^{i\omega t}$), we require h to be extremely small in order to keep the error down to an acceptable size.

2. Perturbation theory and oscillatory integrals

The idea of the proposed method is related to perturbation theory, although not in a standard form. We recall that, roughly speaking, perturbation methods (such as averaging) are designed for problems where the perturbation (or other terms in the differential equation) is multiplied by a small parameter ε, see for instance [4, 25]. Then the general idea is that the solution of the unperturbed system plus corrections in powers of ε yields a good approximation to the solution of the perturbed problem.

In our case the perturbation is not necessarily small in size, but a somewhat related

idea can be applied, because if $g_\omega(t)$ is of the trigonometric type, for instance, then it gives a contribution which is not small but has an average of zero because of major cancellation between positive and negative parts of the forcing term $g_\omega(t)\boldsymbol{f}(\boldsymbol{y}(t))$.

More rigorously, the main idea is to consider the system (1) as a perturbation of

$$\boldsymbol{z}'(t) = \boldsymbol{h}(\boldsymbol{z}(t)), \qquad \boldsymbol{z}(0) = \boldsymbol{y}_0, \tag{6}$$

and then use nonlinear variation of constants [17], in order to relate the solutions of the perturbed and unperturbed systems with the same initial data:

$$\boldsymbol{y}(t) - \boldsymbol{z}(t) = \int_0^t \boldsymbol{\Phi}(t-s)\boldsymbol{f}(\boldsymbol{y}(s))g_\omega(s)ds. \tag{7}$$

Here $\boldsymbol{\Phi}$ is the solution of the so-called variational equation

$$\boldsymbol{\Phi}' = \frac{\partial \boldsymbol{h}(\boldsymbol{z}(t))}{\partial \boldsymbol{z}}\boldsymbol{\Phi}, \qquad \boldsymbol{\Phi}(0) = I.$$

The matrix $\boldsymbol{\Phi}$ is in general not available analytically, yet if the integrand is smooth enough and $g_\omega(t)$ is a trigonometric function (see the examples cited before) then integration by parts gives

$$\boldsymbol{y}(t) - \boldsymbol{z}(t) = \int_0^t \boldsymbol{\Phi}(t-s)\boldsymbol{f}(\boldsymbol{y}(s))g_\omega(s)ds = \mathcal{O}(\omega^{-1}), \qquad \omega \to \infty, \tag{8}$$

since Φ is independent of ω.

This demonstrates that, subject to fairly general assumptions, the difference between both solutions is of order $\mathcal{O}(\omega^{-1})$. Note that here it is essential to suppose that the functions $\boldsymbol{h}$ and $\boldsymbol{f}$ are independent of ω, otherwise the estimation needs to be modified.

This observation makes it possible to use the methodology recently developed for oscillatory integrals, see for instance [19] or [18], as an effective alternative to standard quadrature. More explicitly, in [8, 11] the authors propose the use of Filon quadrature (plus waveform relaxation) to approximate efficiently the oscillatory integral present in (7). This method is numerically effective, but it has the disadvantage of time-stepping, which can be expensive.

A deeper idea, in the spirit of the general theory of oscillatory integrals, is to look for a full expansion of this integral in inverse powers of the oscillatory parameter ω, either by repeated integration by parts or by another method. The main *ansatz* proposed in [9, 10] is precisely that the solution $\boldsymbol{y}(t)$ admits an expansion in inverse powers of ω, and that the terms in this construction adopt the form of modulated Fourier expansions. This can be seen as a correction (in inverse powers of ω this time) of the solution of the unperturbed system, in the spirit of perturbation theory.

If feasible, this approach confers three important computational advantages: firstly, the terms in the expansion can be computed beforehand (symbolically or numerically), and then the desired value of t can be substituted therein, without subdividing the interval of integration. Secondly, the cost of the method is essentially independent of the size of ω, unlike what happens with standard methods, and increasing this parameter will make our method more accurate. Finally, once the (ω-independent) coefficients have been computed, the equation can be solved easily for different frequencies ω.

3. The form of the asymptotic expansion

3.1. Systems of ODEs

If the forcing term in the differential equation can be written as a modulated Fourier expansion, it seems reasonable to suppose that a similar structure holds for the solution of the ODE. More explicitly, we assume that $\boldsymbol{y}(t)$ can be written in the form

$$\boldsymbol{y}(t) \sim \sum_{s=0}^{\infty} \frac{1}{\omega^s} \boldsymbol{\psi}_s(t) \qquad \omega \gg 1, \tag{9}$$

where the functions $\boldsymbol{\psi}_s(t)$ may depend on ω, but in any case $\boldsymbol{\psi}_s(t) = \mathcal{O}(1)$, $\omega \gg 1$, for $s \in \mathbb{Z}_+$. Each of the $\boldsymbol{\psi}_s(t)$ is itself a modulated Fourier expansion:

$$\boldsymbol{\psi}_s(t) = \sum_{m=-\infty}^{\infty} \boldsymbol{p}_{s,m}(t) e^{im\omega t}, \qquad s \geq 1. \tag{10}$$

Once we stipulate that the coefficients of the ODE are independent of ω, the first level of the expansion can be simplified, that is, $\boldsymbol{\psi}_0(t) = \boldsymbol{p}_{0,0}(t)$, i.e. $\boldsymbol{p}_{0,m}(t) \equiv \mathbf{0}$ when $m \neq 0$. This prevents positive powers of ω from appearing in the derivative of $\boldsymbol{y}(t)$.

We also impose $\boldsymbol{\psi}_0(0) = \boldsymbol{y}(0) = \boldsymbol{y}_0$ in order to match the initial condition, which means that $\boldsymbol{\psi}_s(0) = \mathbf{0}$ for $s \geq 1$, or equivalently

$$\sum_{m=-\infty}^{\infty} \boldsymbol{p}_{0,m}(0) = \boldsymbol{y}_0, \qquad \sum_{m=-\infty}^{\infty} \boldsymbol{p}_{s,m}(0) = \mathbf{0}, \qquad s \geq 1.$$

Therefore, the general *ansatz* is

$$\boldsymbol{y}(t) \sim \boldsymbol{p}_{0,0}(t) + \sum_{s=1}^{\infty} \frac{1}{\omega^s} \sum_{m=-\infty}^{\infty} \boldsymbol{p}_{s,m}(t) e^{im\omega t}. \tag{11}$$

We differentiate the function $\boldsymbol{y}(t)$ formally,

$$\boldsymbol{y}' \sim \boldsymbol{p}'_{0,0} + \sum_{m=-\infty}^{\infty} im\boldsymbol{p}_{1,m} e^{im\omega t} + \sum_{s=1}^{\infty} \frac{1}{\omega^s} \sum_{m=-\infty}^{\infty} \left[\boldsymbol{p}'_{s,m} + im\boldsymbol{p}_{s+1,m}\right] e^{im\omega t}.$$

Observe that for simplicity of notation, we have omitted the dependence on t of the $\boldsymbol{p}_{s,m}(t)$ terms. Next, we expand the functions $\boldsymbol{h}, \boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^d$ about the leading term $\boldsymbol{p}_{0,0}$. In this expansion, we group all those terms that multiply equal (inverse) powers of ω, thus separating orders of magnitude:

$$\boldsymbol{h}(\boldsymbol{y}) \sim \boldsymbol{h}(\boldsymbol{p}_{0,0}) + \sum_{s=1}^{\infty} \frac{1}{\omega^s} \sum_{n=1}^{s} \frac{1}{n!} \sum_{\boldsymbol{k} \in \mathbb{I}_{n,s}} \boldsymbol{h}_n(\boldsymbol{p}_{0,0}, \boldsymbol{\chi}_{k_1}, \cdots, \boldsymbol{\chi}_{k_n}), \tag{12}$$

where

$$(\boldsymbol{h}_n(\boldsymbol{p}_{0,0}, \boldsymbol{\theta}, \ldots, \boldsymbol{\theta}))_r = \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} \frac{\partial^n h_r(\boldsymbol{p}_{0,0})}{\partial y_{i_1} \cdots \partial y_{i_n}} \theta_{i_1} \theta_{i_2} \cdots \theta_{i_n}, \qquad r = 1, 2, \ldots, d,$$

$$\boldsymbol{\chi}_k(t) = \sum_{m=-\infty}^{\infty} \boldsymbol{p}_{k,m}(t) e^{im\omega t} \tag{13}$$

and

$$\mathbb{I}_{n,s} = \{(k_1, \ldots, k_n) \in \mathbb{N}^n \,:\, |\boldsymbol{k}| = s\}, \tag{14}$$

with the standard notation for multi-indices $|\boldsymbol{k}| = k_1 + k_2 + \ldots + k_n$. A similar formula applies to the function $\boldsymbol{f}$.

Finally, we collect all those terms that have the same frequency (that is, those terms that multiply $e^{im\omega t}$ for each $m \in \mathbb{Z}$) within each level:

$$\boldsymbol{h}(\boldsymbol{y}) \sim \boldsymbol{h}(\boldsymbol{p}_{0,0}) + \sum_{s=1}^{\infty} \frac{1}{\omega^s} \sum_{m=-\infty}^{\infty} \boldsymbol{b}_{m,s}[\boldsymbol{h}] e^{im\omega t}, \tag{15}$$

where we use the notation

$$\boldsymbol{b}_{m,s}[\boldsymbol{h}] = \sum_{n=1}^{s} \frac{1}{n!} \sum_{\boldsymbol{k} \in \mathbb{I}_{n,s}} \sum_{\boldsymbol{l} \in \mathbb{K}_{n,m}} \boldsymbol{h}_n(\boldsymbol{p}_{0,0}, \boldsymbol{p}_{k_1,l_1}, \cdots, \boldsymbol{p}_{k_n,l_n}) \tag{16}$$

and

$$\mathbb{K}_{n,m} = \{(l_1, \ldots, l_n) \in \mathbb{Z}^n \,:\, |\boldsymbol{l}| = m\}. \tag{17}$$

Equating both sides of the differential equation, from (11) and (15), a general pattern emerges: for $s \geq 1$, we obtain nonoscillatory differential equations for the $\boldsymbol{p}_{s,0}$ terms:

$$\boldsymbol{p}'_{s,0} = \boldsymbol{b}_{s,0}[\boldsymbol{h}] + \sum_{r=-\infty}^{\infty} a_r(t) \boldsymbol{b}_{s,-r}[\boldsymbol{f}] \tag{18}$$

with initial condition

$$\boldsymbol{p}_{s,0}(0) = -\sum_{m \neq 0} \boldsymbol{p}_{s,m}(0), \tag{19}$$

since we have imposed that $\boldsymbol{\psi}_s(0) = \boldsymbol{0}$ for $s \geq 1$. Additionally, we get recursions to compute $\boldsymbol{p}_{s+1,m}(t)$ for $m \neq 0$ from the previous coefficients:

$$\boldsymbol{p}_{s+1,m} = -\frac{i}{m} \left[-\boldsymbol{p}'_{s,m} + \boldsymbol{b}_{s,m}[\boldsymbol{h}] + \sum_{r=-\infty}^{\infty} a_r(t) \boldsymbol{b}_{s,m-r}[\boldsymbol{f}] \right] \tag{20}$$

for $m \neq 0$.

This is the general scheme that we are going to use to deduce the coefficients in the expansion. Note that the differential equations for the $\boldsymbol{p}_{r,0}$ terms can in principle be solved by standard methods, since there is no highly oscillatory behaviour involved. However, in some cases there may be an additional structure to this equation (such as a Hamiltonian formulation, see the example of the inverted pendulum later on), and this calls for more specialised methods.

We remark that modulated Fourier expansions for oscillatory ODEs have been widely analysed as an essential tool in Geometric Numerical Integration and highly

oscillatory ODEs in [6, 7, 16]. In that context, the authors are concerned with equations of the form $\ddot{x} + \Omega x = g(x)$, where

$$\Omega = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \omega^2 I \end{array}\right),$$

$\omega \gg 1$ and I is the identity matrix, both in Hamiltonian and non Hamiltonian settings.

The main difference between this setting and the one proposed in this paper is that from the point of view of perturbation theory, this is no longer a regular perturbation problem but a singular one. Note that our *ansatz* corresponds essentially to a regular perturbation problem: we expand the perturbed solution in inverse powers of ω, so the standard small parameter would be $\varepsilon = 1/\omega$, and we assume that when $\omega \to \infty$ we only have the unperturbed solution.

If powers of ω are allowed in the differential equation, say $\boldsymbol{h}(\boldsymbol{y}), \boldsymbol{f}(\boldsymbol{y}) = \mathcal{O}(\omega^r)$, $r > 0$, then great care is needed, since we may be faced with a singular perturbation problem, and the derivation of the terms in the asymptotic expansion should be carried out in a different way, see [16, XIII.5]. One important case where our setting remains valid, while allowing dependence on ω of the coefficients of the ODE, is that of the inverted pendulum, that we analyse in Section 6.3.

3.2. Second order ODEs

The general setting (11) can be simplified in some circumstances. One particular case of importance is that of second order differential equations with oscillatory forcing terms. This situation has been analysed in [9] for differential equations of the form

$$y''(t) - R(y)y'(t) + S(y(t)) = g_\omega(t), \qquad y(0) = y_0, \quad y'(0) = y_0',$$

which includes equations of Van der Pol and Duffing type. In this case, if we are only interested in the analysis of the solution $y(t)$, and not of the derivative, we may assume that

$$y(t) \sim p_{0,0}(t) + \frac{1}{\omega} p_{1,0}(t) + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} p_{r,m}(t) e^{im\omega t}. \tag{21}$$

We note that this is consistent with the fact that $y(t)$ exhibits small oscillations with amplitude $\mathcal{O}(\omega^{-2})$, increasing to $\mathcal{O}(\omega^{-1})$ for $y'(t)$, see [9]. However, if we analyse the complete system, then we need to use (11), because the derivative will have oscillatory terms at the $\mathcal{O}(\omega^{-1})$ level.

If we are working with a second order differential equation of the form

$$y''(t) + S(y(t)) = g_\omega(t) Q(y(t)), \qquad y(0) = y_0, \quad y'(0) = y_0',$$

then using (21), we obtain the scheme

$$p_{s,0}'' = -b_{s,0}[S] + \sum_{r=-\infty}^{\infty} a_r(t) b_{s,-r}[Q], \tag{22}$$

for $s \geq 1$, together with the initial conditions

$$p_{s,0}(0) = -\sum_{m\neq 0} p_{s,m}(0), \quad s \geq 2, \tag{23}$$
$$p'_{s,0}(0) = -\sum_{m\neq 0} p'_{s,m}(0) - i\sum_{m=-\infty}^{\infty} m p_{s+1,m}(0), \quad s \geq 2,$$

see [9], and also

$$p_{s+2,m} = \frac{1}{m^2}\left\{p''_{s,m} + 2imp'_{s+1,m} + b_{s,m}[S] - \sum_{r=-\infty}^{\infty} a_r(t) b_{s,m-r}[Q]\right\}, \tag{24}$$

for $m \neq 0$. In this case

$$b_{s,m}[Q] = \sum_{n=1}^{s} \frac{Q^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} \prod_{i=1}^{n} p_{k_i,l_i}.$$

In this setting we can also allow dependence on ω in the coefficients of the differential equation. For example, in the case of the inverted pendulum, see Section 6.3, we have $Q(y(t)) = \mathcal{O}(\omega)$. Our approach is still valid in this situation, although we will need to shift the coefficients $b_{s,m}[Q]$ computed above, because of the extra power of ω. Moreover, from (8) we deduce that $y(t) - p_{0,0}(t) = \mathcal{O}(1)$ when ω is large, so we need additional terms in the expansion in order to guarantee that the difference between the perturbed and unperturbed solutions becomes small when ω grows.

4. Construction of the asymptotic expansion

In this section we derive the first few terms of the asymptotic expansion explicitly in a general setting.

4.1. The zeroth term

The term $\boldsymbol{p}_{0,0}(t)$ obeys the following differential equation:

$$\boldsymbol{p}'_{0,0} = \boldsymbol{h}(\boldsymbol{p}_{0,0}) + a_0(t)\boldsymbol{f}(\boldsymbol{p}_{0,0}),$$

which is obtained by equating all the $\mathcal{O}(1)$ terms with zero frecuency in the ODE. Additionally, we impose the initial condition $\boldsymbol{p}_{0,0}(0) = \boldsymbol{y}(0) = \boldsymbol{y}_0$. Equating nonzero terms with the same frequency, we obtain

$$\boldsymbol{p}_{1,m} = -\frac{ia_m(t)}{m}\boldsymbol{f}(\boldsymbol{p}_{0,0}), \qquad m \neq 0.$$

4.2. The first term

When $s = 1$ we obtain

$$\boldsymbol{p}'_{1,0} = \boldsymbol{b}_{1,0}[\boldsymbol{h}] + \sum_{r=-\infty}^{\infty} a_r(t)\boldsymbol{b}_{1,-r}[\boldsymbol{f}]. \tag{25}$$

Here

$$\boldsymbol{b}_{1,m}[\boldsymbol{h}] = \boldsymbol{h}_1(\boldsymbol{p}_{0,0}, \boldsymbol{p}_{1,m}), \qquad m \in \mathbb{Z},$$

and similarly for $\boldsymbol{f}$. Additionally we have the initial condition

$$\boldsymbol{p}_{1,0}(0) = -\sum_{m\neq 0} \boldsymbol{p}_{1,m}(0),$$

which follows from $\boldsymbol{\psi}_1(0) = \mathbf{0}$. Furthermore, we get

$$\boldsymbol{p}_{2,m} = -\frac{i}{m}\left[-\boldsymbol{p}'_{1,m} + \boldsymbol{b}_{1,m}[\boldsymbol{h}] + \sum_{r=-\infty}^{\infty} a_r(t)\boldsymbol{b}_{1,m-r}[\boldsymbol{f}]\right], \qquad m \neq 0, \tag{26}$$

so we can compute $\boldsymbol{p}_{2,m}$ for $m \neq 0$ from (26).

The equation and initial conditions for $\boldsymbol{p}_{2,0}$ are obtained when analysing the $\mathcal{O}(\omega^{-2})$ terms.

4.3. The second term

When $s = 2$ we obtain

$$\boldsymbol{p}'_{2,0} = \boldsymbol{b}_{2,0}[\boldsymbol{h}] + \sum_{r=-\infty}^{\infty} a_r(t)\boldsymbol{b}_{2,-r}[\boldsymbol{f}]. \tag{27}$$

Here

$$\boldsymbol{b}_{2,m}[\boldsymbol{h}] = \boldsymbol{h}_1(\boldsymbol{p}_{0,0}, \boldsymbol{p}_{2,m}) + \frac{1}{2}\sum_{l=-\infty}^{\infty} \boldsymbol{h}_2(\boldsymbol{p}_{0,0}, \boldsymbol{p}_{1,l}, \boldsymbol{p}_{1,m-l}), \qquad m \in \mathbb{Z},$$

and similarly for $\boldsymbol{f}$, together with

$$\boldsymbol{p}_{2,0}(0) = -\sum_{m\neq 0} \boldsymbol{p}_{2,m}(0),$$

and a recursion for the $\boldsymbol{p}_{3,m}$ terms,

$$\boldsymbol{p}_{3,m} = -\frac{i}{m}\left[-\boldsymbol{p}'_{2,m} + \boldsymbol{b}_{2,m}[\boldsymbol{h}] + \sum_{r=-\infty}^{\infty} a_r\boldsymbol{b}_{2,m-r}[\boldsymbol{f}]\right], \qquad m \neq 0. \tag{28}$$

It is clear that the process can be iterated, at the price of increasingly cumbersome expressions.

We point out that when the forcing term has a finite number of nonzero frequencies, then there is a considerable simplification in the construction, see the examples below. Moreover, in many relevant cases the functions $\boldsymbol{h}$ and $\boldsymbol{f}$ are quite simple, for example multivariate polynomials of low degree, and hence many terms involving high order derivatives vanish identically.

5. Bandwidth and stability

We highlight two important aspects of this approach: firstly, if the original forcing term is band-limited, that is, if there exists ϱ such that $a_m = 0$ in (2) if $|m| \geq \varrho + 1$, then is this bandwidth preserved in the $\boldsymbol{\psi}_s(t)$ functions of the expansion? Secondly, if we fix ω, is it possible to say something about the stability of the solution when t grows?

The answer to the first question is in general negative. If the system is nonlinear, one should expect an increase in the number of nonzero frequencies as we move to higher values of s. We call this phenomenon *blossoming*, and it is mainly due to the convolution-type term

$$\sum_{r=-\infty}^{\infty} a_r(t)\boldsymbol{b}_{s,m-r}[\boldsymbol{f}]$$

that appears when computing $\boldsymbol{p}_{s+1,m}$, see the previous section. Nevertheless, the rate of increase can be estimated in general. Namely, we have the following result:

Theorem 1 *For $s \geq 0$, the maximum bandwidth θ_s of the term $\boldsymbol{\psi}_s(t)$ is*

$$\theta_s = s\varrho,$$

where ϱ is the bandwidth of the original forcing term.

Proof. We refer the reader to [10] for the details. □

The actual rate of blossoming can be smaller in some important cases. For instance:

Theorem 2 *Let $\boldsymbol{f}(\boldsymbol{y})$ be constant, then we have $\theta_0 = 0$, $\theta_1 = \varrho$ and the maximum bandwidth θ_s of the term $\boldsymbol{\psi}_s(t)$ is*

$$\theta_s = (s-1)\varrho, \qquad s \geq 2.$$

Proof. See [10]. □

In [9] another particular case is considered: if we are dealing with second order differential equations and we are only interested in blossoming for $y(t)$ (and not for the derivative $y'(t)$), then we have the following

Theorem 3 *If $y(t)$ is a solution of the ODE*

$$y''(t) - R(y(t))y'(t) + S(y(t)) = g_\omega(t),$$

where $R(y)$ and $S(y)$ are analytic, and if θ_s is the maximum bandwidth of $\psi_s(t)$ in the asymptotic expansion of $y(t)$, then it is true that $\theta_0 = \theta_1 = 0$, $\theta_2 = \varrho$ and

$$\theta_s = \left\lfloor \frac{s-1}{2} \right\rfloor \varrho, \qquad s \geq 3. \tag{29}$$

Regarding stability for fixed values of ω, it is possible to carry out an analysis of the method by means of standard linearization of the differential equation around the unperturbed solution. We take a partial sum of the asymptotic expansion $\boldsymbol{z}(t)$, in such

a way that $\boldsymbol{y}(t) - \boldsymbol{z}(t) = \mathcal{O}(1/\omega)$. If the coefficients of the ODE are independent of ω, then it suffices to take $\boldsymbol{z}(t) = \boldsymbol{\psi}_0(t)$, otherwise we need more terms in the asymptotic expansion, according to (8).

Let us write $\boldsymbol{y}(t) = \boldsymbol{z}(t) + \boldsymbol{w}(t)$, assuming that $\boldsymbol{w}(t) = \mathcal{O}(1/\omega)$ for large ω. Then, to leading order we have

$$\boldsymbol{w}' = [A(t) + B(t)g_\omega(t)]\,\boldsymbol{w} + g_\omega(t)\boldsymbol{f}(\boldsymbol{z}), \qquad \boldsymbol{w}(0) = \boldsymbol{0}, \tag{30}$$

where $A(t)$ and $B(t)$ are respectively the Jacobian matrices of $\boldsymbol{h}$ and $\boldsymbol{f}$ evaluated at $\boldsymbol{z}(t)$. Now we split the matrices $A(t) + B(t)g_\omega(t) = U + V(t)$, where U is a constant matrix, and compare (30) with the system

$$\boldsymbol{v}' = U\boldsymbol{v}, \qquad \boldsymbol{v}(0) = \boldsymbol{0},$$

with trivial solution $\boldsymbol{v} \equiv \boldsymbol{0}$. Variation of constants gives

$$\boldsymbol{w}(t) = \boldsymbol{0} + \int_0^t \Phi(t-s)\boldsymbol{F}(s)ds = \int_0^t e^{(t-s)U}\boldsymbol{F}(s)ds, \tag{31}$$

where $\Phi(t) = e^{tU}$ is the fundamental matrix of the system and

$$\boldsymbol{F}(s) = V(s)\boldsymbol{w}(s) + g_\omega(s)\boldsymbol{f}(\boldsymbol{z}(s)).$$

It is clear from (31) that $\boldsymbol{w}(t)$ represents a deviation from the zero solution $\boldsymbol{v}(t)$, and therefore the behaviour of $\boldsymbol{w}(t)$ as $t > 0$ is related to the stability of this zero solution. This in turn is governed by the eigenvalues of the fundamental matrix U, see for instance [25, Ch. 6]. More explicitly:

Theorem 4 *If*

- *all the eigenvalues of the matrix U, say λ_k, $1 \le k \le d$, satisfy that $\Re\lambda_k \le 0$, and those eigenvalues with zero real part are simple, and*
- *it is true that for any $t > 0$ there exist constants $c_1, c_2 > 0$ such that*

$$\int_0^t \|V(s)\|ds < c_1, \qquad \int_0^t \|g_\omega(s)\boldsymbol{f}(\boldsymbol{z}(s))\|ds < c_2,$$

then the zero solution solution $\boldsymbol{v}(t)$ is stable in the sense of Lyapunov, and $\boldsymbol{w}(t)$ is bounded.

Proof. It is a consequence of Gronwall's lemma, see [10] for more details. □

When any of the eigenvalues of the matrix U has a positive real part, then one should expect deterioration of the approximation when t grows, even if one still has the $\mathcal{O}(\omega^{-1})$ estimate. A typical example of this situation is given by systems that can develop chaotic behaviour. In those cases, the solution of the perturbed system may be very different from the solution of the unperturbed one, and actually one of them can be chaotic and the other non chaotic, if the perturbation is used for chaos suppression, see for instance [5]. That being said, it may happen that even in this case the approximation is acceptable for small values of t, depending on the application that we have in mind.

6. Examples

In this section, we present the method applied to several different problems. In all cases, we will compare the approximation given by the first few terms of the asymptotic-numerical solver with the exact solution (which is either analytically available or computed numerically – and laboriously – with standard MATLAB routines up to prescribed accuracy). Although other more sophisticated comparisons are possible, we will normally use the standard ODE solver `ode45` in MATLAB, with absolute and relative tolerance equal to 10^{-12} as an illustration.

We use the notation

$$e_s = \left| \boldsymbol{y}(t) - \sum_{n=0}^{s} \frac{\boldsymbol{\psi}_n(t)}{\omega^n} \right|, \qquad s \geq 0,$$

for the errors, taken componentwise.

We remark that the values of ω that we use in these examples are smaller than the ones normally present in applications. This restriction is essentially imposed by the fact that the comparison with the 'exact' solution produced with MATLAB should be reliable and affordable. Increasing ω will benefit the asymptotic-numerical solver, since the approximation with a fixed number of terms will be more accurate, and the computational cost will be roughly similar.

6.1. A damped harmonic oscillator

As a first example, we consider a simple forced oscillator with damping. This system is modelled by a simple second order ODE:

$$x''(t) + bx'(t) + kx(t) = \mu \cos \omega t, \qquad x(0) = x_0, \qquad x'(0) = x_0', \tag{32}$$

where b is the damping coefficient, k the spring constant and we have set the mass m equal to 1 for simplicity. We introduce a forcing term with amplitude μ and frequency ω, and we suppose that $\omega \gg \omega_0$, where ω_0 is the natural frequency of the unperturbed oscillator in the underdamped case. In matrix form:

$$\boldsymbol{x}'(t) = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mu \cos \omega t, \tag{33}$$

thereby using our notation

$$\boldsymbol{h}(\boldsymbol{x}) = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix} \boldsymbol{x}, \qquad \boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The construction of the asymptotic expansion is particularly simple in this case, since we have after brief computation:

$$\boldsymbol{p}_{0,0}' = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix} \boldsymbol{p}_{0,0}, \qquad \boldsymbol{p}_{0,0}(0) = \boldsymbol{x}(0),$$

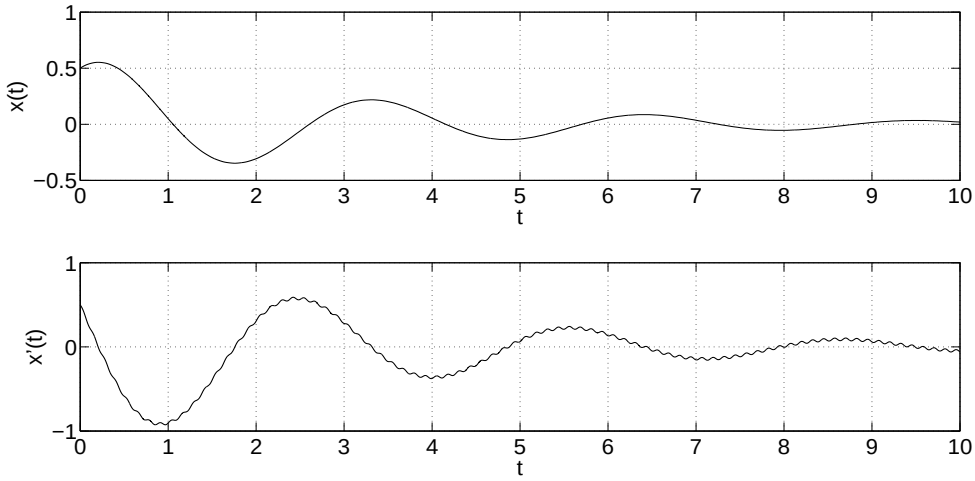

Figure 2: Solution of the perturbed system (32), with parameters $k = 4.2$, $b = 0.6$, $\mu = 0.8$ and $\omega = 50$.

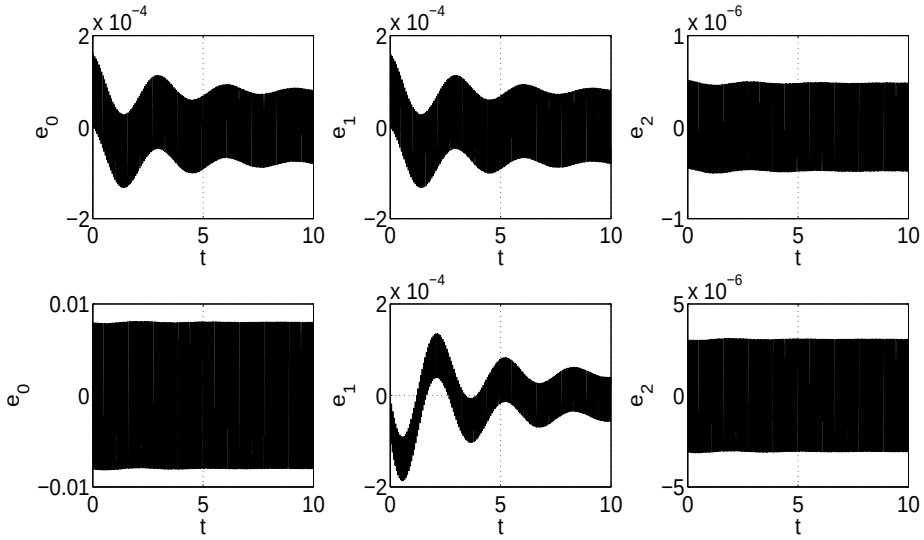

Figure 3: Absolute errors in the approximation of the solution of the perturbed system (33) for $k = 4.2$, $b = 0.6$, $\mu = 0.8$ and $\omega = 100$. Top row, errors in $x(t)$ using the zeroth term (left), using up to the first term of the approximation (centre) and using up to the second term of the approximation (right). Bottom row, same for $x'(t)$.

together with

$$\boldsymbol{p}_{1,-1} = \frac{i\mu}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\boldsymbol{p}_{1,1}.$$

Because the function $\boldsymbol{f}$ is constant, we have from (18) and (19) that for $s \geq 1$

$$\boldsymbol{p}'_{s,0} = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix} \boldsymbol{p}_{s,0}, \qquad \boldsymbol{p}_{s,0}(0) = -\sum_{m\neq 0} \boldsymbol{p}_{s,m}(0).$$

In particular, that means that

$$\boldsymbol{p}'_{1,0} = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix} \boldsymbol{p}_{1,0}, \qquad \boldsymbol{p}_{1,0}(0) = \boldsymbol{0},$$

which leads to $\boldsymbol{p}_{1,0} \equiv \boldsymbol{0}$. Hence we conclude that in this case the first term is simply

$$\boldsymbol{\psi}_1(t) = \begin{pmatrix} 0 \\ \mu \end{pmatrix} \sin \omega t.$$

Note that the $\mathcal{O}(1/\omega)$ term is 0 for the first component of the solution. In other words, $x(t)$ is superimposed with tiny oscillations of amplitude $\mathcal{O}(1/\omega^2)$, whereas in the case of the derivative $x'(t)$, these oscillations have amplitude of order $\mathcal{O}(1/\omega)$. This is intuitively consistent with what can be observed in Figure 2, and also with the *ansatz* taken in [9]. Analogously,

$$\boldsymbol{p}_{2,-1} = -\frac{\mu}{2}\begin{pmatrix} 1 \\ -b \end{pmatrix} = \boldsymbol{p}_{2,1},$$

and since the bandwidth in this example is $\theta_2 = 1$, see Section 5, then we know that $\boldsymbol{p}_{2,m} \equiv \boldsymbol{0}$ if $|m| > 1$. Furthermore

$$\boldsymbol{p}'_{2,0} = U\boldsymbol{p}_{2,0}, \qquad \boldsymbol{p}_{2,0}(0) = -2\boldsymbol{p}_{2,1}(0) = \mu\begin{pmatrix} 1 \\ -b \end{pmatrix},$$

hence

$$\boldsymbol{\psi}_2(t) = \boldsymbol{p}_{2,0} + \mu\begin{pmatrix} -1 \\ b \end{pmatrix} \cos \omega t.$$

Hence, writing everything together, we have

$$\boldsymbol{S}_2(t) = \boldsymbol{p}_{0,0}(t) + \frac{1}{\omega}\begin{pmatrix} 0 \\ \mu \end{pmatrix} \sin \omega t + \frac{1}{\omega^2}\left[\boldsymbol{p}_{2,0}(t) + \mu\begin{pmatrix} -1 \\ b \end{pmatrix} \cos \omega t\right]$$

as our approximation to the solution of the ODE (32) up to order $\mathcal{O}(\omega^{-2})$. In Figure 3 we plot the errors when taking the first few terms of the expansion, compared with the solution of the perturbed system computed directly by the MATLAB standard routine `ode45`.

Regarding stability, since the system is already linear, the matrix U is directly

$$U = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix},$$

with eigenvalues

$$\lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4k}}{2}.$$

Since both $b, k > 0$, then the real part of both eigenvalues is always negative and we have asymptotic stability according to Section 5.

6.2. An expcos oscillator

We include an example that features an oscillator with full spectrum, to show that the method can be equally effective. More precisely, we recall the example of a simple electronic circuit that was presented in [10],

$$x'(t) = -\frac{L}{RC}x(t) + \frac{I_s L}{C}\left[\exp\left(\frac{g_\omega(t) - x(t)}{V_T}\right) - 1\right] - \frac{L}{C}y(t), \qquad y'(t) = x(t),$$

where L, R, C, I_s and V_T are parameters, and the perturbation is $g_\omega(t) = e^{\mu \cos \omega t}$.

Realistic values of the parameters are for instance $L = 10^{-4}$, $R = 100$, $C = 10^{-6}$, $I_s = 10^{-12}$ and $V_T = 0.0259$. The constant term $-I_s L/C$ can be added in the $\mathcal{O}(1)$ level in a straightforward way. The resulting system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} -L/RC & -L/C \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} \beta e^{-x(t)/V_T} \\ 0 \end{pmatrix} \exp\left(\frac{g_\omega(t)}{V_T}\right) - \begin{pmatrix} \beta \\ 0 \end{pmatrix},$$

where $\beta = I_s L/C$. The properties of similar types of oscillator have been analysed in [8] and [11]. As noticed before, see (4), the function $g_\omega(t)$ can be expanded in Fourier series using modified Bessel functions, whose asymptotic behaviour for large orders guarantees convergence for fixed values of μ and t.

It is clear that the coefficients are $a_m = I_m(\mu)$ for $m \in \mathbb{Z}$, using the fact that for integer orders $I_m(\mu) = I_{-m}(\mu)$, see [1, Eq. 9.6.6]. The equation for the zeroth term follows from (18):

$$\boldsymbol{p}_{0,0} = \boldsymbol{h}(\boldsymbol{p}_{0,0}) + I_0(\mu)\boldsymbol{f}(\boldsymbol{p}_{0,0}), \qquad \boldsymbol{p}_{0,0}(0) = \boldsymbol{x}(0),$$

and also

$$\boldsymbol{p}_{1,m} = -\frac{iI_m(\mu)}{m}\boldsymbol{f}(\boldsymbol{p}_{0,0}), \qquad m \neq 0.$$

The differential equation for the nonoscillatory function $\boldsymbol{p}_{0,0}$ cannot be solved explicitly because of the nonlinearity originating in the function $\boldsymbol{f}$, but, being non-oscillatory, can be computed numerically using standard methods.

The differential equation for $\boldsymbol{p}_{1,0}$ reads

$$\boldsymbol{p}'_{1,0} = \boldsymbol{b}_{1,0}[\boldsymbol{h}] + \sum_{r=-\infty}^{\infty} a_r(t)\boldsymbol{b}_{1,-r}[\boldsymbol{f}],$$

which can be simplified to

$$\sum_{r=-\infty}^{\infty} a_r(t)\boldsymbol{b}_{1,-r}[\boldsymbol{f}] = I_0(\mu)\boldsymbol{b}_{1,0}[\boldsymbol{f}] + i\sum_{r\neq 0}\frac{I_r^2(\mu)}{r}\boldsymbol{f}(\boldsymbol{p}_{0,0}) = I_0(\mu)\boldsymbol{b}_{1,0}[\boldsymbol{f}],$$

the last sum being 0 because of the symmetry of the modified Bessel functions with respect to the order. Hence

$$\boldsymbol{p}'_{1,0} = \boldsymbol{b}_{1,0}[\boldsymbol{h}] + I_0(\mu)\boldsymbol{b}_{1,0}[\boldsymbol{f}] = (J[\boldsymbol{h}] + I_0(\mu)J[\boldsymbol{f}])\,\boldsymbol{p}_{1,0}, \qquad \boldsymbol{p}_{1,0}(0) = \boldsymbol{0},$$

which implies that $\boldsymbol{p}_{1,0} \equiv \boldsymbol{0}$. Therefore,

$$\boldsymbol{\psi}_1(t) = 2\boldsymbol{f}(\boldsymbol{p}_{0,0}) \sum_{m=1}^{\infty} \frac{I_m(\mu)}{m} \sin m\omega t.$$

Observe that this last sum converges as well due to the decay of the modified Bessel functions, so its numerical implementation is not problematic.

In a similar way, one can compute the second term $\boldsymbol{\psi}_2(t)$, although the different sums involving modified Bessel functions that appear can be quite expensive to evaluate. In particular, observe that $\mu = 1/V_T$, so if $V_T = 0.0259$ then μ is moderately large and the convergence of the series (4) can be slow. We note, however, that even in this situation the cost of computing the expansion is essentially independent of ω (whereas any standard numerical method will need to reduce the stepsize considerably when ω grows).

6.3. The inverted pendulum

Our main motivation for extending the method to systems of ODEs where the coefficients depend on the parameter ω originates in equations that model mechanical systems, more precisely the equation governing the motion of the inverted pendulum (also known as the *Kapitza pendulum*)

$$\theta''(t) = \frac{1}{\ell}\left[g + a\omega^2 \cos\omega t\right] \sin\theta(t), \qquad \theta(0) = \theta_0, \quad \theta'(0) = \theta'_0, \tag{34}$$

where ℓ is the length of the rod, g is the gravitational acceleration, ω is the frequency of the highly oscillatory motion of the pivot and $\theta(t)$ measures the angle with respect to the vertical direction, $t = 0$ being the top position of the pendulum, see Figure 4. The pivot is subject to a fast vertical movement given by the equation

$$y(t) = \frac{a}{\ell} \cos\omega t.$$

It is known that the top position of a pendulum, which is naturally unstable, can be rendered stable by means of this fast vertical oscillation of the pivot. In order for this stabilization to be possible, we need $a = \mathcal{O}(\omega^{-1})$, so that the forcing input has velocity of order $\mathcal{O}(1)$ with respect to ω, and therefore finite kinetic energy. An alternative justification comes from the stability analysis of a Mathieu equation, which is obtained by supposing that the deviation of the pendulum from the vertical position is small and therefore $\sin\theta(t) \approx \theta(t)$ in (34), see for example [20, §7.7].

We can rewrite the ODE as follows:

$$\theta''(t) = \frac{1}{\ell}\left[g + \sigma\omega \cos\omega t\right] \sin\theta(t), \qquad \theta(0) = \theta_0, \quad \theta'(0) = \theta'_0, \tag{35}$$

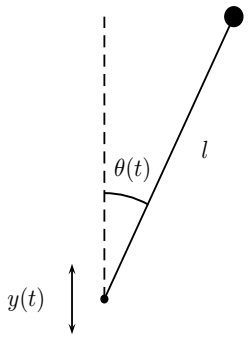

Figure 4: The inverted pendulum. The top position, corresponding to $\theta = 0$, is unstable, but can be rendered stable by adding a fast vertical movement $y(t)$ of the pivot.

where now $\sigma = \mathcal{O}(1)$ with respect to ω. The coefficients of the forcing term are clearly

$$a_1 = a_{-1} = \frac{\sigma}{2\ell}, \qquad a_m = 0,\ |m| \neq 1.$$

We can adapt the ansatz (21) to this case, but since there is a (linear) dependence on ω in the differential equation, it should be now

$$\theta(t) \sim p_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} p_{r,m}(t) e^{im\omega t}. \tag{36}$$

Equating coefficients at the $\mathcal{O}(\omega)$ level, we get

$$-\sum_{m=-\infty}^{\infty} m^2 p_{1,m}(t) e^{im\omega t} = \frac{g}{\ell} \cos\omega t \sin(p_{0,0}(t)),$$

so

$$p_{1,1}(t) = p_{1,-1}(t) = -\frac{\sigma}{2\ell} \sin(p_{0,0}(t)).$$

The $\mathcal{O}(1)$ level gives

$$p''_{0,0}(t) = \frac{g}{\ell} \sin(p_{0,0}(t)) - \frac{\sigma^2}{4\ell^2} \sin(2p_{0,0}(t)),$$

with initial conditions $p_{0,0}(0) = \theta(0)$ and $p'_{0,0}(t) = \theta'(0)$. This is the averaged system proposed in [24], and it can be solved using standard procedures. Alternatively, if we define the potential

$$\mathcal{H}(p,q) = \frac{1}{2}p^2 + \frac{g}{\ell}\cos q - \frac{\sigma^2}{8\ell^2}\cos 2q$$

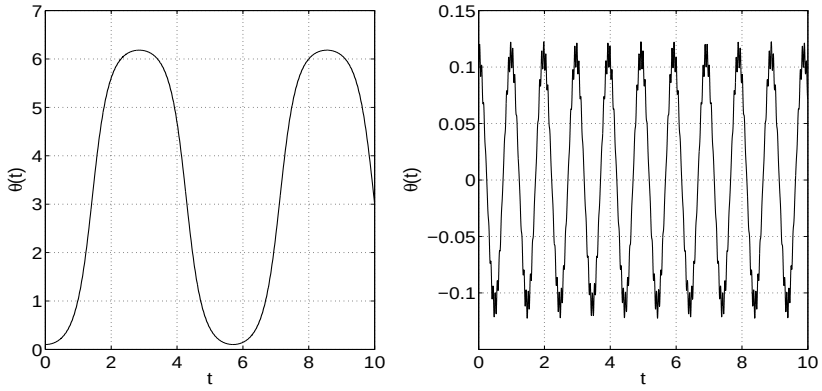

Figure 5: The function $\theta(t)$ with $\ell = 1$, $g = 9.8$, $\omega = 100$ and $\sigma = 1$ (left) and $\sigma = 10$ (right), with initial values $\theta(0) = 0.1$ and $\theta'(0) = 0$. Observe that the top position, $\theta(t) = 0$, becomes stable in the second case.

and set $(x, x') = (q, p)$, then the equation

$$x''(t) = \frac{g}{\ell}\sin(x(t)) - \frac{\sigma^2}{4\ell^2}\sin(2x(t)),$$

is of Hamiltonian type, see [16], and more specific methods can be used. For instance, in [24] the well-known Verlet scheme is proposed, in the context of macrointegration in the Heterogeneous Multiscale Method.

Furthermore, the $\mathcal{O}(1)$ level provides the coefficients $p_{2,m}(t)$ for $m \neq 0$:

$$p_{2,m} = \frac{1}{m^2}\left[2imp'_{1,m} - \frac{\sigma}{2\ell}\cos(p_{0,0})\left[p_{1,m-1} + p_{1,m+1}\right]\right],$$

which are, after brief computation,

$$p_{2,\pm 1} = \mp\frac{i\sigma}{\ell}\cos(p_{0,0})p'_{0,0} - \frac{\sigma}{2\ell}\cos(p_{0,0})p_{1,0}$$
$$p_{2,2} = p_{2,-2} = \frac{\sigma^2}{32\ell^2}\sin(2p_{0,0}).$$

Observe that this is consistent with the bandwidth predicted by the theorem presented in Section 5.

The scheme for computing higher order terms can be obtained from the general formulas (22) and (24). However, since we have a factor ω multiplying the forcing term, we need to shift the coefficients corresponding to the function $Q(y(t))$, namely:

$$p''_{s,0} = -b_{s,0}[S] + \sum_{r=-\infty}^{\infty} a_r(t)b_{s+1,-r}[Q],$$

for $s \geq 1$, and

$$p_{s+2,m} = \frac{1}{m^2}\left[p''_{s,m} + 2imp'_{s+1,m} + b_{s,m}[S] - \sum_{r=-\infty}^{\infty} a_r(t) b_{s+1,m-r}[Q]\right],$$

where in this case

$$S(y) = -\frac{g}{\ell}\sin y, \qquad Q(y) = \sin y.$$

Applying this, we obtain:

$$p''_{1,0} = -b_{1,0}[S] + a_1 b_{2,-1}[Q] + a_{-1} b_{2,1}[Q],$$

and since

$$b_{1,0}[S] = S'(p_{0,0})p_{1,0} = -\frac{g}{\ell}\cos(p_{0,0})p_{1,0},$$
$$b_{2,\pm 1}[Q] = \mp\frac{i\sigma}{\ell}\cos^2(p_{0,0})p'_{0,0} - \frac{\sigma}{2\ell}\cos(2p_{0,0})p_{1,0},$$

then

$$p''_{1,0} = \left[\frac{g}{\ell}\cos(p_{0,0}) - \frac{\sigma^2}{2\ell^2}\cos(2p_{0,0})\right]p_{1,0},$$

with initial conditions

$$p_{1,0}(0) = \frac{\sigma}{\ell}\sin y_0, \qquad p''_{1,0}(0) = -\frac{\sigma}{\ell}y'_0\cos y_0,$$

computed from (23).

Figure 6 illustrate the errors when we take the zeroth and the zeroth plus the first term in the expansion, compared with direct computation of the solution in MATLAB. As expected, and in accordance with (8), the approximation up to order $\mathcal{O}(\omega^{-1})$ gives an error of order $\mathcal{O}(\omega^{-1})$ for the function $\theta(t)$, and of order $\mathcal{O}(1)$ for the derivative.

It is possible to compute higher order terms, although the algebra becomes quite cumbersome due to the terms involving the Q function and the shift due to the dependence on ω. This becomes feasible using a symbolic algebra package, but we omit further details for brevity.

7. Conclusions

This paper has presented a snapshot of the state of the art in a new computational approach to ordinary differential equations with highly oscillatory forcing. Such equations are ubiquitous in numerous applications, while their discretization by standard methods is prohibitively expensive and, for realistic frequencies, often impossible.

The main organising principle of our approach is to combine asymptotic expansions (in inverse powers of the frequency) with numerical solution of non-oscillatory problems. This leads to a number of important advantages.

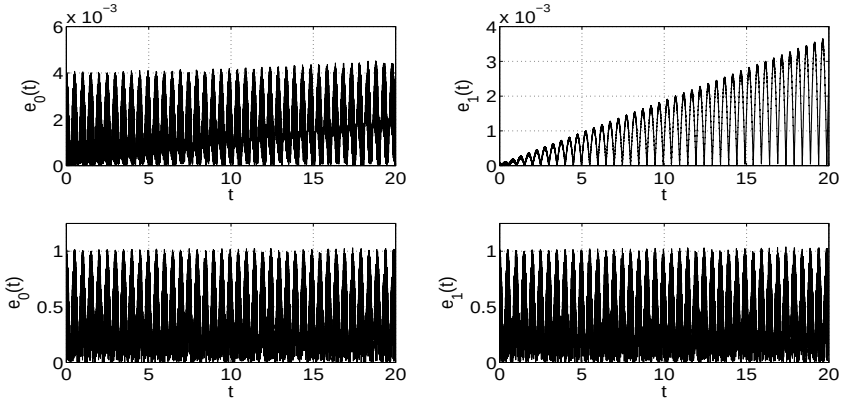

Figure 6: Absolute errors when taking the zeroth (left) and the zeroth plus first terms in the expansion, with parameters $\ell = 1$, $g = 9.8$, $\omega = 1000$ and $\sigma = 10$ and initial values $\theta(0) = 0.1$ and $\theta'(0) = 0$.

1. The entire process consists of 'non-oscillatory' computations: only the final step, a synthesis of different multiscale components, feeds oscillation into the system.

2. The accuracy of the truncated multiscale approximation improves with the growth in frequency. This flies in the face of the usual numerical intuition, which is based upon Taylor expansions, but should not be suprising because of the extensive use of asymptotic information.

3. Once the different multiscale components are in place, they can be used repeatedly with different frequencies: this is an important advantage in engineering design problems when the real challenge is to select the right frequency for a task in hand, e.g. for a stabilisation of a dynamical system.

4. The 'leakage' of oscillations to different bandwidths in nonlinear systems (a process we have termed 'blossoming') can be tracked explicitly and with great precision.

5. Finally, the availability of a truncated asymptotic multiscale expansion represents significantly more than just a long string of floating-point numbers or a graph, the usual output of a standard numerical method. The expansion in question consists of terms which are either solutions of substantively simpler, non-oscillatory ODEs (of which a great deal of useful qualitative information can be derived) or can be obtained by simple recursion. Therefore, such an expansion is a valuable tool in the analysis of the underlying highly oscillatory ODE.

This is the moment to mention that the approach of this paper is just one of the many useful tools that have emerged in the last two decades in our understanding and computation of highly oscillatory phenomena. Of particular relevance to the model of

this paper, an ODE system with highly oscillatory forcing, are homogenization [15] and multiscale methods [14]. Indeed, there are profound connections between our approach and the *heterogeneous multiscale method* [2, 13, 24]. It is fair to expect that the future of highly oscillatory computations, an emerging subject of applied mathematics with an increasing profile, will depend not on a single 'wonder method' but on the understanding and combination of a number of analytic and computational methodologies.

Finally, let us comment on the scope of the approach described in this paper. The underlying motivation to our work is the modelling of high-frequency electronic circuits. The breadth of phenomena modelled in this setting ranges well beyond 'simple' ODEs and it also includes differential–algebraic equations (DAEs), delay differential equations (DDEs) and even partial differential equations (PDEs), all with highly oscillatory forcing. An extension of the methodology of this paper to these settings is a matter of active research. Here we just state that such an extension, although often highly nontrivial, is possible. The next challenge is to integrate the different components – ODE, DAE, DDE and PDE expansions – into an overarching technique for realistic electronic circuits.

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