On the convergence of expansions in polyharmonic eigenfunctions

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Abstract

We consider expansions of smooth functions defined on compact intervals in eigenfunctions of polyharmonic operators equipped with homogeneous Neumann boundary conditions. Having determined asymptotic expressions for both the eigenvalues and eigenfunctions of these operators, we demonstrate how these results can be used in the efficient computation of expansions. Next, we consider the convergence. We establish the key advantage of such expansions over classical Fourier series, namely, both faster and higher-order convergence, and provide a full asymptotic expansion for the error incurred by the truncated expansion. Finally, we obtain conditions that completely determine the convergence rate.

1 Introduction

Modified Fourier expansions have recently been introduced as a minor adjustment of classical Fourier series for the approximation of nonperiodic functions in bounded domains. Developed by Iserles and Nørsett for functions defined in the compact intervals [15], such expansions converge uniformly throughout the domain (including on the boundary), in contrast to Fourier series, which suffer from the well-known Gibbs phenomenon [17]. In fact, when truncated after N terms, the expansion of a (sufficiently smooth) function converges at a rate of $\mathcal{O}\left(N^{-2}\right)$ inside the domain and $\mathcal{O}\left(N^{-1}\right)$ on the boundary [21].

Whilst offering more rapid convergence, such expansions also retain many of the benefits of classical Fourier series. Indeed, in the unit interval $[-1,1]$ the modified Fourier basis is precisely

$$\{\cos n\pi x : n \in \mathbb{N}\} \cup \left\{\sin(n-\tfrac{1}{2})\pi x : n \in \mathbb{N}_+\right\}, \tag{1.1}$$

and thus only differs from the Fourier basis by the shifted argument $n-\frac{1}{2}$ appearing in the sine function. It is known that (1.1) forms an orthogonal basis of $\mathrm{L}^2(-1,1)$ [15], hence any function $f \in \mathrm{L}^2(-1,1)$ may be expressed in terms of its *modified Fourier expansion*

$$f(x) \sim \frac{1}{2}\hat{f}_0^C + \sum_{n=1}^{\infty}\left[\hat{f}_n^C \cos n\pi x + \hat{f}_n^S \sin(n-\tfrac{1}{2})\pi x\right], \quad x \in [-1,1],$$

where $\hat{f}_n^C = \int_{-1}^{1} f(x)\cos n\pi x \,\mathrm{d}x$ and $\hat{f}_n^S = \int_{-1}^{1} f(x)\sin(n-\frac{1}{2})\pi x\,\mathrm{d}x$ are the *modified Fourier coefficients* of f. As regards numerical computation of these coefficients, it has been found to be advantageous to use combinations of highly oscillatory and nonstandard classical quadratures, rather than using the Fast Fourier Transform (which, unsurprisingly, could be exploited in this setting) [15, 16]. This approach allows for more efficient computation of coefficients, with computation of the first N coefficients being possible in only $\mathcal{O}(N)$ operations, as opposed to $\mathcal{O}(N\log N)$ for FFT-based approaches.

To date, modified Fourier expansions have found applications in a number of areas, including the spectral disctetisation of boundary value problems [3, 5] and the computation of spectra of oscillatory integral operators [9]. Benefits over more standard approaches (typically polynomial-based methods) have been documented in [5] and [9].

In this paper, we consider a particular generalisation of the modified Fourier basis (1.1). The aim of this generalisation is to obtain both faster rates and higher degrees of convergence (by the latter, we mean convergence in higher-order norms), whilst retaining the principal benefits of modified Fourier expansions. This topic was originally developed in [6]. The intent of this paper is to provide both a comprehensive theory of such expansions, including resolving a number of conjectures raised therein, and, using theoretical results proved, give a more detailed account of the practical computation of such expansions. First, however, we recap the salient aspects of [6].

1.1 Expansions in polyharmonic eigenfunctions

Modified Fourier expansions can be identified with expansions in eigenfunctions of the Laplace operator equipped with homogeneous Neumann boundary conditions. In the unit interval, (1.1) is precisely the set of eigenfunctions satisfying

$$-\phi''(x) = \mu\phi(x), \quad x \in [-1, 1], \quad \phi'(\pm 1) = 0. \tag{1.2}$$

Interestingly, this observation facilitates the generalisation of modified Fourier expansions to functions defined on certain higher-dimensional domains, including d-variate cubes [16] and particular simplices [13]. As discussed in [15], Neumann boundary conditions are vital to the success enjoyed by such expansions over classical Fourier series. Had Dirichlet boundary conditions $\phi(\pm 1) = 0$ been employed, for example, leading to the basis $\left\{\cos(n - \frac{1}{2})\pi x : n \in \mathbb{N}_+\right\} \cup \{\sin n\pi x : n \in \mathbb{N}_+\}$, slower convergence would be witnessed, as well as a Gibbs-type phenomenon near the endpoints.

The interpretation of the modified Fourier basis in terms of eigenfunctions of the Laplace–Neumann operators indicates how such an approach can be generalised. Seeking more rapidly convergent expansions, we replace the Laplace–Neumann operator with a particular higher-order differential operator equipped with suitably chosen boundary conditions. In [6], it was argued that, amongst all operators of fixed, even order $2q$, $q \in \mathbb{N}_+$, fastest convergence occurs when a function f is expanded in eigenfunctions of the univariate *polyharmonic operator* subject to homogeneous *Neumann boundary conditions*

$$(-1)^q \phi^{(2q)}(x) = \mu\phi(x), \quad x \in [-1, 1], \quad \phi^{(r)}(\pm 1) = 0, \quad r = q, \ldots, 2q - 1. \tag{1.3}$$

In this case, as was shown in [6], the uniform convergence rate is $\mathcal{O}\left(N^{-q}\right)$. This figure improves with increasing q, and exceeds the $\mathcal{O}\left(N^{-1}\right)$ estimate for modified Fourier expansions, which, in view of (1.2), naturally correspond to index $q = 1$.

A significant component of [6] was devoted to the construction of the expansion of a function f in such *polyharmonic–Neumann* eigenfunctions. It was shown that the spectrum of (1.3) consists only of real, nonnegative eigenvalues μ_n, $n \in \mathbb{N}$, with corresponding eigenfunctions ϕ_n that form an orthogonal basis of $\mathrm{L}^2(-1, 1)$. For $q \geq 2$, eigenvalues arise as solutions of a particular transcendental equation and can be easily computed with Newton–Raphson iterations. Moreover, corresponding eigenfunctions always occur in two cases, even and odd, and can be written as sums of products of trigonometric and hyperbolic functions with coefficients that are computed by solving a $q \times q$ algebraic eigenproblem.

Also addressed in [6] was the computation of the expansion coefficients $\hat{f}_n = \int_{-1}^{1} f(x)\phi_n(x)\,\mathrm{d}x$. Using essentially identical techniques to those employed in the modified Fourier case, it was shown that the first N coefficients can be computed in $\mathcal{O}(N)$ operations using only pointwise values of f and certain derivatives.

1.2 Key results and outline

The intent of this paper is to present a more comprehensive study of the eigenfunctions of (1.3) and the corresponding expansion of a function f in such eigenfunctions. The first result we prove concern the precise nature of polyharmonic–Neumann eigenvalues and eigenfunctions. We show that such quantities, whilst not being known explicitly for $q \geq 2$, possess explicit asymptotic representations (in n) that are accurate up to exponentially small remainders. Specifically, having introduced the fundamental properties of polyharmonic–Neumann expansions in Section 2 (and recapped the principal results of [6]), we prove in Section 3 that, if $\mu_n = \alpha_n^{2q}$ is the n^{th} eigenvalue, then

$$\alpha_n = \frac{1}{4}(2n+q-1)\pi + \mathcal{O}\left(\mathrm{e}^{-n\pi\gamma_q}\right), \quad n \gg 1, \tag{1.4}$$

where $\gamma_q = \sin\frac{\pi}{q}$. Moreover, if ϕ_n is the corresponding eigenfunction, we demonstrate that

$$\phi_n(x) = \sum_{s=0}^{q-1} c_s \left[\mathrm{e}^{\frac{1}{4}(2n+q-1)\pi\lambda_s(x-1)} + (-1)^{n+q+1}\mathrm{e}^{-\frac{1}{4}(2n+q-1)\pi\lambda_s(x+1)}\right] + \mathcal{O}\left(\mathrm{e}^{-n\pi\gamma_q}\right), \tag{1.5}$$

where $\lambda_s = -\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}s\pi}{q}}$ and the values c_s are independent of n and known explicitly. Results (1.4) and (1.5) are naturally of theoretical interest. Moreover, the are necessary precursors to a detailed study of the convergence of expansions in polyharmonic–Neumann eigenfunctions, a topic we consider further in Sections 4–6. However, before doing so, we demonstrate how (1.4) and (1.5) provide a simple and effective means to compute the majority of the eigenvalues and eigenfunctions (a necessary first step in constructing the expansion of a function f in such eigenfunctions). Indeed, whilst eigenvalues and eigenfunctions can always be computed by solving an algebraic eigenproblem, we show that this is only necessary for the first handful of values $n = 1, 2, \ldots$. Whenever n is sufficiently large, no computations are required: the estimates (1.4) and (1.5) are exact up to machine epsilon.

Convergence of the polyharmonic–Neumann expansion is considered in Section 4. We prove uniform convergence of this expansion for $f \in \mathrm{H}^1(-1,1)$ (the first classical Sobolev space), and determine the corresponding rate of convergence in Section 5. For smooth f, we derive an asymptotic expansion for the error incurred by its expansion (when truncated after N terms), valid for any point $x \in [-1,1]$. In particular, we show that the rate of convergence is $\mathcal{O}\left(N^{-q}\right)$ uniformly and $\mathcal{O}\left(N^{-q-1}\right)$ in $(-1,1)$. These results generalise those proved in [21] for the modified Fourier ($q = 1$) case. Finally, in Section 6 we discuss the particular factors that determine the convergence rate. Proofs in this paper are largely self-contained: we only assume some basic spectral theory of self-adjoint differential operators.

1.3 Background

The expansion of a function in eigenfunctions of an arbitrary differential operator has been extensively studied. More commonly referred to as a *Birkhoff expansion* [7, 8, 10, 20], much is known in the general case about both convergence and the asymptotic nature of the eigenvalues and eigenfunctions. However, as mentioned in [6], this theory inadequately describes the case of polyharmonic–Neumann expansions. In particular, estimates similar to (1.4) and (1.5) are known to hold for a broad variety of differential operators and boundary conditions, but only with $\mathcal{O}\left(n^{-1}\right)$ remainder terms. To the best of our knowledge, the exponentially-small terms appearing in (1.4) and (1.5) do not currently exist in literature. In addition, though much is known regarding convergence of Birkhoff expansions, in particular as regards the phenomenon of *equiconvergence* [19] (see also [24]), most studies consider only convergence in $(-1,1)$ or assume that the approximated function obeys the same boundary conditions as those prescribed to the linear operator. For polyharmonic–Neumann expansions, such results are of limited use. Nevertheless, the particular nature of the polyharmonic–Neumann operator and its eigenfunctions permits us to compile a far more thorough and accurate theory of the corresponding expansions.

2 Polyharmonic eigenfunction bases

The univariate polyharmonic operator $\mathcal{L} = (-1)^q \frac{\mathrm{d}^{2q}}{\mathrm{d}x^{2q}}$, when equipped with homogeneous Neumann boundary conditions, is semi-positive definite. By standard spectral theory, its spectrum consists of a countable number of nonnegative eigenvalues [18], which we denote μ_n, $n \in \mathbb{N}$. For convenience, we define α_n by $\mu_n = \alpha_n^{2q}$.

Since $\mathcal{L}[\phi] = 0$ if and only if $\phi \in \mathbb{P}_{q-1}$ is a polynomial of degree less than q, $\mu = 0$ is a q-fold eigenvalue. The corresponding orthonormal eigenfunctions are $\phi_{0,n}$, $n = 0, \ldots, q-1$, where $\phi_{0,n} = (n+\frac{1}{2})^{\frac{1}{2}} P_n$ and P_n is the n^{th} Legendre polynomial. All other eigenvalues μ_n are positive, and by standard spectral theory, simple and the collection $\{\mu_n\}$ has no finite limit point. The corresponding eigenfunctions ϕ_n, $n \in \mathbb{N}$, in combination with $\phi_{0,n}$, $n = 0, \ldots, q-1$, form a dense, orthogonal subset of $\mathrm{L}^2(-1,1)$.

An explicit form for the polyharmonic–Neumann eigenfunctions was derived in [6]. In the next section we recap this construction.

2.1 Explicit form of polyharmonic–Neumann eigenfunctions

Let ϕ be a polyharmonic–Neumann eigenfunction with eigenvalue $\mu = \alpha^{2q}$. We first note that

$$\phi(x) = \sum_{r=0}^{2q-1} c_r \mathrm{e}^{\lambda_r \alpha x}, \tag{2.1}$$

where the values $\lambda_r \in \mathbb{C}$ satisfy $\lambda_r^{2q} = (-1)^q$, $r = 0, \ldots, 2q-1$ and the parameters $c_r \in \mathbb{C}$ are determined by the boundary conditions. Simplification of this expression requires separately addressing the two cases corresponding to even and odd q. With q even, the eigenfunction ϕ takes one of two possible forms ϕ^e, ϕ^o corresponding to even or odd functions respectively. These are

$$\begin{aligned}\phi^e(x) = &\sum_{r=0}^{\frac{q}{2}} c_r^e \cos\left(\alpha^e x \sin\frac{\pi r}{q}\right)\cosh\left(\alpha^e x \cos\frac{\pi r}{q}\right) \\ &+ \sum_{r=1}^{\frac{q}{2}-1} d_r^e \sin\left(\alpha^e x \sin\frac{\pi r}{q}\right)\sinh\left(\alpha^e x \cos\frac{\pi r}{q}\right),\end{aligned} \tag{2.2}$$

$$\begin{aligned}\phi^o(x) = &\sum_{r=0}^{\frac{q}{2}-1} c_r^o \cos\left(\alpha^o x \sin\frac{\pi r}{q}\right)\sinh\left(\alpha^o x \cos\frac{\pi r}{q}\right) \\ &+ \sum_{r=1}^{\frac{q}{2}} d_r^o \sin\left(\alpha^o x \sin\frac{\pi r}{q}\right)\cosh\left(\alpha^o x \cos\frac{\pi r}{q}\right),\end{aligned} \tag{2.3}$$

respectively. The parameters c_r^e, d_r^e, α^e and c_r^o, d_r^o, α^o are specified by enforcing the boundary conditions, which results in an algebraic $q \times q$ eigenproblem. The case of q odd is treated in a virtually identical manner [6].

It transpires that eigenfunctions always occur in even and odd cases, regardless of q. Hence, we will occasionally use the notation ϕ_n^e, $\phi_{0,n}^e$ and ϕ_n^o, $\phi_{0,n}^o$ to distinguish the such cases. More frequently, however, we will write $\phi_{0,n}$, ϕ_n and ignore this fact. As with classical Fourier series, splitting into even and odd cases is most convenient for computations (where real numbers are desirable), whereas for analysis, it is simpler not to make this distinction.

The biharmonic ($q = 2$) case warrants further attention. It presents the first significant extension beyond the modified Fourier case, and highlights several features of general polyharmonic–Neumann expansions. In this setting, the eigenfunctions are given by

$$\phi_n^e(x) = \frac{1}{\sqrt{2}}\left(\frac{\cos\alpha_n^e x}{\cos\alpha_n^e} + \frac{\cosh\alpha_n^e x}{\cosh\alpha_n^e}\right), \quad \phi_n^o(x) = \frac{1}{\sqrt{2}}\left(\frac{\sin\alpha_n^o x}{\sin\alpha_n^o} + \frac{\sinh\alpha_n^o x}{\sinh\alpha_n^o}\right), \tag{2.4}$$

and the values α_n^e, α_n^o, $n \in \mathbb{N}$ are precisely the roots of the nonlinear equations $\tanh\alpha^e + \tan\alpha^e = 0$ and $\tanh\alpha^o - \tan\alpha^o = 0$ respectively. These values lie in intervals of exponentially small width. In fact, for all $n \in \mathbb{N}$,

$$\alpha_n^e \in \left((n-\tfrac{1}{4})\pi, (n-\tfrac{1}{4})\pi + c\mathrm{e}^{-2(n-\frac{1}{4})\pi}\right), \quad \alpha_n^o \in \left((n+\tfrac{1}{4})\pi - c\mathrm{e}^{-2(n+\frac{1}{4})\pi}, (n+\tfrac{1}{4})\pi\right), \tag{2.5}$$

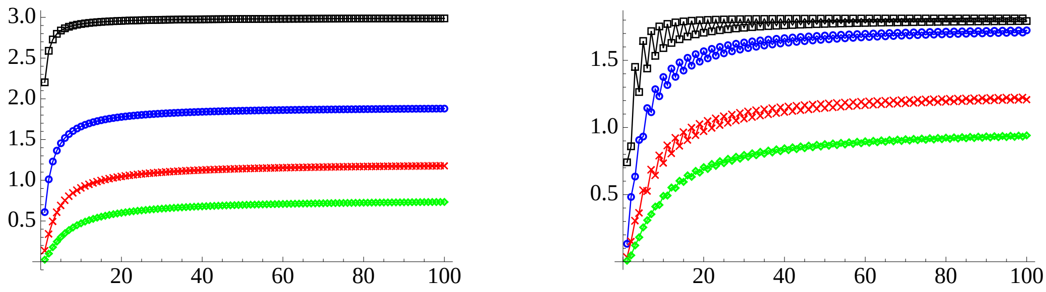

Figure 1: Error in approximating f by f_N for $q = 1$ (squares), $q = 2$ (circles), $q = 3$ (crosses) and $q = 4$ (diamonds). Left: scaled error $N^q \|f - f_N\|_{\mathrm{L}^\infty(-1,1)}$ for $N = 1, \ldots, 100$. Right: scaled error $N^{q+1}\|f - f_N\|_{\mathrm{L}^\infty(-\frac{1}{2},\frac{1}{2})}$.

where $c = \frac{\cos 1+\sin 1}{\sin 1}$. Upon redefining $\alpha_n^e = \alpha_{2n-1}$ and $\alpha_n^o = \alpha_{2n}$, it is readily seen that this establishes the conjecture (1.4) for $q = 2$. Simple arguments, based on (2.4) and (2.5), also verifies (1.5) in this setting. We defer a proof of (1.4) and (1.5) in the general case to Section 3.

2.2 Expansions in polyharmonic–Neumann eigenfunctions

We may express any function $f \in \mathrm{L}^2(-1,1)$ in terms of its expansion in polyharmonic–Neumann eigenfunctions,

$$f(x) \sim \sum_{n=0}^{q-1} \hat{f}_{0,n}\phi_{0,n}(x) + \sum_{n=1}^{\infty} \hat{f}_n \phi_n(x), \tag{2.6}$$

where $\hat{f}_{0,n} = \int_{-1}^{1} f(x)\phi_{0,n}(x)\,\mathrm{d}x$ and $\hat{f}_n = \int_{-1}^{1} f(x)\phi_n(x)\,\mathrm{d}x$ are the coefficients of f in the polyharmonic–Neumann basis. Standard spectral theory verifies convergence of the right hand side of (2.6) to f in the L^2 sense. Moreover, the following Parseval-type characterisation holds,

$$\|f\|^2 = \sum_{n=0}^{q-1} |\hat{f}_{0,n}|^2 + \sum_{n=1}^{\infty} |\hat{f}_n|^2, \quad \forall f \in \mathrm{L}^2(-1,1), \tag{2.7}$$

where $\|g\|^2 = \int_{-1}^{1} |g(x)|^2\,\mathrm{d}x$ is the standard norm on $\mathrm{L}^2(-1,1)$. In practice, the expansion (2.6) is truncated after $N \in \mathbb{N}_+$ terms, leading to the approximation

$$f_N(x) = \sum_{n=0}^{q-1} \hat{f}_{0,n}\phi_{0,n}(x) + \sum_{n=1}^{N} \hat{f}_n \phi_n(x). \tag{2.8}$$

Note that f_N is precisely the orthogonal projection of f onto the space spanned by the first $N+q$ eigenfunctions. In particular, $f_N \to f$ in the L^2 norm. However, it turns out that, for sufficiently smooth f, $f_N \to f$ uniformly on $[-1,1]$ at a rate of $\mathcal{O}\left(N^{-q}\right)$. Moreover, whilst $f(\pm 1) - f_N(\pm 1) = \mathcal{O}\left(N^{-q}\right)$, the error $f(x) - f_N(x) = \mathcal{O}\left(N^{-q-1}\right)$ uniformly in compact subsets of $(-1,1)$. In other words, faster convergence occurs away from the endpoints. Figure 1 demonstrates this observation for $f(x) = \mathrm{e}^{2x}$ and $q = 1,2,3,4$. We devote Sections 4 and 5 to the study of convergence of the approximation f_N, including a proof of these statements.

As mentioned, the purpose of polyharmonic–Neumann expansions is to obtain faster convergence. The aforementioned convergence rates demonstrate the benefit gained by increasing q. Figure 1 also highlights this improvement. For example, with $q = 1$ and $N = 50$, the uniform error in approximating $f(x) = \mathrm{e}^{2x}$ is roughly 6.0×10^{-2}, whereas when q is increased to 4, this value is 1.1×10^{-6}; approximately 5×10^5 times smaller.

This improvement in convergence of the expansion (2.8) with increasing q is a direct consequence of the Neumann boundary conditions. In the next section, we briefly explain why this is the case.

2.3 Neumann boundary conditions

A simple argument to this end was given in [6]. If $\hat{f}_n = \int_{-1}^{1} f(x)\phi_n(x)\,\mathrm{d}x$ is the coefficient of a smooth function f with respect to polyharmonic eigenfunction ϕ_n (for the moment we do not specify boundary conditions), then, upon replacing ϕ_n by $(-1)^q \alpha_n^{-2q} \phi_n^{(2q)}$ and integrating by parts $2q$ times, we obtain the expression

$$\hat{f}_n = \frac{(-1)^q}{\alpha_n^{2q}} \left[\sum_{r=0}^{2q-1} (-1)^r f^{(r)}(x) \phi_n^{(2q-r-1)}(x) \Big|_{x=-1}^{1} + \int_{-1}^{1} f^{(2q)}(x)\phi_n(x)\,\mathrm{d}x \right].$$

It is known in a rather general context that the parameter $\alpha_n = \mathcal{O}(n)$ for large n and the derivative $\phi_n^{(r)} = \mathcal{O}(n^r)$ [20]. Substituting these results into the above expression, a simple argument now demonstrates that, amongst all possible boundary conditions, the fastest possible decay of the coefficient $\hat{f}_n$ is $\mathcal{O}\left(n^{-q-1}\right)$. Moreover, such decay occurs when Neumann boundary conditions are prescribed (in which case, the first q terms of the above sum vanish). Upon the assumption of uniform convergence of f_N to f, this translates into a uniform convergence rate of $\mathcal{O}(N^{-q})$ (see Section 5).

The necessity of such boundary conditions is highlighted upon consideration of the Dirichlet boundary conditions

$$\phi^{(r)}(\pm 1) = 0, \quad r = 0, \ldots, q-1. \tag{2.9}$$

These give the slowest possible coefficient decay: $\hat{f}_n = \mathcal{O}\left(n^{-1}\right)$. In addition, the expansion of a function f in polyharmonic–Dirichlet eigenfunctions does not converge uniformly on $[-1, 1]$, and suffers from a Gibbs-type phenomenon near the endpoints $x = \pm 1$ (a fact we will confirm in Section 4).

It is possible that other boundary conditions yield the same coefficient decay (but no better). For example, when $q = 1$ the Robin boundary conditions $\phi'(\pm 1) + a\phi(\pm 1) = 0$, $a \in \mathbb{R}$, also give $\hat{f}_n = \mathcal{O}\left(n^{-2}\right)$. However, we make the choice of Neumann boundary conditions for their simplicity, thereby making the construction of the approximation f_N easier.

3 Asymptotics for polyharmonic–Neumann eigenvalues and eigenfunctions

This section is devoted to establishing the estimates (1.4) and (1.5). As stated, similar estimates, but with only $\mathcal{O}\left(n^{-1}\right)$ remainder terms, form a central component in the study of general Birkhoff expansions [10, 20]. To the best of our knowledge, estimates for the polyharmonic–Neumann case with exponentially small remainders do not currently exist in literature. As we later discuss, this is doubtless due to that fact that such estimates are only valid under rather specific conditions.

3.1 Polyharmonic–Neumann eigenvalues

Consider an eigenfunction ϕ with eigenvalue $\mu = \alpha^{2q} \neq 0$. By definition $(-1)^q \phi^{(2q)} = \alpha^{2q}\phi$ and $\phi^{(q+r)}(\pm 1) = 0$, $r = 0, \ldots, q-1$. Suppose now that we write ϕ as in (2.1). Then, an application of the boundary conditions yields the following system of equations for the coefficients $c_0, \ldots, c_{2q-1}$:

$$\sum_{s=0}^{2q-1} c_s (\alpha\lambda_s)^{r+q} \mathrm{e}^{\alpha\lambda_s} = \sum_{s=0}^{2q-1} c_s (\alpha\lambda_s)^{r+q} \mathrm{e}^{-\alpha\lambda_s} = 0, \quad r = 0, ..., q-1.$$

As a result, the values α are precisely the roots of the equation $g(\alpha)=0$, where

$$g(\alpha)=\det\begin{pmatrix} e^{\alpha\lambda_0} & e^{\alpha\lambda_1} & \cdots & e^{\alpha\lambda_{2q-1}} \\ \lambda_0 e^{\alpha\lambda_0} & \lambda_1 e^{\alpha\lambda_1} & \cdots & \lambda_{2q-1} e^{\alpha\lambda_{2q-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{q-1} e^{\alpha\lambda_0} & \lambda_1^{q-1} e^{\alpha\lambda_1} & \cdots & \lambda_{2q-1}^{q-1} e^{\alpha\lambda_{2q-1}} \\ e^{-\alpha\lambda_0} & e^{-\alpha\lambda_1} & \cdots & e^{-\alpha\lambda_{2q-1}} \\ \lambda_0 e^{-\alpha\lambda_0} & \lambda_1 e^{-\alpha\lambda_1} & \cdots & \lambda_{2q-1} e^{-\alpha\lambda_{2q-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{q-1} e^{-\alpha\lambda_0} & \lambda_1^{q-1} e^{-\alpha\lambda_1} & \cdots & \lambda_{2q-1}^{q-1} e^{-\alpha\lambda_{2q-1}} \end{pmatrix}. \tag{3.1}$$

Using Cramer's rule, we obtain

$$g(\alpha)=\sum_{\sigma\in S_{2q}} \operatorname{sgn}(\sigma) e^{\alpha\sum_{r=0}^{q-1}[\lambda_{\sigma(r)}-\lambda_{\sigma(q+r)}]}\prod_{r=0}^{q-1}\left[\lambda_{\sigma(r)}\lambda_{\sigma(q+r)}\right]^r, \tag{3.2}$$

where S_{2q} is the set of permutations of the indices $\{0,\dots,2q-1\}$ and $\operatorname{sgn}(\sigma)$ takes value $+1$ if σ is an even permutation and -1 otherwise.

Our interest lies with the asymptotic behaviour $\alpha\to\infty$ (since the eigenvalues μ_n are non-negative and possess no finite limit point, there must be solutions of $g(\alpha)=0$ in this regime). Hence, we scrutinise the sum $\sum_{r=0}^{q-1}[\lambda_{\sigma(r)}-\lambda_{\sigma(q+r)}]$. To do so, we introduce the following ordering on the values $\lambda_0,\dots,\lambda_{2q-1}$. We define $\lambda_0=-i$ and $\lambda_r=\lambda_0\lambda^r$, where $\lambda=e^{\frac{i\pi}{q}}$. Notice that $\lambda_q=i$, and $\lambda_{q+r}=-\lambda_r$. Moreover, $\operatorname{Re}\lambda_r\geq 0$ for $r=0,\dots,q$, and $\operatorname{Re}\lambda_r<0$ otherwise.

Lemma 1. *The quantity* $\operatorname{Re}\sum_{r=0}^{q-1}[\lambda_{\sigma(r)}-\lambda_{\sigma(q+r)}]$ *takes maximal value* $2\cot\frac{\pi}{2q}=2\theta_q$. *This is attained precisely when* $\sigma\in T_{2q}=U_q\cup V_q$, *where*

$$\begin{aligned} U_q&=\{\sigma\in S_{2q}:\{\sigma(r):r=0,\dots,q-1\}=\{0,\dots,q-1\}\},\\ V_q&=\{\sigma\in S_{2q}:\{\sigma(r):r=0,\dots,q-1\}=\{1,\dots,q\}\}. \end{aligned}$$

Moreover, $\sum_{r=0}^{q-1}[\lambda_{\sigma(r)}-\lambda_{\sigma(q+r)}]=2(\theta_q-i)$ *for* $\sigma\in U_q$ *and* $\sum_{r=0}^{q-1}[\lambda_{\sigma(r)}-\lambda_{\sigma(q+r)}]=2(\theta_q+i)$ *for* $\sigma\in V_q$. *Conversely, if* $\sigma\notin T_{2q}$ *then* $\operatorname{Re}\sum_{r=0}^{q-1}[\lambda_{\sigma(r)}-\lambda_{\sigma(q+r)}]\leq 2(\theta_q-\gamma_q)$, *where* $\gamma_q=\sin\frac{\pi}{q}$.

Proof. A simple argument verifies that the maximal value is attained only for $\sigma\in T_{2q}$. Furthermore

$$\sum_{r=0}^{q-1}\lambda_r=\lambda_0\sum_{r=0}^{q-1}\lambda^r=\frac{2i}{e^{\frac{i\pi}{q}}-1}=\theta_q-i,$$

and $\sum_{r=1}^{q}\lambda_r=2i+\sum_{r=0}^{q-1}\lambda_r=\theta_q+i$. For the final part, we merely note that $|\operatorname{Re}\lambda_r|\geq\operatorname{Re}\lambda_1=\gamma_q$ for $r\neq 0,q$. □

This lemma allows us to immediately provide an estimate for the function g:

Lemma 2. *The function* $g(\alpha)$ *defined by (3.1) satisfies*

$$g(\alpha)=e^{2\theta_q\alpha}\det V_0\det V_1\left[e^{-2i\alpha}+(-1)^q e^{2i\alpha}\right]+\mathcal{O}\left(e^{2(\theta_q-\gamma_q)\alpha}\right),\quad \alpha\to\infty,$$

where $V_0,V_1\in\mathbb{C}^{q\times q}$ *are independent of* α *and have* $(r,s)^{\text{th}}$ *entries* λ_s^r *and* λ_{q+s}^r *respectively,* $r,s=0,\dots,q-1$.

Note that both V_0 and V_1 can be expressed in terms of products of diagonal and Vandermonde matrices. Thus, the constant $\det V_0\det V_1$ can be exactly specified [11]. However, since these exact values are of little relevance to the present discussion, we shall not pursue this further.

Proof of Lemma 2. Applying the result of Lemma 1 to (3.2) gives

$$
\begin{aligned}
g(\alpha) =& \mathrm{e}^{2(\theta_q - \mathrm{i})\alpha} \sum_{\sigma \in U_q} \operatorname{sgn}(\sigma) \prod_{r=0}^{q-1} \left(\lambda_{\sigma(r)} \lambda_{\sigma(q+r)} \right)^r \\
&+ \mathrm{e}^{2(\theta_q + \mathrm{i})\alpha} \sum_{\sigma \in V_q} \operatorname{sgn}(\sigma) \prod_{r=0}^{q-1} \left(\lambda_{\sigma(r)} \lambda_{\sigma(q+r)} \right)^r + \mathcal{O}\left(\mathrm{e}^{2(\theta_q - \gamma_q)\alpha} \right), \quad \alpha \to \infty. \qquad (3.3)
\end{aligned}
$$

If $\sigma \in U_q$, we may write

$$
\sigma(r) = \begin{cases} \sigma'(r) & r = 0, \ldots, q-1 \\ q + \sigma''(r-q) & r = q, \ldots, 2q-1, \end{cases}
$$

where $\sigma', \sigma'' \in S_q$. In particular, $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma')\operatorname{sgn}(\sigma'')$. Hence

$$
\sum_{\sigma \in U_q} \operatorname{sgn}(\sigma) \prod_{r=0}^{q-1} \left(\lambda_{\sigma(r)} \lambda_{\sigma(q+r)} \right)^r = \sum_{\sigma', \sigma'' \in S_q} \operatorname{sgn}(\sigma')\operatorname{sgn}(\sigma'') \prod_{r=0}^{q-1} \left(\lambda_{\sigma'(r)} \lambda_{q+\sigma''(r)} \right)^r
$$

and this is precisely $\det V_0 \det V_1$. Similar arguments can be applied to $\sigma \in V_q$. Noting that $\lambda_{2q} = \lambda_0$, we write

$$
\sigma(r) = \begin{cases} 1 + \sigma'(r) & r = 0, \ldots, q-1 \\ q + 1 + \sigma''(r-q) & r = q, \ldots, 2q-1. \end{cases}
$$

In this case $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\sigma')\operatorname{sgn}(\sigma'')$, hence

$$
\sum_{\sigma \in V_q} \operatorname{sgn}(\sigma) \prod_{r=0}^{q-1} \left(\lambda_{\sigma(r)} \lambda_{\sigma(q+r)} \right)^r = -\det V_2 \det V_3,
$$

where $V_2, V_3 \in \mathbb{C}^{q \times q}$ have $(r,s)^{\text{th}}$ entries λ_{1+s}^r and λ_{q+1+s}^r respectively. Observe that $V_2 = DV_0$, $V_3 = DV_1$, where $D \in \mathbb{C}^{q \times q}$ is the diagonal matrix with r^{th} entry λ^r. Hence

$$
\det V_2 \det V_3 = (\det D)^2 \det V_0 \det V_1 = \lambda^{q(q-1)} \det V_0 \det V_1 = \mathrm{e}^{-\mathrm{i}\pi(q-1)} \det V_0 \det V_1,
$$

Substituting this expression into (3.3) now completes the proof. □

We are now able to establish the key result of this section: namely, equation (1.4). We have

Theorem 1. *Suppose that* $\mu_n = \alpha_n^{2q}$, $n \in \mathbb{N}_+$, *is the* n^{th} *eigenvalue of the polyharmonic–Neumann operator. Then* $\alpha_n = \frac{1}{4}(2n + q - 1)\pi + \mathcal{O}\left(\mathrm{e}^{-n\pi\gamma_q}\right)$ *as* $n \to \infty$.

Proof. For an eigenvalue $\mu = \alpha^{2q}$ we have $g(\alpha) = 0$. Hence, $\mathrm{e}^{4\mathrm{i}\alpha} = \mathrm{e}^{i\pi(q-1)} + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right)$. □

The proof of this theorem is similar to that given for general Birkhoff expansions [20]. However, the greatly simplified nature of the linear operator and boundary conditions allows for a more straightforward argument, and in turn, facilitates the greatly improved estimate (1.4).

As mentioned, this result is a vital step towards the effective computation of the values α_n. In Section 3.3 we consider such task. Before doing so, however, we turn our attention to the asymptotic behaviour of the polyharmonic eigenfunctions ϕ_n.

3.2 Polyharmonic–Neumann eigenfunctions

We wish to establish (1.5). To commence, we note that the eigenfunction ϕ corresponding to eigenvalue $\mu = \alpha^{2q} \neq 0$ can be written as

$$\phi(x) = \det \begin{pmatrix} e^{\alpha\lambda_0 x} & e^{\alpha\lambda_1 x} & \cdots & e^{\alpha\lambda_{2q-1} x} \\ \lambda_0^q e^{\alpha\lambda_0} & \lambda_1^q e^{\alpha\lambda_1} & \cdots & \lambda_{2q-1}^q e^{\alpha\lambda_{2q-1}} \\ \lambda_0^{q+1} e^{\alpha\lambda_0} & \lambda_1^{q+1} e^{\alpha\lambda_1} & \cdots & \lambda_{2q-1}^{q+1} e^{\alpha\lambda_{2q-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{2q-1} e^{\alpha\lambda_0} & \lambda_1^{2q-1} e^{\alpha\lambda_1} & \cdots & \lambda_{2q-1}^{2q-1} e^{\alpha\lambda_{2q-1}} \\ \lambda_0^q e^{-\alpha\lambda_0} & \lambda_1^q e^{-\alpha\lambda_1} & \cdots & \lambda_{2q-1}^q e^{-\alpha\lambda_{2q-1}} \\ \lambda_0^{q+1} e^{-\alpha\lambda_0} & \lambda_1^{q+1} e^{-\alpha\lambda_1} & \cdots & \lambda_{2q-1}^{q+1} e^{-\alpha\lambda_{2q-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{2q-2} e^{-\alpha\lambda_0} & \lambda_1^{2q-2} e^{-\alpha\lambda_1} & \cdots & \lambda_{2q-1}^{2q-2} e^{-\alpha\lambda_{2q-1}} \end{pmatrix} = \sum_{s=0}^{2q-1} e^{\alpha\lambda_s x} (-1)^s \det A^{[s]},$$

where $A^{[s]}$ is the corresponding minor

$$A^{[s]} = \begin{pmatrix} \lambda_0^q e^{\alpha\lambda_0} & \cdots & \lambda_{s-1}^q e^{\alpha\lambda_{s-1}} & \lambda_{s+1}^q e^{\alpha\lambda_{s+1}} & \cdots & \lambda_{2q-1}^q e^{\alpha\lambda_{2q-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{2q-1} e^{\alpha\lambda_0} & \cdots & \lambda_{s-1}^{2q-1} e^{\alpha\lambda_{s-1}} & \lambda_{s+1}^{2q-1} e^{\alpha\lambda_{s+1}} & \cdots & \lambda_{2q-1}^{2q-1} e^{\alpha\lambda_{2q-1}} \\ \lambda_0^q e^{-\alpha\lambda_0} & \cdots & \lambda_{s-1}^q e^{-\alpha\lambda_{s-1}} & \lambda_{s+1}^q e^{-\alpha\lambda_{s+1}} & \cdots & \lambda_{2q-1}^q e^{-\alpha\lambda_{2q-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{2q-2} e^{-\alpha\lambda_0} & \cdots & \lambda_{s-1}^{2q-2} e^{-\alpha\lambda_{s-1}} & \lambda_{s+1}^{2q-2} e^{-\alpha\lambda_{s+1}} & \cdots & \lambda_{2q-1}^{2q-2} e^{-\alpha\lambda_{2q-1}} \end{pmatrix}.$$

Using Cramer's rule once more, we deduce that

$$\det A^{[s]} = \sum_{\sigma \in S_{2q,s}} \operatorname{sgn}(\sigma) e^{\alpha\left[\sum_{r=0}^{q-1} \lambda_{\sigma(r)} - \sum_{r=0}^{q-2} \lambda_{\sigma(q+r)}\right]} \prod_{r=0}^{q-1} \lambda_{\sigma(r)}^{q+r} \prod_{r=0}^{q-2} \lambda_{\sigma(q+r)}^{q+r}, \tag{3.4}$$

where $S_{2q,s}$ is the set of bijections from $\{0, \ldots, 2q-2\}$ to $\{0, \ldots, s-1, s+1, \ldots, 2q-1\}$. As in the previous section, we wish to analyse $\det A^{[s]}$ as $\alpha \to \infty$. We have

Lemma 3. *Suppose that $s = 0, \ldots, q$. Then*

$$\det A^{[s]} = e^{(2\theta_q - \lambda_s)\alpha} \det B \det V^{[s]} + \mathcal{O}\left(e^{[2(\theta_q - \gamma_q) - \operatorname{Re} \lambda_s]\alpha}\right), \quad \alpha \to \infty,$$

where $B \in \mathbb{C}^{q \times q}$ has $(r,s)^{\text{th}}$ entry λ_{q+1+s}^{q+r} and

$$V^{[s]} = \begin{pmatrix} \lambda_0^q & \cdots & \lambda_{s-1}^q & \lambda_{s+1}^q & \cdots & \lambda_q^q \\ \lambda_0^{q+1} & \cdots & \lambda_{s-1}^{q+1} & \lambda_{s+1}^{q+1} & \cdots & \lambda_q^{q+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{2q-1} & \cdots & \lambda_{s-1}^{2q-1} & \lambda_{s+1}^{2q-1} & \cdots & \lambda_q^{2q-1} \end{pmatrix}. \tag{3.5}$$

Note that the matrices $V^{[s]}$ are independent of α (as is B). Moreover, each $V^{[s]}$ corresponds to a particular minor of the matrix $V \in \mathbb{C}^{(q+1)\times(q+1)}$ with $(r,s)^{\text{th}}$ entry λ_s^{q+r}. Though not important in our present considerations, this observation will be used later.

Proof of lemma 3. Consider the quantity $\operatorname{Re}\left[\sum_{r=0}^{q-1} \lambda_{\sigma(r)} - \sum_{r=0}^{q-2} \lambda_{\sigma(q+r)}\right]$. Arguing as in Lemma 1, we find that this is maximised precisely when $\sigma \in T_{q,s}$, where

$$T_{q,s} = \{\sigma \in S_{2q,s} : \{\sigma(r) : r = 0, \ldots, q-1\} = \{0, \ldots, s-1, s+1, \ldots, q\}\},$$

in which case $\sum_{r=0}^{q-1} \lambda_{\sigma(r)} - \sum_{r=0}^{q-2} \lambda_{\sigma(q+r)} = 2\theta_q - \lambda_s$. For $\sigma \notin T_{q,s}$, we have

$$\mathrm{Re}\left[\sum_{r=0}^{q-1} \lambda_{\sigma(r)} - \sum_{r=0}^{q-2} \lambda_{\sigma(q+r)}\right] \leq 2(\theta_q - \gamma_q) - \mathrm{Re}\,\lambda_s.$$

Substituting this into (3.4), we obtain

$$\det A^{[s]} = \mathrm{e}^{(2\theta_q - \lambda_s)\alpha} \sum_{\sigma \in T_{q,s}} \mathrm{sgn}(\sigma) \prod_{r=0}^{q-1} \lambda_{\sigma(r)}^{q+r} \prod_{r=0}^{q-2} \lambda_{\sigma(q+r)}^{q+r} + \mathcal{O}\left(\mathrm{e}^{[2(\theta_q - \gamma_q) - \mathrm{Re}\,\lambda_s]\alpha}\right).$$

In an identical manner to Lemma 1, we deduce that this sum is precisely $\det B \det V^{[s]}$. □

This lemma suggests that it is prudent to renormalise the eigenfunction ϕ by dividing by $\mathrm{e}^{2\theta_q \alpha} \det B$. This gives the expression

$$\phi(x) = \sum_{s=0}^{q-1} \left[(-1)^s \det V^{[s]} \mathrm{e}^{\lambda_s \alpha(x-1)} - b_s \mathrm{e}^{-\lambda_s \alpha(x+1)}\right] + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q \alpha}\right), \quad \alpha \to \infty, \qquad (3.6)$$

where the constants $b_0, \dots, b_{q-1}$ are to be determined. We have

Lemma 4. *The constants b_s, $s = 0, \dots, q-1$ appearing in (3.6) satisfy*

$$b_s = (-1)^s \det V^{[s]} \mathrm{i}^{q-1} \mathrm{e}^{2\mathrm{i}\alpha} + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q \alpha}\right), \quad \alpha \to \infty, \quad s = 0, \dots, q-1.$$

Proof. Consider the boundary condition $\phi^{(q+r)}(-1) = 0$, $r = 0, \dots, q-1$. Substituting (3.6) gives

$$0 = \alpha^{-q-r} \phi^{(q+r)}(-1) = \sum_{s=0}^{q-1} \left[(-1)^s \det V^{[s]} \lambda_s^{q+r} \mathrm{e}^{-2\lambda_s \alpha} - (-1)^{q+r} \lambda_s^{q+r} b_s\right] + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q \alpha}\right).$$

Suppose that $\tilde{D} \in \mathbb{R}^{q \times q}$ is the diagonal matrix with r^{th} entry $(-1)^{q+r}$. Then, written in matrix form, the above expression is

$$\begin{aligned}
\tilde{D} V^{[q]} \{b_r\}_{r=0}^{q-1} &= V^{[q]} \{(-1)^r \det V^{[r]} \mathrm{e}^{-2\lambda_r \alpha}\}_{r=0}^{q-1} + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q \alpha}\right) \\
&= \left(\det V^{[0]} \mathrm{e}^{-2\lambda_0 \alpha}\right) V^{[q]} \{1, 0, \dots, 0\}^\top + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q \alpha}\right) \\
&= \det V^{[0]} \mathrm{e}^{2\mathrm{i}\alpha} \{\lambda_0^{q+r}\}_{r=0}^{q-1} + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q \alpha}\right).
\end{aligned}$$

The matrix $\tilde{D}$ is self-inverse. Moreover, $\tilde{D}\{\lambda_0^{q+r}\}_{r=0}^{q-1} = \{(-1)^{q+r} \lambda_0^{q+r}\}_{r=0}^{q-1} = \{\lambda_q^{q+r}\}_{r=0}^{q-1}$. Hence

$$V^{[q]} \{b_r\}_{r=0}^{q-1} = \det V^{[0]} \mathrm{e}^{2\mathrm{i}\alpha} \{\lambda_q^{q+r}\}_{r=0}^{q-1} + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q \alpha}\right),$$

and, using a standard relation, $b_r = \det V^{[0]} \mathrm{e}^{2\mathrm{i}\alpha} \frac{\det \tilde{V}^{[r]}}{\det V^{[q]}} + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q \alpha}\right)$, where

$$\tilde{V}^{[r]} = \begin{pmatrix} \lambda_0^q & \cdots & \lambda_{r-1}^q & \lambda_q^q & \lambda_{r+1}^q & \cdots & \lambda_{q-1}^q \\ \lambda_0^{q+1} & \cdots & \lambda_{r-1}^{q+1} & \lambda_q^{q+1} & \lambda_{r+1}^{q+1} & \cdots & \lambda_{q-1}^{q+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{2q-1} & \cdots & \lambda_{r-1}^{2q-1} & \lambda_q^{2q-1} & \lambda_{r+1}^{2q-1} & \cdots & \lambda_{q-1}^{2q-1} \end{pmatrix}.$$

This matrix is obtained from the matrix $V^{[r]}$ by interchanging precisely $q - r - 1$ columns. Hence $\det \tilde{V}^{[r]} = (-1)^{q+r+1} \det V^{[r]}$. Moreover, it is trivial to show that $V^{[0]} = D V^{[q]}$, where $D \in \mathbb{C}^{q \times q}$ is the diagonal matrix with r^{th} entry λ^{q+r}. Substituting these observations into the expression for b_r, we deduce that $b_r = \mathrm{e}^{2\mathrm{i}\alpha}(-1)^{q+r+1} \det D \det V^{[r]} + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q \alpha}\right)$. Since $\det D = \lambda^{q^2 + \frac{1}{2}q(q-1)} = (-1)^q \mathrm{i}^{q-1}$, we obtain the result. □

Using this lemma, we obtain the expression

$$\phi(x)=\sum_{s=0}^{q-1}(-1)^s \det V^{[s]}\left[\mathrm{e}^{\lambda_s\alpha(x-1)}+\mathrm{i}^{q-1}\mathrm{e}^{2\mathrm{i}\alpha}\mathrm{e}^{-\lambda_s\alpha(x+1)}\right]+\mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right),\quad \alpha\to\infty, \tag{3.7}$$

for the eigenfunction ϕ. Equation (1.5) now follows after an application of Theorem 1:

Theorem 2. *Suppose that $\mu_n=\alpha_n^{2q}$, $n\in\mathbb{N}_+$, is the n^{th} eigenvalue of the polyharmonic–Neumann operator with corresponding eigenfunction ϕ_n. Then*

$$\phi_n(x)=\sum_{s=0}^{q-1}c_s\left[\mathrm{e}^{\frac{1}{4}(2n+q-1)\pi\lambda_s(x-1)}+(-1)^{n+q+1}\mathrm{e}^{-\frac{1}{4}(2n+q-1)\pi\lambda_s(x+1)}\right]+\mathcal{O}\left(\mathrm{e}^{-n\pi\gamma_q}\right),$$

uniformly in $x\in[-1,1]$, where $c_s=(-1)^s\det V^{[s]}$ and the matrix $V^{[s]}$ is given by (3.5).

This theorem establishes (1.5). A simple consequence of this result concerns the asymptotic behaviour of polyharmonic–Neumann eigenfunctions in the interior $(-1,1)$. As the following corollary indicates, such eigenfunctions are exponentially close to regular oscillators away from the endpoints $x=\pm1$:

Corollary 1. *Suppose that ϕ_n is as in Theorem 2. Then*

$$\phi_n(x)=c(-1)^{\frac{n+q-1}{2}}\cos\tfrac{1}{4}(2n+q-1)\pi x+\mathcal{O}\left(\mathrm{e}^{-\frac{1}{2}n\pi\gamma_q(1-|x|)}\right),\quad n+q \text{ odd},$$
$$\phi_n(x)=-c(-1)^{\frac{n+q}{2}}\sin\tfrac{1}{4}(2n+q-1)\pi x+\mathcal{O}\left(\mathrm{e}^{-\frac{1}{2}n\pi\gamma_q(1-|x|)}\right),\quad n+q \text{ even},$$

uniformly for x in compact subsets of $(-1,1)$, where $c=2\mathrm{e}^{-\frac{q-1}{4}\pi\mathrm{i}}\det V^{[0]}$.

Proof. Since $\operatorname{Re}\lambda_s\geq\gamma_q$ for $s=1,\ldots,q-1$, an application of Theorem 2 gives

$$\begin{aligned}\phi_n(x)&=\det V^{[0]}\left[\mathrm{e}^{-\frac{1}{4}(2n+q-1)\pi\mathrm{i}(x-1)}+(-1)^{n+q+1}\mathrm{e}^{\frac{1}{4}(2n+q-1)\pi\mathrm{i}(x+1)}\right]+\mathcal{O}\left(\mathrm{e}^{-\frac{1}{2}n\pi\gamma_q(1-|x|)}\right).\\&=\det V^{[0]}\mathrm{e}^{\frac{1}{4}(2n+q-1)\pi\mathrm{i}}\left[\mathrm{e}^{-\frac{1}{4}(2n+q-1)\pi\mathrm{i}x}+(-1)^{n+q+1}\mathrm{e}^{\frac{1}{4}(2n+q-1)\pi\mathrm{i}x}\right]+\mathcal{O}\left(\mathrm{e}^{-\frac{1}{2}n\pi\gamma_q(1-|x|)}\right).\end{aligned}$$

The result now follows from considering the two cases separately and rearranging. □

In Figure 2 we exhibit this result for $q=3$. Note the very rapid onset of the asymptotic behaviour away from the endpoints.

A central component of the study of general Birkhoff expansions is the phenomenon of equiconvergence [19, 24]: inside the domain eigenfunctions approach regular oscillators in the limit $n\to\infty$ (though, in general, only at a rate of $\mathcal{O}\left(n^{-1}\right)$). For this reason, pointwise convergence of Birkhoff expansions may be studied using standard tools of Fourier analysis. This classical approach, however, is unsuitable for the study of polyharmonic–Neumann expansions. As we prove in Section 5, such expansions converge much more rapidly than classical Fourier series. Moreover, our interest also lies with uniform convergence throughout $[-1,1]$, which cannot be easily established through such means.

To connect these results to the explicit example of biharmonic eigenfunctions (see Section 2.1), we note that Theorem 2, when applied with $q=2$, gives

$$\phi_n(x)=(1-\mathrm{i})\mathrm{e}^{\mathrm{i}\alpha_n}\left[\mathrm{e}^{-\mathrm{i}\alpha_n x}+(-1)^{n+1}\mathrm{e}^{\mathrm{i}\alpha_n x}\right]-2\mathrm{i}\mathrm{e}^{-\alpha_n}\left[\mathrm{e}^{\alpha_n x}+(-1)^{n+1}\mathrm{e}^{-\alpha_n x}\right]+\mathcal{O}\left(\mathrm{e}^{-n\pi}\right).$$

Suppose, for example, that $n=2m-1$ (the case $n=2m$ is identical). Then $\alpha_n=(m-\frac{1}{4})\pi+\mathcal{O}\left(\mathrm{e}^{-n\pi}\right)$, and we obtain

$$\begin{aligned}\phi_{2m-1}(x)&=2\sqrt{2}(-1)^{m+1}\mathrm{i}\cos(m-\tfrac{1}{4})\pi x-2\mathrm{i}\frac{\cosh(m-\frac{1}{4})\pi x}{\cosh(m-\frac{1}{4})\pi}+\mathcal{O}\left(\mathrm{e}^{-2m\pi}\right)\\&=-2\mathrm{i}\left[\frac{\cos(m-\frac{1}{4})\pi x}{\cos(m-\frac{1}{4})\pi}+\frac{\cosh(m-\frac{1}{4})\pi x}{\cosh(m-\frac{1}{4})\pi}\right]+\mathcal{O}\left(\mathrm{e}^{-2m\pi}\right).\end{aligned}$$

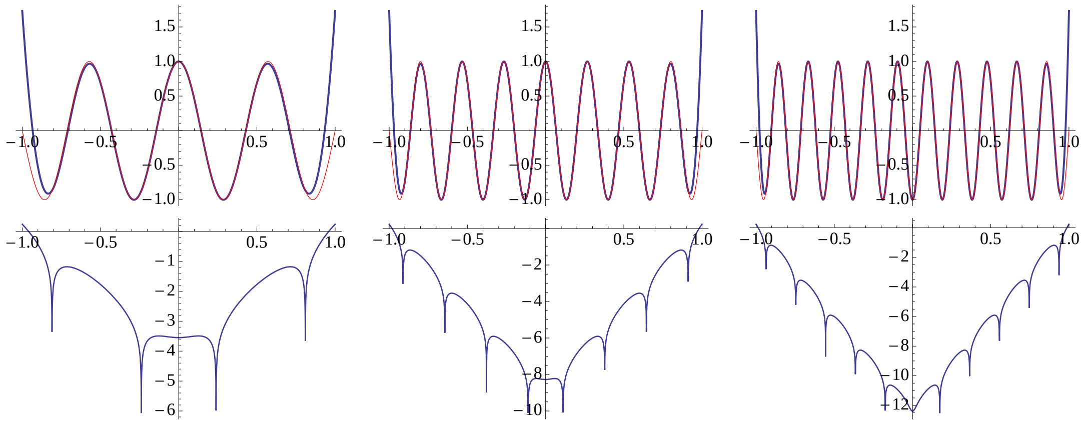

Figure 2: Top row: the triharmonic eigenfunctions ϕ_n (thicker line) and approximations $\cos\frac{1}{2}(n+1)\pi x$ (thinner line) for $n = 6, 14, 20$ (left to right). Bottom row: the error $\log_{10}|\phi_n(x) - \cos\frac{1}{2}(n+1)\pi x|$.

Upon comparison of this formula with (2.4), we confirm Theorem 2 in this case (up to renormalisation by a factor of $-2\mathrm{i}$).

Returning to the general case, one consequence of exponentially decaying remainder terms of Theorems 1 and 2 is that we can perform an extremely detailed study of polyharmonic–Neumann expansions. In particular, as we detail in Section 5, we are able to provide an asymptotic expansion for the error $f(x) - f_N(x)$ in inverse powers of N at any point $x \in [-1, 1]$. Moreover, the constants appearing in such an expansion are known explicitly explicitly. To do so, it is first useful to provide an expression for the term $\det V^{[0]}$ appearing in Corollary 1. We have

Lemma 5. *The quantity* $\det V^{[0]}$ *is given explicitly by*

$$\det V^{[0]} = \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi(-q^2+5q-2)} \prod_{0\le r<s<q} (\lambda^s - \lambda^r)\,.$$

Proof. The matrix $V^{[0]}$ has $(r,s)^{\text{th}}$ entry λ_{s+1}^{q+r}. We note that $\lambda_{s+1}^{q+r} = (\lambda_0\lambda^{s+1})^{q+r} = \lambda_1^{q+r}\lambda^{rs}\lambda^{qs}$, and therefore $V^{[0]} = D^{[0]}WD^{[1]}$, where W is the Vandermonde matrix with $(r,s)^{\text{th}}$ entry λ^{rs} and $D^{[0]}$ and $D^{[1]}$ are diagonal matrices with r^{th} entries λ_1^{q+r} and $\lambda^{qr} = (-1)^r$ respectively. In particular, $\det D^{[0]} = \lambda_1^{\frac{1}{2}q(3q-1)} = \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi(3q-1)(q-2)}$ and $\det D^{[1]} = \mathrm{e}^{\frac{1}{2}\mathrm{i}\pi q(q-1)}$. After simplification, this gives $\det D^{[0]}\det D^{[1]} = \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi(-q^2+5q-2)}$. The result now follows after applying known results for the determinant of a Vandermonde matrix [11] to W. □

As discussed in [6], much is known regarding the zeros of polyharmonic–Neumann eigenfunctions. For example, the n^{th} eigenfunction possesses precisely $n + q$ simple zeros in $(-1, 1)$ and zeros of consecutive eigenfunctions interlace [22]. As a direct result of Theorem 2, we are able to precisely determine the distribution of such zeros in the limit $n \to \infty$. Unsurprisingly, given that ϕ_n is exponentially close to a regular oscillator in $(-1, 1)$, this distribution is uniform:

Corollary 2. *The zeros of* ϕ_n *are asymptotically uniformly distributed as* $n \to \infty$.

Proof. Suppose that $I = [a, b] \subseteq (-1, 1)$ is a closed interval. Let $Z_n(I)$ be the number of zeros of ϕ_n in I. It follows from Theorem 2 that $Z_n(I) = \frac{1}{2}(b-a)n + \mathcal{O}(1)$ as $n \to \infty$. Since ϕ_n has precisely $n+q$ simple zeros in $[-1, 1]$, the proportion of zeros in I is $\frac{1}{2}|I| + \mathcal{O}\left(n^{-1}\right)$ for large n (note that $|I| = 2$ for $I = [-1, 1]$, which explains the factor of $\frac{1}{2}$).

It remains to show that the same result holds for intervals I containing at least one of the endpoints $x = \pm 1$. For this, we first note that ϕ_n is either even or odd. Hence, it suffices to consider $I = [a, 1] \subseteq (-1, 1]$. If $a > 0$, then

$$Z(I) = \frac{1}{2}Z\left([-1,-a]\cup[a,1]\right) = \frac{1}{2}\left\{Z([-1,1]) - Z([-a,a])\right\} = \frac{1}{2}(1-a)n + \mathcal{O}\left(1\right),$$

as required. If $a<0$, then $Z(I)=Z([-1,1])-Z([-a,1])$, and the result follows. $\square$

Though this result is of interest, it is included only as a simple example of usefulness of Theorem 2 and will not be needed in subsequent analysis. Conversely, the remainder of this section is devoted to establishing several further estimates for polyharmonic–Neumann eigenfunctions that are required in Section 4–6, where we study the convergence of expansions in such eigenfunctions.

The first of these results concerns the growth of derivatives of the eigenfunctions ϕ_n. Intuitively, it feels correct that $\|\phi^{(r)}\|_\infty=\mathcal{O}(\alpha^r)$ for large α. This is indeed the case:

Lemma 6. *Suppose that ϕ is a polyharmonic–Neumann eigenfunction with corresponding eigenvalue $\mu=\alpha^{2q}\neq 0$. Then $\|\phi^{(r)}\|_\infty=\mathcal{O}(\alpha^r)$ for large α and any $r\in\mathbb{N}$. Moreover,*

$$\phi(1)=d_r\alpha^r+\mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right),\quad \phi(-1)=(-1)^r\mathrm{i}^{q-1}\mathrm{e}^{2\mathrm{i}\alpha}\phi(1)+\mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right),\quad \alpha\to\infty,$$

where $d_r=c_0(-\mathrm{i})^{q-r-1}+\sum_{s=0}^{q-1}c_s\lambda_s^r$.

Proof. Consider equation (3.7). This expression is uniform in $x\in[-1,1]$, therefore

$$\phi^{(r)}(x)=\alpha^r\sum_{s=0}^{q-1}c_s\lambda_s^r\left[\mathrm{e}^{\lambda_s\alpha(x-1)}+(-1)^r\mathrm{i}^{q-1}\mathrm{e}^{2\mathrm{i}\alpha}\mathrm{e}^{-\lambda_s\alpha(x+1)}\right]+\mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right).$$

Since $\operatorname{Re}\lambda_s\geq 0$, the functions $\mathrm{e}^{\lambda_s\alpha(x-1)}$ and $\mathrm{e}^{-\lambda_s\alpha(x+1)}$ are bounded by 1 on $[-1,1]$. Hence, the first result now follows immediately. For the second, substituting $x=1$ (for example) into the above expression gives

$$\phi^{(r)}(1)=\alpha^r\left[c_0\lambda_0^r(-1)^r\mathrm{i}^{q-1}\mathrm{e}^{4\mathrm{i}\alpha}+\sum_{s=0}^{q-1}c_s\lambda_s^r\right]+\mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right).$$

Since $\mathrm{e}^{4\mathrm{i}\alpha}=(-1)^{q-1}+\mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right)$ (see Theorem 1), the result now follows. $\square$

To form the polyharmonic–Neumann expansion of a function f, it is necessary to normalise the eigenfunctions ϕ. For this, it is useful to have an asymptotic estimate for $\|\phi\|$. We have

Lemma 7. *Suppose that ϕ is a polyharmonic–Neumann eigenfunction with corresponding eigenvalue $\mu=\alpha^{2q}\neq 0$. Then*

$$\|\phi\|=c+\mathcal{O}\left(\mathrm{e}^{-\gamma_q\alpha}\right),\quad \alpha\to\infty,\tag{3.8}$$

where $c=2^{\frac{1}{2}q(q-1)+1}\prod_{0\leq r<s<q}\sin\frac{\pi(r-s)}{2q}$.

Proof. Suppose that we write $b=\mathrm{i}^{q-1}\mathrm{e}^{2\mathrm{i}\alpha}$, so that

$$\phi(x)=\sum_{s=0}^{q-1}c_s\left[\mathrm{e}^{\alpha\lambda_s(x-1)}+b\mathrm{e}^{-\lambda_s\alpha(x+1)}\right]+\mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right).$$

Hence

$$\|\phi\|^2=\sum_{r,s=0}^{q-1}c_r\bar{c}_s\int_{-1}^{1}\left[\mathrm{e}^{\alpha\lambda_r(x-1)}+b\mathrm{e}^{-\lambda_r\alpha(x+1)}\right]\left[\mathrm{e}^{\alpha\bar{\lambda}_s(x-1)}+\bar{b}\mathrm{e}^{-\bar{\lambda}_s\alpha(x+1)}\right]\mathrm{d}x+\mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right).$$

Consider the constant b. Since $\mathrm{e}^{2\mathrm{i}\alpha}=(-1)^{q-1}\mathrm{e}^{-2\mathrm{i}\alpha}+\mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right)$, we deduce that b is real in the limit $\alpha\to\infty$. Specifically, $b=\bar{b}+\mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right)$. Expanding the previous expression and simplifying now gives

$$\|\phi\|^2=2\sum_{r,s=0}^{q-1}c_r\bar{c}_s\mathrm{e}^{-\alpha(\lambda_r+\bar{\lambda}_s)}\left[\int_{-1}^{1}\cosh\alpha(\lambda_r+\bar{\lambda}_s)x\,\mathrm{d}x+b\int_{-1}^{1}\cosh\alpha(\lambda_r-\bar{\lambda}_s)x\,\mathrm{d}x\right]+\mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right).$$

Note that $\int_{-1}^{1} \cosh zx \, \mathrm{d}x = \frac{2}{z} \sinh z$ for $z \neq 0$ and 2 otherwise. Moreover, for $r, s = 0, \ldots, q-1$, $\lambda_r + \bar{\lambda}_s = 0$ if and only if $r = s = 0$, and $\lambda_r - \bar{\lambda}_s = 0$ only when $r + s = q$. Hence

$$\|\phi\|^2 = 4|c_0|^2 + \sum_{\substack{r,s=0 \\ (r,s)\neq(0,0)}}^{q-1} \frac{4c_r\bar{c}_s}{\alpha(\lambda_r + \bar{\lambda}_s)} \mathrm{e}^{-\alpha(\lambda_r+\bar{\lambda}_s)} \sinh \alpha(\lambda_r + \bar{\lambda}_s)$$
$$+ \sum_{\substack{r,s=0 \\ r+s\neq q}}^{q-1} \frac{4bc_r\bar{c}_s}{\alpha(\lambda_r - \bar{\lambda}_s)} \mathrm{e}^{-\alpha(\lambda_r+\bar{\lambda}_s)} \sinh \alpha(\lambda_r - \bar{\lambda}_s) + 4b \sum_{r=1}^{q-1} c_r \bar{c}_{q-r} \mathrm{e}^{-2\lambda_r \alpha} + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q \alpha}\right). \quad (3.9)$$

The final sum is $\mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right)$ and hence can be discarded. For the second sum, we notice that $2\mathrm{e}^{-\alpha(\lambda_r+\bar{\lambda}_s)} \sinh \alpha(\lambda_r + \bar{\lambda}_s) = 1 + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right)$ for $(r,s) \neq (0,0)$. Therefore

$$\sum_{\substack{r,s=0 \\ (r,s)\neq(0,0)}}^{q-1} \frac{4c_r\bar{c}_s}{\alpha(\lambda_r + \bar{\lambda}_s)} \mathrm{e}^{-\alpha(\lambda_r+\bar{\lambda}_s)} \sinh \alpha(\lambda_r + \bar{\lambda}_s) = \frac{2}{\alpha} \sum_{\substack{r,s=0 \\ (r,s)\neq(0,0)}}^{q-1} \frac{c_r\bar{c}_s}{\lambda_r + \bar{\lambda}_s} + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right).$$

Now consider the third sum in (3.9). Since $2\mathrm{e}^{-\alpha(\lambda_r+\bar{\lambda}_s)} \sinh \alpha(\lambda_r - \bar{\lambda}_s) = \mathrm{e}^{-2\alpha\bar{\lambda}_s} - \mathrm{e}^{-2\alpha\lambda_r}$, and

$$\mathrm{e}^{-2\alpha\bar{\lambda}_s} - \mathrm{e}^{-2\alpha\lambda_r} = \begin{cases} -\mathrm{e}^{2\mathrm{i}\alpha} & r = 0,\ s = 1, \ldots, q-1 \\ \mathrm{e}^{-2\mathrm{i}\alpha} & s = 0,\ r = 1, \ldots, q-1 \\ \mathrm{e}^{-2\mathrm{i}\alpha} - \mathrm{e}^{2\mathrm{i}\alpha} & r = s = 0 \\ 0 & \text{otherwise} \end{cases}$$

up to a term of order $\mathrm{e}^{-2\gamma_q\alpha}$, it follows that

$$2b \sum_{\substack{r,s=0 \\ r+s\neq q}}^{q-1} \frac{c_r\bar{c}_s}{\alpha(\lambda_r - \bar{\lambda}_s)} \mathrm{e}^{-\alpha(\lambda_r+\bar{\lambda}_s)} \sinh \alpha(\lambda_r - \bar{\lambda}_s)$$
$$= b\frac{c_0\bar{c}_0}{\lambda_0 - \bar{\lambda}_0}\left(\mathrm{e}^{-2\mathrm{i}\alpha} - \mathrm{e}^{2\mathrm{i}\alpha}\right) - b\sum_{s=1}^{q-1} \frac{c_0\bar{c}_s}{\alpha(\lambda_0 - \bar{\lambda}_s)} \mathrm{e}^{2\mathrm{i}\alpha} + b\sum_{r=1}^{q-1} \frac{c_r\bar{c}_0}{\alpha(\lambda_r - \bar{\lambda}_0)} \mathrm{e}^{-2\mathrm{i}\alpha} + \mathcal{O}\left(\mathrm{e}^{-2\alpha\gamma_q}\right)$$
$$= -b\sum_{s=0}^{q-1} \frac{c_0\bar{c}_s}{\alpha(\lambda_0 - \bar{\lambda}_s)} \mathrm{e}^{2\mathrm{i}\alpha} + b\sum_{r=0}^{q-1} \frac{c_r\bar{c}_0}{\alpha(\lambda_r - \bar{\lambda}_0)} \mathrm{e}^{-2\mathrm{i}\alpha} + \mathcal{O}\left(\mathrm{e}^{-2\alpha\gamma_q}\right).$$

Recall from the proof of Lemma 4 that $\det V^{[0]} = \det D \det V^{[q]}$ and $\det D = (-1)^q \mathrm{i}^{q-1}$. Hence $c_0 = \det V^{[0]} = (-1)^q \mathrm{i}^{q-1} \det V^{[q]} = \mathrm{i}^{q-1} c_q$, and therefore

$$bc_0\mathrm{e}^{2\mathrm{i}\alpha} = \mathrm{i}^{2(q-1)}\mathrm{e}^{4\mathrm{i}\alpha} c_q = c_q + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right),$$

since $\mathrm{e}^{4\mathrm{i}\alpha} = \mathrm{e}^{i\pi(q-1)} + \mathcal{O}\left(\mathrm{e}^{-\gamma_q\alpha}\right)$ (see Theorem 1). Since b is real in the limit $\alpha \to \infty$, we also find that $b\bar{c}_0\mathrm{e}^{-2\mathrm{i}\alpha} = \bar{c}_q + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right)$. Substituting these observations into the previous expression, we obtain

$$4b \sum_{\substack{r,s=0 \\ r+s\neq q}}^{q-1} \frac{c_r\bar{c}_s}{\alpha(\lambda_r - \bar{\lambda}_s)} \mathrm{e}^{-\alpha(\lambda_r+\bar{\lambda}_s)} \sinh \alpha(\lambda_r - \bar{\lambda}_s) = \frac{2}{\alpha}\sum_{s=0}^{q-1} \frac{c_q\bar{c}_s}{\lambda_q + \bar{\lambda}_s} + \frac{2}{\alpha}\sum_{r=0}^{q-1} \frac{c_r\bar{c}_q}{\lambda_r + \bar{\lambda}_q} + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right),$$

for the third term of (3.9). Combining this with the expression for the second term, now gives

$$\|\phi\|^2 = 4|c_0|^2 + \frac{2}{\alpha} \sum_{\substack{r,s=0 \\ (r,s)\neq(0,0),(q,q)}}^{q} \frac{c_r\bar{c}_s}{\lambda_r + \bar{\lambda}_s} + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q\alpha}\right).$$

To establish (3.8), we first need to show that the sum vanishes. To prove this result, it suffices to show that

$$\sum_{r=0}^{t} \frac{c_r \bar{c}_{t-r}}{\lambda_r + \bar{\lambda}_{t-r}} = 0, \quad t = 1, \ldots, q, \quad \sum_{r=t-q}^{q} \frac{c_r \bar{c}_{t-r}}{\lambda_r + \bar{\lambda}_{t-r}} = 0, \quad t = q+1, \ldots, 2q-1.$$

Moreover, since $\lambda_r + \bar{\lambda}_{t-r} = -\mathrm{i}\lambda^r(1-\lambda^{-t})$, these conditions reduce to

$$\sum_{r=0}^{t} c_r \bar{c}_{t-r} \lambda^{-r} = 0, \quad t = 1, \ldots, q, \quad \sum_{r=t-q}^{q} c_r \bar{c}_{t-r} \lambda^{-r} = 0, \quad t = q+1, \ldots, 2q-1. \tag{3.10}$$

Suppose that we define the matrix $V \in \mathbb{C}^{(q+1)\times(q+1)}$ with $(r,s)^{\text{th}}$ entry λ_s^{q+r}, $r, s = 0, \ldots, q$. It is readily seen that $(-1)^{q+r} \det V^{[r]} = \det V (V^{-1})_{r,q}$. Hence

$$\{c_r\}_{r=0}^{q} = (-1)^q (\det V) V^{-1} \{0, \ldots, 0, 1\}^\top.$$

Consider the matrix V. Since $\lambda_s^{q+r} = \lambda_0^{q+r} \lambda^{rs} \lambda^{qs}$, we may write $V = D^{[0]} W D^{[1]}$, where W is the Vandermonde matrix with $(r,s)^{\text{th}}$ entry λ^{rs}, and $D^{[0]}$ and $D^{[1]}$ are the diagonal matrices with r^{th} entries λ_0^{q+r} and $\lambda^{qr} = (-1)^r$ respectively. Simple arguments now give that

$$\frac{(-1)^q}{\det V} \{(-1)^r c_r\}_{r=0}^{q} = W^{-1}\{0, \ldots, 0, 1\}^\top.$$

Set $e_r = \frac{(-1)^{q+r}}{\det V} c_r$. To prove (3.10), it suffices to show the result with the values c_r replaced by e_r. Note that $W\{e_r\}_{r=0}^{q} = \{0, \ldots, 0, 1\}^\top$, and this is equivalent to the polynomial interpolation conditions $p(\lambda^r) = \delta_{r,q}$, $r = 0, \ldots, q$, where $p \in \mathbb{P}^q$ is the polynomial $\sum_{r=0}^{q} e_r x^r$. Trivially, p can written in terms of the q^{th} Lagrange polynomial:

$$p(x) = \prod_{r=0}^{q-1} \frac{x - \lambda^r}{\lambda^q - \lambda^r}.$$

Now consider the polynomial

$$q(x) = \overline{p(\overline{x})} p(\lambda^{-1} x) = \sum_{r,s=0}^{q} \bar{e}_s e_r \lambda^{-r} x^{r+s} = \sum_{t=0}^{2q} \gamma_t x^{2t},$$

where $\gamma_t = \sum_{r=0}^{t} e_r \bar{e}_{t-r} \lambda^{-r}$ for $t = 0, \ldots, q$ and $\gamma_t = \sum_{r=t-q}^{q} e_r \bar{e}_{t-r} \lambda^{-r}$ for $t = q+1, \ldots, 2q-1$. Therefore, it suffices to show that the polynomial $q(x)$ involves only 1 and x^{2q} and no other powers of x. We have

$$\overline{p(\overline{x})} p(\lambda^{-1} x) = \frac{1}{|\det V|^2} \prod_{r=0}^{q-1} (x - \bar{\lambda}^r)(x\lambda^{-1} - \lambda^r) = -\frac{1}{|\det V|^2} \prod_{r=0}^{q-1} (x - \lambda^{2q-r})(x - \lambda^{r+1}).$$

The product may be written as $\prod_{r=1}^{2q} (x - \lambda^r)$. Since λ is a $2q^{\text{th}}$ root of unity, this reduces to $x^{2q} - 1$. Hence $q(x) = -|\det V|^{-2}(x^{2q} - 1)$, as required.

We conclude that $\|\phi\|^2 = 4|c_0|^2 + \mathcal{O}\left(\mathrm{e}^{-2\gamma_q \alpha}\right)$. To complete the proof, we recall Lemma 5. Since $c_0 = \det V^{[0]}$ and

$$|\det V^{[0]}|^2 = \prod_{0 \le r < s < q} |\lambda^r - \lambda^s|^2 = \prod_{0 \le r < s < q} 2\left[1 - \cos \tfrac{\pi(r-s)}{q}\right] = 2^{q(q-1)} \prod_{0 \le r < s < q} \sin^2 \tfrac{\pi(r-s)}{2q},$$

we obtain the result. □

Much like Theorems 1 and 2, this result is of both theoretical interest and practical use, since it can be use to construct polyharmonic–Neumann expansions, as we next consider.

	n	1	2	3	4	5	10	15	20	25	30
$q=2$	e_n	2.43	4.00	5.16	6.99	8.44	15.5	22.5	29.5	36.4	43.3
	a_n	3	3	2	2	2	1	0	0	0	0
$q=3$	e_n	—	3.62	—	6.20	—	13.6	—	25.7	—	37.7
	a_n	0	3	0	2	0	1	0	0	0	0
$q=4$	e_n	2.35	4.63	4.42	5.44	6.97	11.6	16.8	21.5	26.5	31.4
	a_n	4	3	3	2	2	1	1	0	0	0

Table 1: Numerical computation of α_n for $q=2,3,4$. The value $e_n = -\log_{10}\left(|\alpha_n - \frac{1}{4}(2n+q-1)|/\alpha_n\right)$ measures the number of significant digits (a dash indicates where $\alpha_n = \frac{1}{4}(2n+q-1)$ exactly) and a_n is the number of Newton–Raphson iterations required to obtain machine epsilon.

n	1	2	3	4	5	10	15	20
$q=2$	$(7.6,2)$	$(4.2,4)$	$(1.9,5)$	$(8.8,7)$	$(3.9,8)$	$(6.3,15)$	$(9.6,22)$	$(1.5,28)$
$q=3$	$(1.5,2)$	$(2.9,3)$	$(6.5,5)$	$(1.5,5)$	$(2.8,7)$	$(1.3,12)$	$(4.3,19)$	$(2.2,24)$
$q=4$	$(1.0,2)$	$(5.0,3)$	$(9.3,4)$	$(7.2,5)$	$(3.9,6)$	$(1.9,10)$	$(9.9,16)$	$(4.4,20)$

Table 2: Uniform error in approximating ϕ_n using Theorem 2 for $q=2,3,4$. Here $(c,n) = c\times 10^{-n}$ for $c\in\mathbb{R}$ and $n\in\mathbb{N}$.

The estimates proved in this section, namely the exponential asymptotics for polyharmonic eigenvalues and eigenfunctions, improve known results in the literature of Birkhoff expansions. We speculate that the principal reason for their omission is due to the fact that such estimates are only valid under very specific conditions. In fact, there is evidence to suggest that only the polyharmonic operator with particularly simple boundary conditions will admit such estimates. A proof of such result requires further study, most likely along similar lines to [20], and is beyond the scope of this paper.

As we now address, such exponential asymptotics are of great use in the computation of the expansion f_N. In [6], the polyharmonic operator was chosen, out of all possible $2q^{\text{th}}$ order operators, for its simplicity. The previous comments indicate another reason for this choice.

3.3 Computation of polyharmonic–Neumann expansions

In [6] it was shown how to construct the eigenfunctions ϕ_n in a systematic manner (see also Section 2.1). Once the values α_n have been computed, the coefficients of such functions are found by solving a $q\times q$ algebraic eigenproblem. Computation of the values α_n involves solving a transcendental equation, which can be performed with standard iterative techniques, e.g. Newton–Raphson.

However, the exponential asymptotics of this section mean that such procedure is only necessary for small values of the parameter n. Once n is sufficiently large, we may use the approximations given in Theorems 1 and 2 instead (note that Theorem 2 gives an expression involving complex parameters. It is a simple, but tedious, exercise to translate this result into a real form, thereby giving an expression better suited to computations). To highlight this, in Tables 1 and 2 we consider the error in approximating α_n and ϕ_n by their asymptotic estimates. As is evident, such estimates are accurate to within machine epsilon whenever $n>15$, meaning that only the first 15 eigenvalues and eigenfunctions require numerical computation. Moreover, for the values α_n only at most four Newton–Raphson iterations are required for convergence, a fact which is easily explained from the exponential asymptotics. We remark in passing that had the estimates in Theorems 1 and 2 only been accurate up to $\mathcal{O}\left(n^{-1}\right)$ (as is the case for the majority of Birkhoff expansions), then computation of both α_n and ϕ_n would have been significantly harder.

The other main task in constructing the expansion f_N involves computing the coefficients $\hat{f}_n$. We shall not dwell on this issue, since it has been dealt with more thoroughly in [6], aside from mentioning that the basic approach is to replace the function f by a certain interpolating polynomial p and approximate the coefficient $\hat{f}_n$ by $\hat{p}_n$. This is a so-called Filon-type method (see also [14]). High asymptotic accuracy is guaranteed by interpolating certain derivatives of

f at the endpoints $x = \pm 1$, whilst high classical order (in the sense of numerical quadrature) is obtained by interpolating the function f at a number of nodes in $[-1, 1]$.

4 Convergence of polyharmonic–Neumann expansions

In the final three sections of this paper we consider the convergence of expansions in polyharmonic–Neumann eigenfunctions. To this end, we address the following three questions:

1. In what sense (i.e. norm) does f_N converge to f?
2. What is the rate of convergence in terms of N?
3. What factors determine both the degree and rate of convergence?

In particular, we wish to determine conditions under which $f_N \to f$ uniformly on $[-1, 1]$, thereby confirming the advantage of polyharmonic–Neumann expansions over both Fourier series and expansions in polyharmonic–Dirichlet eigenfunctions, for example. Moreover, we also seek to fully confirm the advantage of increasing the parameter q: namely, both a faster rate and higher degree of convergence of the expansion f_N.

Since polyharmonic–Neumann eigenfunctions form an orthogonal basis of $\mathrm{L}^2(-1, 1)$, the approximation f_N converges to f in the L^2 norm (as mentioned in Section 2.2). Our main focus of this section is the question of convergence in higher-order Sobolev norms H^r, $r \in \mathbb{N}$. In turn, this study allows uniform convergence to by verified, using standard imbedding theorems.

As mentioned in Section 1, much is known about the convergence of general Birkhoff expansions, especially as regards the phenomenon of equiconvergence. However, these results typically insufficiently describe the case of polyharmonic–Neumann expansions. In the forthcoming sections we present a largely self-contained convergence analysis of such expansions.

4.1 Duality under differentiation

In [3], it was shown that modified Fourier expansions (polyharmonic–Neumann expansions with $q = 1$) form an orthogonal basis for not just $\mathrm{L}^2(-1, 1)$, but also the space $\mathrm{H}^1(-1, 1)$. In particular, f_N converges to $f \in \mathrm{H}^1(-1, 1)$ in the H^1 norm. This proof was generalised in [6]: polyharmonic–Neumann expansions form an orthogonal basis $\mathrm{H}^q(-1, 1)$, provided this space is equipped with the inner product

$$(f, g)_q = \int_{-1}^{1} \left[f(x)g(x) + f^{(q)}(x)g^{(q)}(x) \right] \mathrm{d}x, \quad f, g \in \mathrm{H}^q(-1, 1). \tag{4.1}$$

Central to this proof is the following lemma:

Lemma 8. *If we apply the operator* $\frac{\mathrm{d}^q}{\mathrm{d}x^q}$ *to the set of polyharmonic–Neumann eigenfunctions* ϕ_n*, we obtain, up to scalar multiples, the set of polyharmonic eigenfunctions that satisfy the Dirichlet boundary conditions (2.9). Such eigenfunctions are dense and orthogonal in* $\mathrm{L}^2(-1, 1)$*. Moreover, for* $f \in \mathrm{H}^q(-1, 1)$*,* $(f_N)^{(q)}$ *is precisely the truncated expansion of* $f^{(q)}$ *in such eigenfunctions.*

Proof. Though this proof is found in [6], it is useful to repeat it here, since similar techniques will be used later.

It is clear that q-fold differentiation yields the set of polyharmonic–Dirichlet eigenfunctions (note that the polyharmonic–Dirichlet operator has no zero eigenvalue). Density and orthogonality now follow directly from standard spectral theory [18]. For the second result, we first note that, for $f \in \mathrm{H}^r(-1, 1)$, $r = 0, \ldots, q$,

$$\int_{-1}^{1} f(x)\phi(x)\,\mathrm{d}x = \frac{(-1)^{q+r}}{\alpha^{2q}} \int_{-1}^{1} f^{(r)}(x)\phi^{(2q-r)}(x)\,\mathrm{d}x, \tag{4.2}$$

where ϕ is a polyharmonic–Neumann eigenfunction with corresponding eigenvalue $\mu = \alpha^{2q}$. This follows from the equality $\phi^{(2q)} = (-1)^q \alpha^{2q} \phi$ and repeated integration by parts. Now, suppose that $\phi^{(q)} = c\psi$, where ψ is the corresponding normalised polyharmonic–Dirichlet eigenfunction and c is a constant. Using (4.2) with $r = q$ gives

$$c^2 = c^2 \int_{-1}^{1} \psi(x)\psi(x)\,\mathrm{d}x = \int_{-1}^{1} \phi^{(q)}(x)\phi^{(q)}(x)\,\mathrm{d}x = \alpha^{2q}.$$

Moreover, we have

$$\int_{-1}^{1} f(x)\phi(x)\,\mathrm{d}x = \frac{1}{\alpha^{2q}} \int_{-1}^{1} f^{(q)}(x)\phi^{(q)}(x)\,\mathrm{d}x = \frac{1}{c}\int_{-1}^{1} f^{(q)}(x)\psi(x)\,\mathrm{d}x,$$

so that $(f,\phi)\phi^{(q)}(x) = (f^{(q)},\psi)\psi(x)$, where $(\cdot,\cdot)$ is the standard Euclidean inner product. The result now follows. □

This so-called *duality under differentiation* of polyharmonic–Neumann and polyharmonic–Dirichlet expansions immediately provides the main result:

Theorem 3. *The set of polyharmonic–Neumann eigenfunctions forms an orthogonal basis for the space* $\mathrm{H}^q(-1,1)$ *equipped with the inner product (4.1). In particular,* f_N *converges to* f *in the* H^q *norm and we have the Parseval-type characterisation*

$$|||f|||_q^2 = \sum_{n=0}^{q-1} |\hat{f}_{0,n}|^2 + \sum_{n=1}^{\infty} (1+\mu_n)|\hat{f}_n|^2, \quad \forall f \in \mathrm{H}^q(-1,1), \tag{4.3}$$

where $|||f||| = \sqrt{(f,f)_q}$ *is the norm induced from (4.1).*

This theorem indicates that polyharmonic–Neumann expansions contrast strongly with, for example, Fourier series, which only converge in the L^2 sense (as we later consider, the same is true for polyharmonic–Dirichlet expansions). This higher degree of convergence translates into a faster convergence rate, as we demonstrate in Section 5.

Theorem 3 also provokes the following question: for which values of $r \neq 0, q$ does f_N converge to $f \in \mathrm{H}^r(-1,1)$ in the H^r norm? As we will show in Section 4.3, this holds for all $r = 1,\ldots,q-1$. To do so, much as in Lemma 8, we first need to describe the r^{th} derivative $f_N^{(r)}$ in terms of an expansion in certain polyharmonic eigenfunctions.

4.2 Biorthogonal pairs of polyharmonic–Neumann eigenfunctions

For $r = 1,\ldots,q-1$, the derivative $f_N^{(r)}$ can no longer be expressed as an orthogonal series. Instead, it can be written in terms of a certain biorthogonal pair of polyharmonic eigenfunctions.

Let us first recall some theory of Birkhoff expansions (see [20], for example). Suppose that the polyharmonic operator $\mathcal{L} = (-1)^q \frac{\mathrm{d}^{2q}}{\mathrm{d}x^{2q}}$ is equipped with boundary conditions $\mathcal{B}_r\phi = 0$, $r = 1,\ldots,2q$. The *adjoint* boundary conditions $\mathcal{B}_r^*[\phi] = 0$, $r = 1,\ldots,2q$, are defined so that

$$\int_{-1}^{1} \mathcal{L}\phi(x)\bar{\psi}(x)\,\mathrm{d}x = \int_{-1}^{1} \phi(x)\overline{\mathcal{L}\psi}(x)\,\mathrm{d}x,$$

for all $2q$-times continuously differentiable, comlex valued functions ϕ,ψ satisfying $\mathcal{B}_r\phi = 0$ and $\mathcal{B}_r^*\psi = 0$. We say that the operator $\mathcal{L}$, when equipped with boundary conditions $\mathcal{B}_r$ (which we write as $\{\mathcal{L},\mathcal{B}_r\}$), is self-adjoint provided $\mathcal{B}_r = \mathcal{B}_r^*$ (up to reordering).

Under some assumptions on the $\mathcal{B}_r$, the spectrum of $\{\mathcal{L},\mathcal{B}_r\}$ is countable with real eigenvalues $\{\mu_n\}$ and eigenfunctions $\{\phi_n\}$ [20]. Moreover, the spectrum of $\{\mathcal{L},\mathcal{B}_r^*\}$ consists of precisely the values μ_n, with corresponding eigenfunctions $\{\psi_n\}$ that satisfy $(\phi_n,\psi_m) = \delta_{n,m}$ (after appropriate renormalisation). For this reason, we refer to the pair $\{\phi_n,\psi_n\}$ as a *biorthogonal pair* of polyharmonic eigenfunctions. Such biorthogonality signals that a function f may be expanded in the formal series

$$f(x) \sim \sum_{n=1}^{\infty} (f,\psi_n)\phi_n(x).$$

Note that we do not make an assumptions regarding convergence of this series at this point.

It is evident that, when prescribed either Neumann $\phi^{(q+r)}(\pm 1) = 0$, $r = 0, \ldots, q-1$, or Dirichlet $\phi^{(r)}(\pm 1) = 0$ boundary conditions, the operator $\mathcal{L}$ is self-adjoint. We now catalogue the nature of the polyharmonic operator under a variety of other boundary conditions:

Lemma 9. *Suppose that $p = 1, \ldots, q-1$ and that the polyharmonic operator $\mathcal{L} = (-1)^q \frac{\mathrm{d}^{2q}}{\mathrm{d}x^{2q}}$ is equipped with boundary conditions*

$$\phi^{(q+r-p)}(\pm 1) = 0, \quad r = 0, \ldots, q-1. \tag{4.4}$$

Then the adjoint boundary conditions are

$$\psi^{(r)}(\pm 1) = 0, \quad r = 0, \ldots, p-1, \quad \psi^{(2q-r-1)}(\pm 1) = 0, \quad r = 0, \ldots, q-p-1. \tag{4.5}$$

In particular, the corresponding pair of polyharmonic eigenfunctions subject to boundary conditions (4.4) and (4.5) are biorthogonal.

Proof. We have

$$\int_{-1}^{1} \mathcal{L}\phi(x)\bar{\psi}(x)\,\mathrm{d}x = (-1)^q \sum_{r=0}^{2q-1} (-1)^{r+1} \phi^{(r)}(x)\bar{\psi}^{(2q-r-1)}(x)\Big|_{-1}^{1} + \int_{-1}^{1} \phi(x)\overline{\mathcal{L}\psi}(x)\,\mathrm{d}x.$$

If ϕ satisfies boundary conditions (4.4), then this sum vanishes precisely when ψ obeys the conditions (4.5). $\square$

In subsequent analysis, it is necessary to understand the nature of the zero eigenfunction of the operator $\mathcal{L}$ when equipped with boundary conditions (4.4) or (4.5). Recall that the polyharmonic–Neumann operator has a zero eigenvalue of multiplicity q. The corresponding eigenspace is $\mathbb{P}_{q-1}$, the space of polynomials of degree $q-1$. Trivial calculations verify that the polyharmonic operator with boundary conditions (4.4) or (4.5) has a $(q-p)$-fold zero eigenvalue. The corresponding eigenspaces are $\mathbb{P}_{q-p-1}$ and $\left\{g \in \mathbb{P}_{q+p-1} : g^{(r)}(\pm 1) = 0,\ r = 0, \ldots, p-1\right\}$ respectively.

We are now in position to prove the main result of this section:

Theorem 4. *If we apply the differentiation operator $\frac{\mathrm{d}^p}{\mathrm{d}x^p}$, $p = 1, \ldots, q-1$, to the set of polyharmonic–Neumann eigenfunctions, we obtain, up to scalar multiples, the set of polyharmonic eigenfunctions that satisfy the boundary conditions (4.4). Furthermore, for $f \in \mathrm{H}^p(-1,1)$, $(f_N)^{(p)}$ is the truncated expansion of $f^{(p)}$ in the biorthogonal pair of polyharmonic eigenfunctions corresponding to boundary conditions (4.4) and (4.5).*

Proof. The first result is trivial. For the second, suppose that ϕ_n is the n^{th} polyharmonic–Neumann eigenfunction with eigenvalue $\mu_n = \alpha_n^{2q} \neq 0$. Let $\phi_n^{(p)} = c_n \psi_n$ and $\phi_n^{(2q-p)} = d_n \chi_n$, where $\{\psi_n, \chi_n\}$ is the biorthogonal pair corresponding to boundary conditions (4.4) and (4.5). Assume that such eigenfunctions are normalised so that $(\psi_n, \chi_m) = \delta_{n,m}$. Setting $r = p$, $\phi = \phi_n$ and $f = \phi_n$ in (4.2) immediately gives

$$1 = \frac{(-1)^{q+p}}{\alpha_n^{2q}} c_n d_n \int_{-1}^{1} \psi_n(x)\chi_n(x)\,\mathrm{d}x.$$

Hence, $c_n d_n = (-1)^{q+p}\alpha_n^{2q}$. Moreover, using (4.2) once more,

$$\hat{f}_n \phi_n^{(p)}(x) = \frac{(-1)^{q+p}}{\alpha_n^{2q}} c_n d_n \int_{-1}^{1} f^{(p)}(x)\chi_n(x)\,\mathrm{d}x \psi_n(x) = \left(f^{(p)}, \chi_n\right)\psi_n(x).$$

It follows that

$$\frac{\mathrm{d}^p}{\mathrm{d}x^p} \sum_{n=1}^{N} \hat{f}_n \phi_n(x) = \sum_{n=1}^{N} \left(f^{(p)}, \chi_n\right)\psi_n(x), \tag{4.6}$$

for any $N \in \mathbb{N}$. To complete the proof, we need to consider the component of the expansion f_N corresponding to the q-fold zero eigenvalue. To this end, suppose that we write $\{\psi_{0,n} : n = 0, \ldots, q-p-1\}$ and $\{\chi_{0,n} : n = 0, \ldots, q-p-1\}$ for the sets of polyharmonic eigenfunctions corresponding to the zero eigenvalue and subject to boundary conditions (4.4) and (4.5) respectively. It now suffices to show that

$$\frac{\mathrm{d}^p}{\mathrm{d}x^p} \sum_{n=0}^{q-1} \hat{f}_{0,n}\phi_{0,n}(x) = \sum_{n=0}^{q-p-1} \left(f^{(p)}, \chi_{0,n}\right) \psi_{0,n}(x). \tag{4.7}$$

Since $\{\psi_{0,n}\}$ is a basis for $\mathbb{P}_{q-p-1}$, we have $\frac{\mathrm{d}^p}{\mathrm{d}x^p} \sum_{n=0}^{q-1} \hat{f}_{0,n}\phi_{0,n}(x) = \sum_{n=0}^{q-p-1} a_n\psi_{0,n}(x)$ for values $a_n \in \mathbb{R}$. Due to the biorthogonality relation $(\psi_{0,n}, \chi_{0,m}) = \delta_{n,m}$, we have

$$a_n = \left(\frac{\mathrm{d}^p}{\mathrm{d}x^p} \sum_{m=0}^{q-1} \hat{f}_{0,m}\phi_{0,m}, \chi_{0,n} \right).$$

In view of (4.6) and the fact that $(\psi_n, \chi_{0,m}) = 0$, we may write

$$a_n = \left(\frac{\mathrm{d}^p}{\mathrm{d}x^p} \left\{ \sum_{m=0}^{q-1} \hat{f}_{0,m}\phi_{0,m} + \sum_{m=1}^{N} \hat{f}_n\phi_m \right\}, \chi_{0,n} \right) = \left(\frac{\mathrm{d}^p}{\mathrm{d}x^p} f_N, \chi_{0,n} \right),$$

for any $N \in \mathbb{N}_+$. We now note that, since $\chi_{0,n}^{(r)}(\pm 1) = 0$ for $r = 0, \ldots, p-1$, integration by parts p times gives the relation

$$\left(g^{(p)}, \chi_{0,n}\right) = \left(g, \chi_{0,n}^{(p)}\right) \tag{4.8}$$

for any function $g \in \mathrm{H}^p(-1,1)$. In particular, $a_n = \left(f_N, \chi_{0,n}^{(p)}\right)$. Since N was arbitrary and $f_N \to f$ in the $\mathrm{L}^2(-1,1)$ norm, it follows that $a_n = \left(f, \chi_{0,n}^{(p)}\right)$. An application of (4.8) now gives $a_n = (f^{(p)}, \chi_{0,n})$, hence verifying (4.7). □

Well-known results for general Birkhoff expansions can now be used to establish convergence of $f_N^{(r)}$ to $f^{(r)}$ in L^2, and hence convergence of f_N to $f \in \mathrm{H}^r(-1,1)$ in the H^r norm. However, the particular nature of polyharmonic–Neumann eigenfunctions allows us to present an alternative, simpler proof of this conjecture in a completely self-contained manner.

4.3 Convergence in H^r norm, $r = 1, \ldots, q-1$

Throughout this section we write c for a positive constant, independent of f and N.

Our technique of proof will be based on known results for the cases $r = 0, q$ and interpolation therein for the intermediate values $r = 1, \ldots, q-1$. To do so, we first need to establish a Bessel-type inequality in the H^r norm for polyharmonic–Neumann expansions. Specifically, we shall prove that $\|f_N\|_r \leq c\|f\|_r$ for $f \in \mathrm{H}^r(-1,1)$ and $N \in \mathbb{N}_+$.

We commence by stating the following lemma, found in a virtually identical form in [10, p.2332]:

Lemma 10. *Suppose that $a = (a_1, a_2, \ldots)$, where $a_n = \int_{-1}^{1} \mathrm{e}^{zn(1\pm x)} f(x)\,\mathrm{d}x$ (with the same sign for all n) and $f \in \mathrm{L}^2(-1,1)$. Suppose further that $z \neq 0$ and $\mathrm{Re}\, z \leq 0$. Then $a \in l^2(\mathbb{N})$ and $\|a\| \leq c\|f\|$, where $\|a\|^2 = \sum_{n=1}^{\infty} |a_n|^2$.*

This lemma possesses the following converse, also found in [10]:

Lemma 11. *Suppose that $b = (b_1, b_2, \ldots) \in l^2(\mathbb{N})$. Then, for $\mathrm{Re}\, z \leq 0$ and $z \neq 0$, the family of all finite sums of terms of the form $b_n\mathrm{e}^{zn(1\pm x)}$ is uniformly bounded in $\mathrm{L}^2(-1,1)$ with norm bounded by $c\|b\|$.*

With these lemmas in hand, we now return to the polyharmonic problem:

Lemma 12. *Suppose that $\{\psi_n, \chi_n\}$ are a biorthogonal pair of polyharmonic eigenfunctions, with ψ_n and χ_n subject to boundary conditions (4.4) and (4.5) respectively. Then, the family of all finite sums of terms $(f, \chi_n)\psi_n$ is uniformly bounded in* $\mathrm{L}^2(-1,1)$ *with norm bounded by* $c\|f\|$.

Proof. Recall Theorem 2. We may write χ_n as

$$\chi_n(x) = \sum_{s=0}^{q-1} \left[a_s \mathrm{e}^{\alpha_n \lambda_s (x-1)} + b_s \mathrm{e}^{-\alpha_n \lambda_s (x+1)} \right] + \mathcal{O}\left(\mathrm{e}^{-n\pi\gamma_q}\right), \tag{4.9}$$

with constants a_s and b_s independent of n. Since $\alpha_n = \frac{1}{4}(2n+q-1)\pi + \mathcal{O}\left(\mathrm{e}^{-n\pi\gamma_q}\right)$ and $\operatorname{Re}\lambda_s \leq 0$, χ_n is a finite sum of exponentials of the form $\mathrm{e}^{zn(1\pm x)}$ with $\operatorname{Re} z \leq 0$ and $z \neq 0$. Hence, for $f \in \mathrm{L}^2(-1,1)$, it follows from Lemma 10 that the sequence (f, χ_n) is in $l^2(\mathbb{N})$ with norm bounded by $c\|f\|$. Since we may also write ψ_n in the form (4.9) (with different constants a_s and b_s), the full result is now a simple consequence of Lemma 11. ☐

We are now able to prove the aforementioned Bessel-type inequality for polyharmonic–Neumann expansions:

Lemma 13. *Suppose that $f \in \mathrm{H}^r(-1,1)$, $r = 0, \ldots, q$, and that f_N is the truncated expansion of f in polyharmonic–Neumann eigenfunctions. Then $\|f_N\|_r \leq c\|f\|_r$ for all $N \in \mathbb{N}_+$.*

Proof. By Theorem 4, the function $f_N^{(r)}$ is a finite sum of terms of the form $(f^{(r)}, \chi_n)\psi_n$. An application of Lemma 12 now gives the result. ☐

Having established this inequality, we may now prove the key result of this section:

Theorem 5. *Suppose that $f \in \mathrm{H}^r(-1,1)$, $r = 0, \ldots, q$, and that f_N is the truncated expansion of f in polyharmonic–Neumann eigenfunctions. Then f_N converges to f in the* $\mathrm{H}^r(-1,1)$ *norm.*

Proof. Since we have already proved the result for $r = 0, q$, we assume that $r = 1, \ldots, q-1$. In this case, given $\epsilon > 0$, there exists $g \in \mathrm{H}^q(-1,1)$ with $\|f - g\|_r < \epsilon$ [2]. In view of Lemma 13, $\|f_N - g_N\|_r < c\epsilon$. Hence

$$\|f - f_N\|_r \leq \|g - g_N\|_r + \|f - g\|_r + \|f_N - g_N\|_r < \|g - g_N\|_q + (1 + c)\epsilon.$$

Since $g \in \mathrm{H}^q(-1,1)$, $\|g - g_N\|_q < \epsilon$ for large N (by Theorem 3), completing the proof. ☐

An immediate consequence of this theorem is uniform convergence of polyharmonic–Neumann expansions:

Corollary 3. *Suppose that $f \in \mathrm{H}^r(-1,1)$, $r = 1, \ldots, q$, and that f_N is the truncated polyharmonic–Neumann expansion of f. Then $f_N^{(s)}$ converges uniformly to $f^{(s)}$ for $s = 0, \ldots, r-1$.*

Proof. This follows immediately from the Sobolev imbedding $\mathrm{H}^s(-1,1) \hookrightarrow \mathrm{C}^{s-1}[-1,1]$, $s \in \mathbb{N}$, (see, e.g. [2]) and Theorem 5. ☐

In particular, this corollary establishes that f_N converges uniformly to $f \in \mathrm{H}^1(-1,1)$. Note that this improves upon a result proved in [6], which assumed $\mathrm{H}^q(-1,1)$-regularity.

We remark in passing that, as a consequence of Theorem 5, the expansion of a function in any biorthogonal pair of polyharmonic eigenfunctions with boundary conditions (4.4) and (4.5) converges in the L^2 norm. This result, as mentioned, is known in a more general context. The (somewhat circuitous) method of proof presented above cannot be extended to arbitrary Birkhoff expansions (except in certain cases), since it relies both on the particular duality of polyharmonic eigenfunctions and known results for the Dirichlet and Neumann cases (themselves consequences of standard spectral theory for self-adjoint differential operators).

Theorem 5 and Corollary 3 clearly demonstrate the advantage gained from increasing the parameter q: namely, higher orders of convergence. As we consider in Section 5, this in turn corresponds to faster convergence rates. In addition, these results provide criteria for both the best and worst boundary conditions to prescribe to the polyharmonic operator in terms of the convergence of the truncated expansion f_N, as opposed to the arguments of Section 2.2 based on the decay of the coefficients $\hat{f}_n$. Specifically, it is easily established that the expansion based on polyharmonic eigenfunctions subject to boundary conditions (4.4) converges maximally in the $\mathrm{H}^{q-p}(-1,1)$ norm, $p=0,\ldots,q$. Correspondingly, for boundary conditions (4.5), only $\mathrm{L}^2(-1,1)$ convergence occurs. Hence, choosing $p=0$ for the highest possible degree of convergence, we once more arrive at Neumann boundary conditions. Conversely, Dirichlet boundary conditions ($p=q$) give the worst degree of convergence.

4.4 Pointwise convergence

Corollary 3 verifies that f_N and its first $(q-1)$ derivatives converge uniformly to the corresponding derivatives of f. In this section, we prove that the q^{th} derivative of f_N, whilst not converging uniformly on $[-1,1]$, does in fact converge to $f^{(q)}$ uniformly in compact subsets of $(-1,1)$.

To prove this result, we first note that the expression (4.2) for the coefficient $\hat{f}_n$ can be repeatedly integrated by parts to give

$$\begin{aligned}\hat{f}_n =& \frac{1}{\alpha_n^{2q}}\sum_{s=0}^{p-1}(-1)^s\left[f^{(q+s)}(1)\phi_n^{(q-s-1)}(1)-f^{(q+s)}(-1)\phi_n^{(q-s-1)}(-1)\right]\\ &+\frac{(-1)^p}{\alpha_n^{2q}}\int_{-1}^{1}f^{(q+p)}(x)\phi_n^{(q-p)}(x)\,\mathrm{d}x,\end{aligned}\tag{4.10}$$

provided $f\in\mathrm{H}^{q+p}(-1,1)$, $p=0,\ldots,q$. In particular, since $\alpha_n=\mathcal{O}(n)$ and $\phi_n^{(q-1)}(\pm1)=(\pm1)^n d_{q-1}\alpha_n^{q-1}+\mathcal{O}\left(n^{q-1}\mathrm{e}^{-n\pi\gamma_q}\right)$ by Lemma 6, we have

$$\begin{aligned}\hat{f}_n &= \frac{1}{\alpha_n^{2q}}\left[f^{(q)}(1)\phi_n^{(q-1)}(1)-f^{(q)}(-1)\phi_n^{(q-1)}(-1)\right]+\mathcal{O}\left(n^{-q-2}\right).\\ &=\frac{d_{q-1}}{\alpha_n^{q+1}}\left[f^{(q)}(1)+(-1)^{n+1}f^{(q)}(-1)\right]+\mathcal{O}\left(n^{-q-2}\right),\end{aligned}\tag{4.11}$$

for $f\in\mathrm{H}^{q+2}(-1,1)$. Furthermore, for $x\in(-1,1)$, it follows from Theorem 2 that

$$\phi_n^{(q)}(x)=\alpha_n^q(-1)^q c_0\left[\mathrm{e}^{-\mathrm{i}\alpha_n(x-1)}+(-1)^{n+1}\mathrm{e}^{\mathrm{i}\alpha_n(x+1)}\right]+\mathcal{O}\left(n^q\mathrm{e}^{-\frac{1}{2}n\pi\gamma_q(1-|x|)}\right).\tag{4.12}$$

We are now in a position to establish pointwise convergence of $f_N^{(q)}$ to f:

Theorem 6. *Suppose that $f\in\mathrm{H}^{q+2}(-1,1)$ and that f_N is the truncated expansion of f in polyharmonic–Neumann eigenfunctions. Then $f_N^{(q)}$ converges to $f^{(q)}$ uniformly in compact subsets of $(-1,1)$.*

Proof. First consider the sequences

$$a_N^{\pm}(x)=\sum_{n=1}^{N}\frac{(\pm1)^n}{\alpha_n}\mathrm{e}^{-\mathrm{i}\alpha_n(x-1)},\quad b_N^{\pm}(x)=\sum_{n=1}^{N}\frac{(\pm1)^n}{\alpha_n}\mathrm{e}^{\mathrm{i}\alpha_n(x+1)},\quad N\in\mathbb{N}.$$

We claim that these sequences converge uniformly in compact subsets of $(-1,1)$. Note that $\mathrm{e}^{-\mathrm{i}\alpha_{N+n}(x-1)}=\mathrm{e}^{-\mathrm{i}\alpha_N(x-1)}(-1)^n\mathrm{e}^{-\mathrm{i}\frac{1}{2}n\pi(x+1)}+\mathcal{O}\left(\mathrm{e}^{-n\pi\gamma_q}\right)$. Hence, for example,

$$a_{N+M}^{+}-a_N^{+}=\frac{2\mathrm{e}^{-\mathrm{i}\alpha_N(x-1)}}{\pi}\sum_{n=0}^{M}\frac{(-1)^n}{n+N+\frac{1}{2}(q-1)}\mathrm{e}^{-\mathrm{i}\frac{1}{2}n\pi(x+1)}+\mathcal{O}\left(N\mathrm{e}^{-N\pi\gamma_q}\right).$$

Much like the tail of a Fourier series, this sum tends to zero as $N\to\infty$ for every $M\in\mathbb{N}$. Hence a_N^{+} forms a Cauchy sequence, and thus converges uniformly in compact subsets of $(-1,1)$ to a continuous function $a^{+}(x)$. Similar arguments confirm convergence of a_N^{-} and $b_N^{\pm}$. Using (4.11), (4.12) and this result, we deduce convergence of $f_N^{(q)}(x)$ to a continuous function $g(x)$. Since $f_N^{(q)}\to f^{(q)}$ in the L^2 norm and g is continuous, we conclude that $g\equiv f^{(q)}$, as required. □

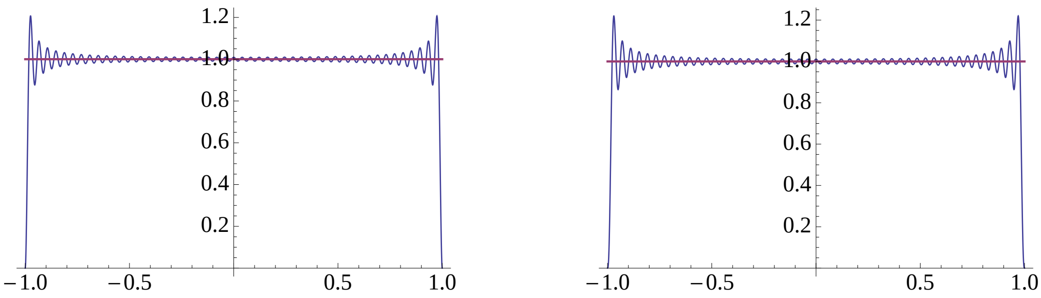

Figure 3: The Gibbs phenomenon for polyharmonic–Dirichlet expansions. Graph of $f(x) = 1$ and $f_{50}(x)$ for $-1 \leq x \leq 1$, where $q = 2$ (left), $q = 3$ (right) and f_N is the expansion of f in polyharmonic–Dirichlet eigenfunctions.

As mentioned, the expansion of a function f in polyharmonic–Dirichlet eigenfunctions does not converge uniformly on $[-1, 1]$. However, in view of Lemma 8, the previous theorem equivalently states that such expansions converge away from the endpoints $x = \pm 1$. Near the endpoints, however, they suffer from a Gibbs-type phenomenon. In Figure 3 we exhibit this effect for the approximation of the function $f(x) = 1$ by polyharmonic–Dirichlet eigenfunctions: the presence of $\mathcal{O}(1)$ oscillations near $x \pm 1$ highlighting the Gibbs phenomenon in this case.

5 Rate of convergence

The intent of this section is to provide estimates for the rate of convergence of the approximation f_N. We first derive results in various Sobolev norms. However, the exponential asymptotics of Section 3 can be used to provide precise expressions for the pointwise error $f(x) - f_N(x)$ at any point $x \in [-1, 1]$. In turn, this allows us to derive not only the stated $\mathcal{O}\left(N^{-q-1}\right)$ estimate for the convergence rate in $(-1, 1)$, but also an exact expression for the leading order error term as a function of x. We dedicate Section 5.2 to this topic.

5.1 Convergence rate in various norms

Standard techniques of Fourier analysis are used to derive the first result of this section:

Lemma 14. *Suppose that* $f \in \mathrm{H}^r(-1, 1)$. *Then* $\|f - f_N\|_r \leq cN^{r-s}\|f\|_s$ *for* $s = r, \ldots, q$.

Proof. Consider the case $r = 0$. By (2.7), we have $\|f - f_N\|^2 = \sum_{n>N} |\hat{f}_n|^2$. Note that $\alpha_n^{2s}|\hat{f}_n|^2 = \left(f^{(s)}, \psi_n\right)$, where ψ_n is a polyharmonic eigenfunction equipped with boundary conditions (4.4) and $p = q - s$. It now follows from the proof of Lemma 12 that $\sum_{n>N} \alpha_n^{2s}|\hat{f}_n|^2 \leq c\|f\|_s^2$. Using this result and the fact that $\alpha_n = \mathcal{O}(n)$, we obtain

$$\|f - f_N\|^2 = \sum_{n>N} \frac{\alpha_n^{2s}}{\alpha_n^{2s}}|\hat{f}_n|^2 \leq N^{-2s} \sum_{n>N} \alpha_n^{2s}|\hat{f}_n|^2 \leq N^{-2s}\|f\|_s^2,$$

which completes the proof for $r = 0$. Now suppose that $r = 1, \ldots, s$. Recall the interpolation inequality (see, for example [2])

$$\|g\|_r \leq c\|g\|^{1-\frac{r}{s}}\|g\|_s^{\frac{r}{s}}, \quad \forall g \in \mathrm{H}^s(-1, 1). \tag{5.1}$$

Setting $g = f - f_N$ and using the previously derived result, we obtain

$$\|f - f_N\|_r \leq cN^{-s(1-\frac{r}{s})}\|f\|_s^{1-\frac{r}{s}}\|f - f_N\|_s^{\frac{r}{s}} = cN^{r-s}\|f\|_s^{1-\frac{r}{s}}\|f - f_N\|_s^{\frac{r}{s}}.$$

Note that $\|f - f_N\|_s \leq \|f\|_s + \|f_N\|_s$. An application of Lemma 13 now gives $\|f - f_N\|_s \leq c\|f\|_s$, thus completing the proof. □

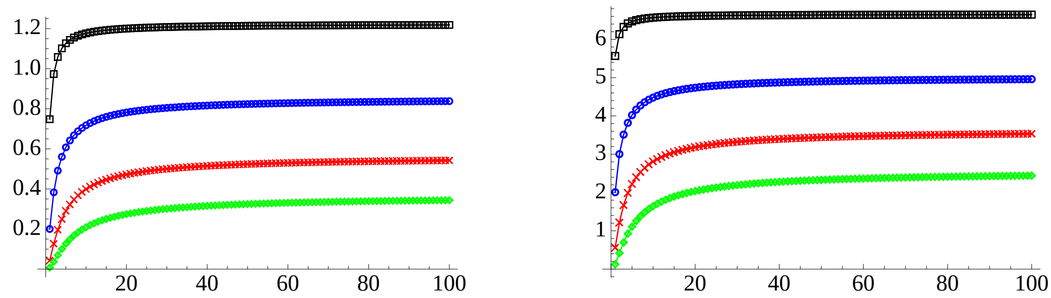

Figure 4: Error in approximating $f(x) = \mathrm{e}^{2x}$ by $\mathcal{F}_N[f](x)$ for $q = 1$ (squares), $q = 2$ (circles), $q = 3$ (crosses) and $q = 4$ (diamonds). Left: scaled error $N^{q+\frac{1}{2}}\|f - \mathcal{F}_N[f]\|$ for $N = 1, \ldots, 100$. Right: scaled error $N^{q-\frac{1}{2}}\|f - \mathcal{F}_N[f]\|_1$.

This lemma gives estimates for the convergence rate of f_N in various Sobolev norms. However, for smooth functions f, it leads to the conclusion that $\|f - f_N\| = \mathcal{O}\left(N^{-q}\right)$. This turns out not to be the case, the convergence rate is in fact $\mathcal{O}(N^{-q-\frac{1}{2}})$, as the following result demonstrates:

Theorem 7. *Suppose that* $f \in \mathrm{H}^{q+1}(-1,1)$. *Then* $\|f - f_N\|_r \leq cN^{r-q-\frac{1}{2}}\|f\|_{q+1}$ *for* $r = 0, \ldots, q$. *Moreover,* $\|(f - f_N)^{(r)}\|_\infty \leq cN^{r-q}\|f\|_{q+1}$ *for* $r = 0, \ldots, q-1$.

Proof. From (4.10) we find that $|\hat{f}_n| \leq cn^{-q-1}\|f\|_{q+1}$. Hence, using (2.7), we have

$$\|f - f_N\|^2 \leq c\|f\|^2_{q+1} \sum_{n>N} n^{-2q-2} \leq cN^{-2q-1}\|f\|^2_{q+1},$$

which gives the result for $r = 0$. By an identical argument, using (4.3) instead of (2.7), we also obtain the corresponding result for $r = q$. The full proof now follows after an application of (5.1) with $g = f - f_N$ and $s = q$. To derive the estimate for the uniform error, we use Theorem 7 and the Sobolev interpolation inequality $\|g\|_\infty \leq c\sqrt{\|g\|\|g\|_1}$, $\forall g \in \mathrm{H}^1(-1,1)$, with $g = (f - f_N)^{(r)}$. $\square$

The first part of Theorem 7 is verified in Figure 4. The result for the uniform error, i.e. $\|f - f_N\|_\infty = \mathcal{O}\left(N^{-q}\right)$ was confirmed in Figure 1.

5.2 The error $f(x) - f_N(x)$

The exponential asymptotics of Section 3 allow us to determine an explicit asymptotic expansion for the error $f(x) - f_N(x)$ in inverse powers of N. This expansion involves only certain derivatives of f evaluated at the endpoints $x = \pm 1$. A particular consequence of this result is the aforementioned estimate $f(x) - f_N(x) = \mathcal{O}\left(N^{-q-1}\right)$ for $-1 < x < 1$. However, we may also give an exact expression for the leading order behaviour of the error as a function of both N and x. This was originally established in [21] for the modified Fourier ($q = 1$) case. Our result, proved in a similar manner, extends this result to arbitrary $q \geq 2$.

For simplicity, we assume that $f \in \mathrm{C}^\infty[-1,1]$ throughout this section. Minor modifications can be made to the results proved herein to deal with lower regularity. To commence, recall that

$$\phi_n(x) = \sum_{s=0}^{q-1} c_s \left[\mathrm{e}^{\lambda_s \alpha_n (x-1)} + (-1)^{n+q+1}\mathrm{e}^{-\lambda_s \alpha_n (x+1)}\right] + \mathcal{O}\left(\mathrm{e}^{-n\pi\gamma_q}\right), \qquad (5.2)$$

by Theorem 2. For convenience, we disregard all exponentially small terms from this point onwards. Note also that $\|\phi_n\| \sim c$, where the constant c is known explicitly (Lemma 7). Suppose now that we define

$$\Theta^\pm(r, N; x) = \frac{1}{c^2} \sum_{n \geq N} \frac{(\pm 1)^n}{\alpha_n^r} \phi_n(x), \quad r > 1, \quad N \in \mathbb{N}_+. \qquad (5.3)$$

Note that the functions $\Theta^{\pm}$ are well-defined and continuous (as functions of x) for all values $r > 1$, since the infinite sum converges uniformly on $[-1,1]$ (though we have not explicitly shown it, it is a simple exercise to verify that $\|\phi_n\|_\infty = \mathcal{O}(1)$ for all n). We seek explicit expressions for $\Theta^{\pm}$. In [21] it was shown for the case $q=1$ that $\Theta^{\pm}$ can be written in terms of a particular special function, the Lerch transcendental function $\Phi(z,s,a)$ [23], defined by

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} s > 1, \quad |s| \leq 1. \tag{5.4}$$

As we now demonstrate, Lerch functions are also used to express $\Theta^{\pm}$ for arbitrary $q \geq 1$:

Lemma 15. *The function $\Theta^{\pm}(r,N;x)$ satisfies*

$$\Theta^{\pm}(r,N;x) = \frac{2^r(\pm 1)^N}{\pi^r c^2} \sum_{s=0}^{q-1} c_s \Big[\mathrm{e}^{\lambda_s \alpha_N (x-1)} \Phi\left(\pm \mathrm{e}^{\frac{1}{2}\lambda_s (x-1)\pi}, r, \tfrac{1}{2}(2N+q-1)\right) \\ + (-1)^q \mathrm{e}^{-\lambda_s \alpha_N (x+1)} \Phi\left(\mp \mathrm{e}^{-\frac{1}{2}\lambda_s (x+1)\pi}, r, \tfrac{1}{2}(2N+q-1)\right) \Big],$$

where Φ is the Lerch transcendental function (5.4).

Proof. Consider the sum $\sum_{n \geq N} \frac{\mathrm{e}^{\lambda \alpha_n}}{\alpha_n^r}$. Using the asymptotic expressions for α_n, we have

$$\sum_{n\geq N} \frac{\mathrm{e}^{\lambda\alpha_n}}{\alpha_n^r} = \sum_{n \geq N} \frac{\mathrm{e}^{\lambda \frac{1}{4}(2n+q-1)\pi}}{[\frac{1}{4}(2n+q-1)\pi]^r} \\ = \frac{\mathrm{e}^{\lambda\alpha_N}}{(\frac{\pi}{2})^r} \sum_{m=0}^{\infty} \frac{(\mathrm{e}^{\frac{1}{2}\lambda\pi})^m}{[m+\frac{1}{2}(2N+q-1)]^r} = \left(\tfrac{2}{\pi}\right)^r \mathrm{e}^{\lambda\alpha_N} \Phi\left(\mathrm{e}^{\frac{1}{2}\lambda\pi}, r, \tfrac{1}{2}(2N+q-1)\right). \tag{5.5}$$

Next, consider the sum $\sum_{n\geq N}(-1)^n \frac{\mathrm{e}^{\lambda\alpha_n}}{\alpha_n^r}$. In an identical manner, we derive

$$\sum_{n\geq N} (-1)^n \frac{\mathrm{e}^{\lambda\alpha_n}}{\alpha_n^r} = \left(\tfrac{2}{\pi}\right)^r (-1)^N \mathrm{e}^{\lambda\alpha_N} \Phi\left(-\mathrm{e}^{\frac{1}{2}\lambda\pi}, r, \tfrac{1}{2}(2N+q-1)\right).$$

We conclude that

$$\sum_{n\geq N} (\pm 1)^n \frac{\mathrm{e}^{\lambda\alpha_n}}{\alpha_n^r} = \left(\tfrac{2}{\pi}\right)^r (\pm 1)^N \mathrm{e}^{\lambda\alpha_N} \Phi\left(\pm\mathrm{e}^{\frac{1}{2}\lambda\pi}, r, \tfrac{1}{2}(2N+q-1)\right). \tag{5.6}$$

With this to hand, we replace ϕ_n by (5.2) in (5.3), giving

$$\Theta^{\pm}(r,N;x) = \frac{1}{c^2} \sum_{s=0}^{q-1} c_s \left[\sum_{n\geq N} (\pm 1)^n \frac{\mathrm{e}^{\lambda_s \alpha_n (x-1)}}{\alpha_n^r} + (-1)^{q+1} \sum_{n \geq N} (\mp 1)^n \frac{\mathrm{e}^{-\lambda_s\alpha_n(x+1)}}{\alpha_n^r} \right] \\ = c^{-2} \left(\frac{2}{\pi}\right)^r \sum_{s=0}^{q-1} c_s \Big[(\pm 1)^N \mathrm{e}^{\lambda_s\alpha_N(x-1)} \Phi\left(\pm \mathrm{e}^{\frac{1}{2}\lambda(x-1)\pi}, r, \tfrac{1}{2}(2N+q-1)\right) \\ + (-1)^{q+1}(\mp 1)^N \mathrm{e}^{-\lambda_s\alpha_N(x+1)} \Phi\left(\mp \mathrm{e}^{-\frac{1}{2}\lambda_s(x+1)\pi}, r, \tfrac{1}{2}(2N+q-1)\right) \Big],$$

as required. □

The functions $\Theta^{\pm}$ appear explicitly in the asymptotic expansion of $f(x) - f_N(x)$. To derive such result we first recall the expression (4.10) for the coefficient $\hat{f}_n$. Setting $p = q$ and iterating we arrive at (see also [6])

$$\hat{f}_n \sim \sum_{r=0}^{\infty} \sum_{s=0}^{q-1} \frac{(-1)^{rq+s}}{\alpha_n^{2(r+1)q}} \left[f^{((2r+1)q+s)}(1) \phi_n^{(q-s-1)}(1) - f^{((2r+1)q+s)}(-1) \phi_n^{(q-s-1)}(-1) \right].$$

Since $\alpha_n = \mathcal{O}(n)$ and $\phi_n^{(r)} = \mathcal{O}(n^r)$, this is an asymptotic expansion (in the Poincaré sense) for the coefficient $\hat{f}_n$ in inverse powers of n. Moreover, recalling that $\phi_n^{(r)}(\pm 1) \sim (\pm 1)^{r+n+q+1} d_r \alpha_n^r$ (see Lemma 6), we have

$$\hat{f}_n \sim \sum_{r=0}^{\infty} \sum_{s=0}^{q-1} \frac{(-1)^{rq+s} d_{q-s-1}}{\alpha_n^{(2r+1)q+s+1}} \left[f^+_{(2r+1)q+s} + (-1)^{n+s+1} f^-_{(2r+1)q+s} \right], \tag{5.7}$$

where $f^\pm_r = f^{(r)}(\pm 1)$. With (5.7) in hand, we now obtain the main result of this section:

Theorem 8. *For large N, the error $f(x) - f_N(x)$ has the following asymptotic expansion*

$$f(x) - f_N(x) \sim \sum_{r=0}^{\infty} \sum_{s=0}^{q-1} (-1)^{rq+s} d_{q-s-1} \bigg[f^+_{(2r+1)q+s} \Theta^+((2r+1)q+s+1, N; x) \\ + (-1)^{s+1} f^-_{(2r+1)q+s} \Theta^-((2r+1)q+s+1, N; x) \bigg]. \tag{5.8}$$

Proof. We may write $f(x) - f_N(x) = \sum_{n \geq N} \|\phi_n\|^{-2} \hat{f}_n \phi_n(x)$. Substituting the asymptotic expansion (5.7) and replacing the various infinite sums with $\Theta^\pm$ now yields the result. □

Note that it is not clear *a priori* that (5.8) is an asymptotic expansion for $f(x) - f_N(x)$ in the usual Poincaré sense. However, this turns out to be the case, since the functions $\Theta^\pm(r, N; x)$ satisfy $\Theta^\pm(r, N; x) = \mathcal{O}(N^{-r})$ for $-1 < x < 1$ and $\Theta^\pm(r, N; x) = \mathcal{O}(N^{1-r})$ when $x = \pm 1$. In fact, not only can we derive such estimates, we may also exactly determine the leading order asymptotic behaviour of the functions $\Theta^\pm(r, N; \cdot)$ in these cases:

Lemma 16. *The function $\Theta^\pm(r, N; x)$ satisfies*

$$\Theta^\pm(r, N; x) = c_0 c^{-2} (\pm 1)^N \left[\frac{e^{-i\alpha_N x}}{1 \mp i e^{-\frac{1}{2} i\pi x}} + (-1)^{q+1} \frac{e^{i\alpha_N x}}{1 \pm i e^{\frac{1}{2} i\pi x}} \right] e^{i\alpha_N} \alpha_N^{-r} + \mathcal{O}\left(N^{-r-1}\right),$$

uniformly for x in compact subsets of $(-1, 1)$. In particular, $\Theta^\pm(r, N; x) = \mathcal{O}(N^{-r})$.

Proof. For $x \in (-1, 1)$ we have $\operatorname{Re} \lambda_s(x-1) < 0$ and $\operatorname{Re} \lambda_s(x+1) > 0$, $s = 1, \ldots, q-1$. Hence, up to exponentially small terms,

$$\Theta^\pm(r, N; x) = \frac{2^r (\pm 1)^N c_0}{\pi^r c^2} \bigg[e^{-i\alpha_N(x-1)} \Phi\left(\pm e^{-i\frac{1}{2}(x-1)\pi}, r, \tfrac{1}{2}(2N+q-1)\right) \\ + (-1)^{q+1} e^{i\alpha_N(x+1)} \Phi\left(\mp e^{i\frac{1}{2}(x+1)\pi}, r, \tfrac{1}{2}(2N+q-1)\right) \bigg].$$

In [21] an asymptotic expansion for the Lerch function $\Phi(-e^{i\pi z}, r, M)$ was derived. In particular,

$$\Phi(-e^{i\pi z}, r, M) = M^{-r} \left(1 + e^{i\pi z}\right)^{-1} + \mathcal{O}\left(M^{-(r+1)}\right), \quad M \to \infty, \quad -1 < x < 1.$$

We now consider the four Lerch functions appearing in the previous expression. Setting $M = \frac{1}{2}(2N+q-1)$, we have

$$\begin{aligned} \Phi\left(e^{-i\frac{1}{2}(x-1)\pi}, r, \tfrac{1}{2}(2N+q-1)\right) &= \Phi\left(-e^{-i\frac{1}{2}(x+1)\pi}, r, \tfrac{1}{2}(2N+q-1)\right) \\ &= M^{-r}\left(1 - i e^{-\frac{1}{2} i\pi x}\right)^{-1} + \mathcal{O}\left(M^{-(r+1)}\right). \end{aligned}$$

Similarly

$$\Phi\left(-e^{i\frac{1}{2}(x+1)\pi}, r, \tfrac{1}{2}(2N+q-1)\right) = M^{-r}\left(1 + i e^{\frac{1}{2} i\pi x}\right)^{-1} + \mathcal{O}\left(M^{-(r+1)}\right).$$

Hence

$$\Theta^+(r, N; x) = c_0 c^{-2} \left[\frac{e^{-i\alpha_N x}}{1 - i e^{-\frac{1}{2} i\pi x}} + (-1)^{q+1} \frac{e^{i\alpha_N x}}{1 + i e^{\frac{1}{2} i\pi x}} \right] e^{i\alpha_N} \alpha_N^{-r} + \mathcal{O}\left(N^{-r-1}\right).$$

In a similar manner, we find an expression for $\Theta^-(r, N; x)$, giving the result. □

It remains to determine the behaviour of $\Theta^{\pm}(r,N;x)$ when $x=\pm 1$. For this, we have

Lemma 17. *The functions* $\Theta^{\pm}(r,N;x)$ *satisfies* $\Theta^{\pm}(r,N;\mp 1)=\mathcal{O}\left(N^{-r}\right)$ *and*

$$\Theta^{\pm}(r,N;\pm 1)=\frac{2(\pm 1)^{q+1}d_0}{c^2\pi(r-1)}\alpha_N^{1-r}+\mathcal{O}\left(N^{-r}\right).$$

Proof. By the definition of $\Theta^{\pm}$, we have

$$\Theta^{\pm}(r,N;\mp 1)=\frac{1}{c^2}\sum_{n\geq N}\frac{(\pm 1)^n}{\alpha_n^r}\phi_n(\mp 1).$$

Since $\phi(\mp 1)=(\mp 1)^{n+q+1}d_0$ by Lemma 6, it follows that

$$\Theta^{\pm}(r,N;\mp 1)=\frac{d_0(\mp 1)^{q+1}}{c^2}\sum_{n\geq N}\frac{(-1)^n}{\alpha_n^r}=\frac{d_0 2^r(\mp)^{q+1}(-1)^N}{c^2\pi^r}\Phi\left(-1,r,\tfrac{1}{2}(2N+q-1)\right),$$

and this is $\mathcal{O}\left(N^{-r}\right)$. Now consider $\Theta^{\pm}(r,N;\pm 1)$. By identical arguments

$$\Theta^{\pm}(r,N;\pm 1)=\frac{2^r(\pm 1)^{q+1}d_0}{c^2\pi^r}\sum_{m=0}^{\infty}\frac{1}{[m+\frac{1}{2}(2N+q-1)]^r}.$$

The right hand size is precisely $\zeta(r,\frac{1}{2}(2N+q-1))$, where ζ is the Hurwitz zeta function [1]. The result now follows immediately, since $\zeta(r,M)\sim\frac{1}{r-1}M^{1-r}$ for large M. □

As shown in [21], it is also possible to provide a full asymptotic expansion for the Lerch function Φ. Hence, we could have given a complete asymptotic expansion for $\Theta^{\pm}$ in inverse powers of N. However, our interest lies primarily with the leading order behaviour of $\Theta^{\pm}$, and in turn the error $f(x)-f_N(x)$, for which we have the following theorem:

Theorem 9. *The error* $f(x)-f_N(x)$ *satisfies*

$$f(x)-f_N(x)=\frac{d_{q-1}c_0\mathrm{e}^{\mathrm{i}\alpha_N}}{c^2\alpha_N^{q+1}}\left[f_q^{+}+(-1)^{N+q}f_q^{-}\right]\left[(-1)^{q+1}G^{+}(N;x)+G^{-}(N;x)\right]+\mathcal{O}\left(N^{-q-1}\right),$$

uniformly for x *in compact subsets of* $(-1,1)$*, where* $G^{\pm}(N;x)=\mathrm{e}^{\pm\mathrm{i}\alpha_N x}(1\pm\mathrm{i}\mathrm{e}^{\frac{1}{2}\mathrm{i}\pi x})^{-1}$*. In particular,* $f(x)-f_N(x)=\mathcal{O}\left(N^{-q-1}\right)$ *for* $-1<x<1$*. Moreover,*

$$f(\pm 1)-f_N(\pm 1)=\frac{2d_{q-1}d_0(\pm 1)^q}{c^2\pi q}\alpha_N^{-q}+\mathcal{O}\left(N^{-q-1}\right)=\mathcal{O}\left(N^{-q}\right).$$

Proof. We combine Lemmas 16 and 17 with (5.8). □

This theorem is verified in Figure 5. In particular, the oscillations (at a frequency of $\mathcal{O}(N)$) present in the diagrams are due to the $\mathrm{e}^{\pm\mathrm{i}\alpha_N x}$ terms appearing in the functions $G^{\pm}$. Moreover, the envelope curve, which grows large as $|x|\to 1$, is explained by the denominators $1\pm\mathrm{i}\mathrm{e}^{\frac{1}{2}\mathrm{i}\pi x}$.

6 Derivative conditions and higher-order convergence

Closer inspection of the asymptotic expansion (5.8) reveals that the rate of convergence of the approximation f_N is completely determined by the values of certain derivatives of the function f evaluated at $x=\pm 1$. As proved, for arbitrary functions with no vanishing derivatives, the uniform error is $\mathcal{O}\left(N^{-q}\right)$. However, whenever a finite number of such derivatives are zero, we can expect faster convergence of the approximation.

To properly detail this effect, we define the finite set $D_m\subseteq\mathbb{N}$ by

$$D_m=\left\{l\in\mathbb{N}:l=(2r+1)q+s<m,\ r\in\mathbb{N},\ s=0,\ldots,q-1\right\},\quad m\in\mathbb{N},\tag{6.1}$$

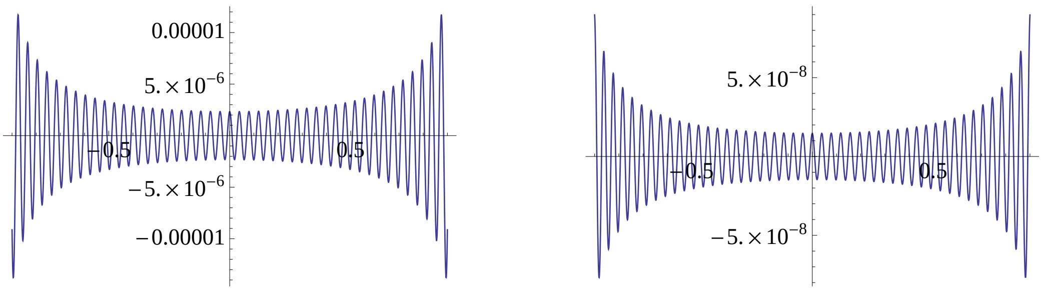

Figure 5: Pointwise error $f(x) - f_{50}(x)$ for $|x| \leq \frac{9}{10}$ with $q = 2$ (left), $q = 3$ (right) and $f(x) = x^2 \cos 2x$.

and, for $p = 0, \ldots, q-1$ and $k \in \mathbb{N}$ we let

$$\rho_{k,0} = 2kq, \quad \rho_{k,p} = (2k+1)q + p, \quad p = 1, \ldots, q-1. \tag{6.2}$$

Note that the derivative $f^{(l)}(\pm 1)$ appears in (5.8) if and only if $l \in D_{\rho_{k,p}}$ for some k, p. For this reason, we say that a function f obeys *the first* $\rho_{k,p}$ *derivative conditions* if $f^{(l)}(\pm 1) = 0$, $\forall l \in D_{\rho_{k,p}}$.

For example, when $q = 1$ this condition is equivalent to $f^{(2r+1)}(\pm 1) = 0$, $r = 0, \ldots, k-1$. The properties of modified Fourier expansions of functions obeying such derivative conditions have been detailed in [3, 5].

Returning to the general case, we have

Theorem 10. *Suppose that f obeys the first $\rho_{k,p}$ derivative conditions. If $p \neq 0$, then the error $\|f - f_N\|_\infty = \mathcal{O}\left(N^{-\rho_{k,p}}\right)$ and $f(x) - f_N(x) = \mathcal{O}\left(N^{-\rho_{k,p}-1}\right)$ uniformly for x in compact subsets of $(-1, 1)$. For $p = 0$ these values are $\mathcal{O}\left(N^{-(2k+1)q}\right)$ and $\mathcal{O}\left(N^{-(2k+1)q-1}\right)$ respectively.*

Proof. This follows immediately after substituting the derivative conditions into the expression (5.8) and using the estimates of Lemmas 16 and 17 for the functions $\Theta^\pm$. □

This theorem demonstrates the effect of derivative conditions on the convergence rate of polyharmonic–Neumann expansions. For example, when $q = 1$ a function obeying the first $2k = \rho_{k,0}$ conditions has an $\mathcal{O}\left(N^{-2k-1}\right)$ uniform error; a result which is also found in [3, 21].

Throughout this and the previous section we have assumed that the approximated function is smooth. This is not necessary, and results could have been also derived under lower smoothness assumptions. Naturally, derivative conditions only makes sense for functions of sufficient smoothness. However, as the following theorem attests, whenever this is the case they also endow the approximation f_N with a higher degree of convergence:

Theorem 11. *Suppose that f obeys the first $\rho_{k,p}$ derivative conditions and that $f \in \mathrm{H}^{\rho_{k,p}}(-1, 1)$ for $p \neq 0$ or $f \in \mathrm{H}^{2kq+l}(-1, 1)$ when $p = 0$, where $l = 0, \ldots, q$. Then, the approximation f_N converges to f in the H^r norm for $r = 0, \ldots, \rho_{k,p}$ or $r = 0, \ldots, 2kq + l$ respectively.*

For the sake of brevity, we omit the proof of this result, which follows along similar lines to that of Theorem 5, making necessary adjustments for the particular derivative conditions.

7 Conclusions

The aim of the paper was to study expansions in polyharmonic eigenfunctions equipped with homogeneous Neumann boundary conditions. First, we have obtained exponential asymptotics for both the eigenvalues and eigenfunctions, and, using these results, determined a full asymptotic expansion for the error in approximating a smooth function by its truncated expansion. In doing so, we have resolved several conjectures raised in [6]. Moreover, we have detailed how such asymptotic estimates can be used to efficiently construct the truncated expansion.

The main drawback of polyharmonic–Neumann expansions is that, though it is theoretically possible to obtain arbitrarily high orders of convergence, as q increases so does the computational cost in forming the approximation f_N. Therefore, it seems inadvisable to use values of q much greater than $q = 4$. Nevertheless, as mentioned in Section 1, slowly convergent modified Fourier expansions have been found to offer a number of advantage over more rapidly convergent methods in a number of applications. Polyharmonic–Neumann expansions may also possess benefits in these regards, and this remains a question for future research.

If rapid convergence were required, however, it may be better to use a small value of q in combination with a technique to accelerate convergence. Since the factors that control convergence are now well-understood (see Section 6), such a device can be developed. In [12] and [4], the convergence acceleration of modified Fourier expansions has been addressed. Techniques presented therein could also be extended to this setting.

Modified Fourier expansions were generalised in [16] to d-variate cubes, and their convergence studied in [5]. This presents an obvious extension of polyharmonic–Neumann expansions. However, care must be taken. Polyharmonic eigenfunctions in cubes cannot be expressed in terms of simple functions, and thus are of little use in practical computations. However, it can be shown that the eigenfunctions of the *subpolyharmonic* operator $\mathcal{L} = (-1)^q \left(\partial_{x_1}^{2q} + \ldots + \partial_{x_d}^{2q}\right)$ arise precisely as Cartesian products of the univariate polyharmonic eigenfunctions studied in this paper. Hence, this provides a potential means to generalise such expansions to higher dimensions.

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