First-order trace formulas for the iterates of the Fox–Li operator

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To the memory of Israel Gohberg, one of the pioneers of present–day Wiener–Hopf theory

Abstract. The paper is devoted to first-order trace formulas for the iterates of the Fox–Li and related Wiener–Hopf integral operators. Such formulas provide first insight into the asymptotic behaviour of the eigenvalues and can be used to test whether a specific guess for the eigenvalue distribution is acceptable or not. The main technical problem consists in obtaining the asymptotics of a multivariate oscillatory integral whose stationary points constitute a line.

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1. Introduction and main results

The Fox–Li operator is the integral operator on $L^2(-1,1)$ given by

$$(F_\omega f)(x) := \sqrt{\frac{\omega}{\pi \mathrm{i}}} \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega(x-y)^2} f(y)\, \mathrm{d}y, \quad x \in (-1,1),$$

where $\omega > 0$ is a large parameter. Here and in the following, $\sqrt{\mathrm{i}}$ stands for $\mathrm{e}^{\mathrm{i}\pi/4}$. The spectrum of this operator is of great importance in laser engineering [12], [13], [15], [18], [19]. Physical aspects of the Fox–Li spectrum are also studied in the recent papers [2], [3], [4]. A very recent paper devoted to numerical methods for the approximation of the spectrum is [11]. These works indicate that the spectrum of F_ω is composed of points on a spiral commencing at 1 and rotating clockwise to the origin.

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However, rigorous results are still very sparse. Landau and Widom [16], [20] established a second–order result for the asymptotic distribution of the singular values of F_ω, that is, for the square roots of the eigenvalues of $F_\omega F_\omega^*$. In [7], the regularized operator given by

$$(F_{\omega,\varepsilon} f)(x) := \sqrt{\frac{\omega}{\pi \mathrm{i}}} \int_{-1}^{1} \mathrm{e}^{(\mathrm{i}-\varepsilon)\omega(x-y)^2} f(y)\,\mathrm{d}y, \quad x \in (-1,1),$$

was considered, and it was proved that, for each fixed $\varepsilon > 0$, the eigenvalues of $F_{\omega,\varepsilon}$ converge to the logarithmic spiral

$$\left\{ \frac{1}{\sqrt{1+\varepsilon \mathrm{i}}} \exp\left(-\frac{\varepsilon x^2}{4(1+\varepsilon^2)} - \mathrm{i}\frac{x^2}{4(1+\varepsilon^2)} \right) : x \in (0,\infty) \right\}$$

in the Hausdorff metric as $\omega \to \infty$.

We here prove the following first–order trace formula for the eigenvalues of the Fox–Li operator F_ω itself.

Theorem 1.1. *The operator F_ω is a trace class operator, all eigenvalues are contained in the open unit disk, and, for each fixed natural number $k \geq 1$,*

$$\operatorname{tr} F_\omega^k = \frac{2\sqrt{\omega}}{\sqrt{\pi \mathrm{i} k}} + o(\sqrt{\omega}) \quad \textit{as} \quad \omega \to \infty. \tag{1}$$

We do not know a rigorous argument that shows that F_ω has infinitely many eigenvalues. However, Theorem 1.1 for $k = 1$ implies the following. Since it tells us that F_ω is of trace class, we may compute the trace by integrating the kernel along the diagonal [14, Corollary III.10.2]. On the other hand, the trace is the sum of the eigenvalues $\lambda_1, \lambda_2, \ldots$ of F_ω, repeated according to algebraic multiplicity. Consequently,

$$\sum_n \lambda_n = \operatorname{tr} F_\omega = \sqrt{\frac{\omega}{\pi \mathrm{i}}} \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega \cdot 0^2}\,\mathrm{d}x = \frac{2\sqrt{\omega}}{\sqrt{\pi \mathrm{i}}},$$

and since $|\lambda_n| < 1$ for all n, it follows that the number N of eigenvalues satisfies

$$\frac{2\sqrt{\omega}}{\sqrt{\pi}} = \left| \sum_{n=1}^{N} \lambda_n \right| \leq \sum_{n=1}^{N} |\lambda_n| < N,$$

which leastwise reveals that F_ω has at least $2\sqrt{\omega/\pi}$ eigenvalues.

Given a family of functions $b_\omega : (0,\infty) \to \mathbb{C}$, we say that the eigenvalues of F_ω are asymptotically distributed as the values of b_ω in the weak sense if, for each natural number $k \geq 1$,

$$\operatorname{tr} F_\omega^k = \int_0^\infty b_\omega^k(x)\,\mathrm{d}x + o(\sqrt{\omega}) \quad \text{as} \quad \omega \to \infty. \tag{2}$$

Of course, saying so is motivated by the formulae

$$\operatorname{tr} F_\omega^k = \sum_n \lambda_n^k, \qquad \int_0^\infty b_\omega^k(x)\,\mathrm{d}x \approx \sum_n b_\omega^k(n).$$

We remark that equal asymptotic distribution in the strong sense would mean something like

$$\operatorname{tr}\varphi(F_\omega) = \int_0^\infty \varphi(b_\omega(x))\,\mathrm{d}x + o(\sqrt{\omega}) \quad \text{as} \quad \omega \to \infty$$

for every function $\varphi \in C^\infty(\mathbb{C})$ satisfying $\varphi(0) = 0$ or for every φ of the form $\varphi = \chi_\Omega$, where $\Omega \subset \mathbb{C} \setminus \{0\}$ is a nice set and χ_Ω stands for the characteristic function of Ω. In the latter case, it would follow that

$$\#\{n : \lambda_n \in \Omega\} = |\{x \in (0,\infty) : b(x) \in \Omega\}| + o(\sqrt{\omega}) \quad \text{as} \quad \omega \to \infty,$$

where $\#S$ and $|S|$ denote the cardinality and the Lebesgue measure of S, respectively. By Theorem 1.1, formula (2) is equivalent to saying that

$$\int_0^\infty b_\omega^k(x)\,\mathrm{d}x = \frac{2\sqrt{\omega}}{\sqrt{\pi \mathrm{i} k}} + o(\sqrt{\omega}) \quad \text{as} \quad \omega \to \infty. \tag{3}$$

Using solely (3) we will show the following.

Theorem 1.2. *Let $b_\omega(x) = \exp(-\alpha(\omega)x^\nu - \mathrm{i}\beta(\omega)x^\nu)$ with positive real numbers $\alpha(\omega)$, $\beta(\omega)$, ν. Then the eigenvalues of F_ω are asymptotically distributed as the values of b_ω in the weak sense if and only if*

$$\nu = 2, \quad \alpha(\omega) = o\left(\frac{1}{\omega}\right), \quad \beta(\omega) = \frac{\pi^2}{16\omega} + o\left(\frac{1}{\omega}\right).$$

This result may be viewed as a first step toward establishing with mathematical rigour Vainshtein's formula

$$\nu = 2, \quad \alpha(\omega) \approx \frac{\zeta(1/2)\pi^{3/2}}{16\sqrt{2}\,\omega^{3/2}}, \quad \beta(\omega) \approx \frac{\pi^2}{16\omega},$$

which, to quote Cochran and Hinds [12], was obtained by Vainshtein [19] "using a distinctly physical approach, based upon wave-guide theory." Here $\zeta(1/2)$ is Riemann's zeta function at the point $1/2$.

The Fox–Li operator F_ω is easily seen to be unitarily similar to the operator on $L^2(0, 2\sqrt{\omega})$ that acts by the rule

$$(Wf)(x) := \frac{1}{\sqrt{\pi \mathrm{i}}} \int_0^{2\sqrt{\omega}} \mathrm{e}^{\mathrm{i}(x-y)^2} f(y)\,\mathrm{d}y, \quad x \in (0, 2\sqrt{\omega}). \tag{4}$$

In this way we are entering the realm of Wiener–Hopf operators. Given a function $a \in L^\infty(\mathbb{R})$, the so-called symbol, the convolution operator $C(a)$ on $L^2(\mathbb{R})$ is defined by

$$(C(a)f)(x) := \frac{1}{2\pi} \int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}\xi x} a(\xi) \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}\xi y} f(y)\,\mathrm{d}y\,\mathrm{d}\xi, \quad \xi \in \mathbb{R}.$$

Thus, $C(a)$ takes the Fourier transform, multiplies the result by a, and then applies the inverse Fourier transform. The boundedness of a guarantees (and is in fact equivalent to) the boundedness of $C(a)$. The Wiener–Hopf operator $W(a)$ is the compression of $C(a)$ to $L^2(0,\infty)$, that is, $W(a) = PC(a)|L^2(0,\infty)$, where $(Pf)(x)$

is zero for $x < 0$ and $f(x)$ for $x > 0$. Finally, for a real number $\tau > 0$, the truncated Wiener–Hopf operator $W_\tau(a)$ is the compression of $W(a)$ to $L^2(0,\tau)$, i.e., $W_\tau(a) = P_\tau W(a)|L^2(0,\tau)$, where $(P_\tau f)(x) = f(x)$ for $0 < x < \tau$ and $(P_\tau f)(x) = 0$ for $x > \tau$. If

$$a(\xi) = \hat{\ell}(\xi) := \int_{-\infty}^{\infty} \ell(t) \mathrm{e}^{\mathrm{i}\xi t}\, \mathrm{d}t, \quad t \in \mathbb{R},$$

the Fourier transform being understood in the usual sense for $\ell \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$ and in the sense of distributions in more general situations, the convolution $C(a)$ can be written as

$$(C(a)f)(x) = \int_{-\infty}^{\infty} \ell(x-y) f(y)\, \mathrm{d}y, \quad x \in \mathbb{R},$$

while the operators $W(a)$ and $W_\tau(a)$ are given by the same formula with integration over $(-\infty,\infty)$ replaced by integration over $(0,\infty)$ and $(0,\tau)$, respectively. Because

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t^2} \mathrm{e}^{\mathrm{i}\xi t}\, \mathrm{d}t = \mathrm{e}^{-\mathrm{i}\xi^2/4}, \quad \xi \in \mathbb{R}$$

in the sense of distributions, we may identify the operator (3) as $W_{2\sqrt{\omega}}(\sigma)$ with $\sigma(\xi) := \mathrm{e}^{-\mathrm{i}\xi^2/4}$. We remark that $\sigma(\xi)$ has oscillating discontinuities as $\xi \to \pm\infty$ and that this function does not belong to the classes of symbols with a well developed theory of their Wiener–Hopf operators, such as $C(\dot{\mathbb{R}}) + H^\infty(\mathbb{R})$, $PC(\overline{\mathbb{R}})$, $SO(\mathbb{R})$, $SAP(\mathbb{R})$; see [8] and [10]. In terms of Wiener–Hopf operators, Theorem 1.1 becomes formula (5) in the following result.

Theorem 1.3. *Let $\sigma(\xi) := \mathrm{e}^{-\mathrm{i}\xi^2/4}$. The spectra of the operators $C(\sigma)$ and $W(\sigma)$ are the unit circle $\mathbb{T}$ and the closed unit disc $\overline{\mathbb{D}}$, respectively. The spectrum of $W_\tau(\sigma)$ is contained in the open unit disc $\mathbb{D}$, and for every natural number $k \geq 1$, the operators $W_\tau^k(\sigma) := [W_\tau(\sigma)]^k$ and $W_\tau(\sigma^k)$ are trace class operators and*

$$\operatorname{tr} W_\tau^k(\sigma) = \operatorname{tr} W_\tau(\sigma^k) + o(\tau) \quad \text{as} \quad \tau \to \infty. \tag{5}$$

Denoting by $\ell_k(t)$ the kernel of the convolution integral operator $W_\tau(\sigma^k)$, we have

$$\begin{aligned} \operatorname{tr} W_\tau(\sigma^k) &= \int_0^\tau \ell_k(x-x)\, \mathrm{d}x = \tau \ell_k(0) = \frac{\tau}{2\pi} \int_{-\infty}^{\infty} \sigma^k(\xi)\, \mathrm{d}\xi \\ &= \frac{\tau}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}k\xi^2/4}\, \mathrm{d}\xi = \frac{\tau}{2\pi} \sqrt{\frac{4\pi}{\mathrm{i}k}} = \frac{\tau}{\sqrt{\pi \mathrm{i}k}}, \end{aligned}$$

and taking into account that F_ω is unitarily similar to $W_{2\sqrt{\omega}}(\sigma)$, we see that (5) is indeed the same as (1).

The discrete analogues of Wiener–Hopf operators are Toeplitz matrices. Given $a \in L^\infty(\mathbb{T})$, the $n \times n$ Toeplitz matrix $T_n(a)$ is the matrix $(a_{j-k})_{j,k=1}^n$ where a_j is the jth Fourier coefficient of a,

$$a_j := \frac{1}{2\pi} \int_0^{2\pi} a(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{e}^{-\mathrm{i}j\theta}\, \mathrm{d}\theta, \quad j \in \mathbb{Z}.$$

It is well known and not difficult to prove (see, e.g., [9, Lemma 5.16 and Theorem 5.17]) that if a is an arbitrary function in $L^\infty(\mathbb{T})$, then

$$\operatorname{tr} T_n^k(a) = \operatorname{tr} T_n(a^k) + o(n) = (a^k)_0 + o(n) \quad \text{as} \quad n \to \infty, \tag{6}$$

which is the discrete counterpart of (5). A finite Toeplitz matrix is automatically a trace class operator, but a truncated Wiener–Hopf operator need not be of trace class. Therefore the continuous analogue of (6) does not make sense for arbitrary $a \in L^\infty(\mathbb{R})$. What is known is the following, and we will include a proof for the reader's convenience.

Theorem 1.4. *If $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, then the operators $W_\tau^k(a)$ and $W_\tau(a^k)$ are of trace class for every natural number $k \geq 1$ and every real number $\tau > 0$, and*

$$\operatorname{tr} W_\tau^k(a) = \operatorname{tr} W_\tau(a^k) + o(\tau) = \frac{\tau}{2\pi} \int_{-\infty}^{\infty} a^k(\xi)\, \mathrm{d}\xi + o(\tau) \quad \textit{as} \quad \tau \to \infty. \tag{7}$$

The function $\sigma(\xi) = \mathrm{e}^{-\mathrm{i}\xi^2/4}$ in Theorem 1.3 is not in $L^1(\mathbb{R})$ and hence Theorem 1.3 cannot be deduced from Theorem 1.4. The actual value of Theorems 1.1 and 1.3 is that they show that (7) nevertheless remains true for $a(\xi) = \sigma(\xi) = \mathrm{e}^{-\mathrm{i}\xi^2/4}$.

The following theorem unites (5) and (7).

Theorem 1.5. *Let $a(\xi) = c(\xi)\sigma(\xi)$ where $\sigma(\xi) := \mathrm{e}^{-\mathrm{i}\xi^2/4}$ and $c \in C^3(\mathbb{R})$ is a function having finite limits $c(-\infty) = c(+\infty) =: c(\infty)$. Set $u(\xi) := c(\xi) - c(\infty)$ and suppose that the functions $\xi^4 u(\xi)$, $\xi^3 u'(\xi)$, $\xi^2 u''(\xi)$, $u'''(\xi)$ belong to $L^1(\mathbb{R})$ and have zero limits as $\xi \to \pm\infty$. Then, for every natural number $k \geq 1$ and every real number $\tau > 0$, the operators $W_\tau^k(a)$ and $W_\tau(a^k)$ are of trace class and (7) holds.*

The remaining sections of the paper are devoted to the proofs of the theorems. In Section 2, we prove Theorem 1.4 and the portion of Theorem 1.3 concerning spectra. Proposition 2.4 addresses the pseudospectra of F_ω and shows that, for each $\varepsilon > 0$, the ε-pseudospectrum of F_ω contains the closed unit disk $\overline{\mathbb{D}}$ whenever ω is sufficiently large. Theorem 1.1 and the (equivalent) trace formula of Theorem 1.3 are proved in Section 3 by determining the first-order asymptotics of the oscillatory multivariate integral $\int_{-1}^{1} m_k(x,x)\, \mathrm{d}x$ where $m_k(x,y)$ is the kernel of the integral operator F_ω^k; note that $m_k(x,y)$ is a $(k-1)$-fold integral. Sections 4 and 5 contain the proofs of Theorems 1.2 and 1.5, respectively.

2. Wiener–Hopf operators

We begin with the proof of Theorem 1.4. Let $\mathcal{C}_p$ denote the pth Schatten–von Neumann class and $\|\cdot\|_p$ the norm in $\mathcal{C}_p$, that is, the ℓ^p norm of the singular values of the operator. In particular, $\|\cdot\|_1$ is the trace norm, $\|\cdot\|_2$ is the Hilbert–Schmidt norm (= Frobenius norm), and $\|\cdot\|_\infty$ coincides with the usual operator norm on L^2. It is well known that $W_\tau(a) \in \mathcal{C}_1$ whenever $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$; see, e.g., [10, Section 10.83]. This implies that $W_\tau^k(a)$ and $W_\tau(a^k)$ are also in $\mathcal{C}_1$ for $k \ge 1$.

Lemma 2.1. *If $b, c \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, then*

$$\|W_\tau(b)W_\tau(c) - W_\tau(bc)\|_1 = o(\tau) \quad \text{as} \quad \tau \to \infty.$$

Proof. If $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, then $a \in L^2(\mathbb{R})$ and $a = \hat{\ell}$ with $\ell \in C(\dot{\mathbb{R}}) \cap L^2(\mathbb{R})$, where $\dot{\mathbb{R}}$ is the one-point compactification of $\mathbb{R}$. We denote by $H(a)$ the Hankel operator generated by a. This is the operator that acts on $L^2(0,\infty)$ by the rule

$$(H(a)f)(x) := \int_0^\infty \ell(x+y) f(y)\,\mathrm{d}y, \quad x \in (0,\infty).$$

Letting $\widetilde{a}(\xi) := a(-\xi)$, we have

$$(H(\widetilde{a})f)(x) = \int_0^\infty \ell(-x-y) f(y)\,\mathrm{d}y, \quad x \in (0,\infty).$$

A formula by Widom says that

$$W_\tau(bc) - W_\tau(b)W_\tau(c) = P_\tau H(b) H(\widetilde{c}) P_\tau + R_\tau H(\widetilde{b}) H(c) R_\tau, \tag{8}$$

where P_τ is as in Section 1 and $R_\tau : L^2(0,\infty) \to L^2(0,\tau)$ is the operator that is given by $(R_\tau f)(x) := f(\tau - x)$ for $0 < x < \tau$ and $(R_\tau f)(x) := 0$ for $x > \tau$; see, for example, [10, Section 9.7(d)]. Since $\|BC\|_1 \le \|B\|_2 \|C\|_2$, it suffices to prove that

$$\frac{\|P_\tau H(a)\|_2^2}{\tau} \to 0 \quad \text{as} \quad \tau \to \infty$$

for $a = \hat{\ell} \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. We have

$$\begin{aligned} \frac{\|P_\tau H(a)\|_2^2}{\tau} &= \frac{1}{\tau}\int_0^\tau \int_0^\infty |\ell(x-y)|^2\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{\tau}\int_0^\tau \int_x^\infty |\ell(t)|^2\,\mathrm{d}t\,\mathrm{d}x \\ &= \frac{1}{\tau}\int_0^\tau \int_x^\tau |\ell(t)|^2\,\mathrm{d}t\,\mathrm{d}x + \frac{1}{\tau}\int_0^\tau \int_\tau^\infty |\ell(t)|^2\,\mathrm{d}t\,\mathrm{d}x, \end{aligned}$$

and the second term on the right is

$$\int_\tau^\infty |\ell(t)|^2\,\mathrm{d}t = o(1).$$

The first term equals

$$\frac{1}{\tau}\int_0^\tau \int_0^t |\ell(t)|^2\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{\tau}\int_0^\tau t|\ell(t)|^2\,\mathrm{d}t,$$

and we write this as

$$\frac{1}{\tau}\int_0^{\tau_0} t|\ell(t)|^2\,\mathrm{d}t + \frac{1}{\tau}\int_{\tau_0}^\tau t|\ell(t)|^2\,\mathrm{d}t \tag{9}$$

where $\tau_0 = \tau_0(\varepsilon)$ is chosen so that

$$\frac{1}{\tau}\int_{\tau_0}^\tau t|\ell(t)|^2\,\mathrm{d}t \le \int_{\tau_0}^\tau |\ell(t)|^2\,\mathrm{d}t \le \int_{\tau_0}^\infty |\ell(t)|^2\,\mathrm{d}t < \frac{\varepsilon}{2}.$$

Since then

$$\frac{1}{\tau}\int_0^{\tau_0} t|\ell(t)|^2\,\mathrm{d}t \le \frac{\tau_0}{\tau}\int_0^{\tau_0} |\ell(t)|^2\,\mathrm{d}t \le \frac{\tau_0}{\tau}\int_0^{\infty} |\ell(t)|^2\,\mathrm{d}t < \frac{\varepsilon}{2}$$

if only τ is large enough, we see that (9) is smaller than any prescribed $\varepsilon > 0$ whenever τ is sufficiently large. □

Lemma 2.2. *If $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $k \ge 1$ is a natural number, then*

$$\|W_\tau^k(a) - W_\tau(a^k)\|_1 = o(\tau) \quad \textit{as} \quad \tau \to \infty.$$

Proof. This is trivial for $k = 1$. Assume that the assertion is true for some $k \ge 1$. We write $W_\tau^{k+1}(a) - W_\tau(a^{k+1})$ as

$$\Big(W_\tau^k(a) - W_\tau(a^k)\Big)W_\tau(a) + W_\tau(a^k)W_\tau(a) - W_\tau(a^{k+1})$$

and have

$$\Big\|\Big(W_\tau^k(a) - W_\tau(a^k)\Big)W_\tau(a)\Big\|_1 \le \|W_\tau^k(a) - W_\tau(a^k)\|_1 \|W_\tau(a)\|_\infty.$$

Clearly, $\|W_\tau(a)\|_\infty \le \|a\|_\infty$. Furthermore, $\|W_\tau^k(a) - W_\tau(a^k)\|_1 = o(\tau)$ by assumption, and $\|W_\tau(a^k)W_\tau(a) - W_\tau(a^{k+1})\|_1 = o(\tau)$ due to Lemma 2.1. Thus, the assertion is valid for $k + 1$. □

As $|\mathrm{tr}\, A| \le \|A\|_1$ for every trace class operator A, Theorem 1.4 is an obvious consequence of Lemma 2.2.

The following result proves part of Theorem 1.3. We denote the spectrum of an operator A by $\mathrm{sp}\, A$. The essential spectrum $\mathrm{sp}_{\mathrm{ess}} A$ is the set of all $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is not Fredholm, that is, not invertible modulo compact operators. Clearly, $\mathrm{sp}_{\mathrm{ess}} A \subset \mathrm{sp}\, A$.

Proposition 2.3. *If $\sigma(\xi) := \mathrm{e}^{-i\xi^2/4}$ then*

$$\mathrm{sp}\, C(\sigma)) = \mathbb{T}, \quad \mathrm{sp}_{\mathrm{ess}} W(\sigma) = \mathrm{sp}\, W(\sigma) = \overline{\mathbb{D}}, \quad \mathrm{sp}\, W_\tau(\sigma) \subset \mathbb{D}.$$

Proof. Throughout this proof, a denotes an arbitrary function in $L^\infty(\mathbb{R})$. The spectrum of $C(a)$ is the essential range $\mathcal{R}(a)$ of a. Hence $\mathrm{sp}\, C(\sigma) = \mathbb{T}$. To prove the assertion for the spectra of $W(\sigma)$, we have recourse to known results on Toeplitz operators. The passage from Wiener–Hopf operators on $L^2(0,\infty)$ to Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ and back can be performed by a standard unitary similarity; see, for example, Section 9.5(e) of [10]. The Hartman–Wintner and Brown–Halmos theorems, which can be found, for instance, as Theorems 2.30 and 2.33 in [10], yield the spectral inclusions

$$\mathcal{R}(a) \subset \mathrm{sp}\, W(a) \subset \mathrm{conv}\, \mathcal{R}(a),$$

where conv denotes the convex hull. Consequently, $\mathbb{T} \subset \mathrm{sp}\, W(\sigma) \subset \overline{\mathbb{D}}$. To show that $\mathrm{sp}_{\mathrm{ess}} W(\sigma)$ is all of $\overline{\mathbb{D}}$, fix some $\lambda \in \mathbb{D}$. We have $\sigma(\xi) - \lambda = |\sigma(\xi) - \lambda|\mathrm{e}^{-\mathrm{i}\varphi(\xi)}$ with a function φ that can be written as $\varphi = \psi + \delta$ where $\psi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\delta \in C(\mathbb{R})$ is monotonous on $(-\infty, 0$ and $(0,\infty)$ with $\delta(\pm\infty) = +\infty$. Now we can employ a result of [5], which is also cited and proved as Theorem 6.4 of [8] and, reduced to a necessary invertibility criterion, went as Proposition 2.26(d) into [10]. This result says that if $a-\lambda$ has an argument as just described, then for $W(a-\lambda)$ to be Fredholm it is necessary that $|\varphi(\xi)| = O(\log|\xi|)$ as $|\xi| \to \infty$. Because in our case $|\varphi(\xi)|$ increases as $|\xi|^2$, it follows that $W(\sigma) - \lambda I = W(\sigma - \lambda)$ cannot be Fredholm. Thus, $\lambda \in \mathrm{sp}_{\mathrm{ess}} W(\sigma)$.

Finally, in [7, Theorem 1.1], it is shown that $\|W_\tau(\sigma)\|_\infty < 1$. We therefore arrive at the conclusion that $\mathrm{sp}\, W_\tau(\sigma) \subset \mathbb{D}$. □

As $W_\tau(\sigma)$ is not a normal operator, one could ask whether we should rather study the ε-pseudospectrum

$$\mathrm{sp}_\varepsilon W_\tau(\sigma) := \{\lambda \in \mathbb{C} : 1/\varepsilon \le \|(W_\tau(\sigma) - \lambda I)^{-1}\|_\infty \le \infty\}$$

than the spectrum $\mathrm{sp}\, W_\tau(\sigma)$. See [18]. It is known that, for each $\varepsilon > 0$, the sets $\mathrm{sp}_\varepsilon W_\tau(a)$ converge to $\mathrm{sp}_\varepsilon W(a)$ as $\tau \to \infty$ in the Hausdorff metric if a is piecewise continuous [6]. The symbol $\sigma(\xi) = \mathrm{e}^{-\mathrm{i}\xi^2/4}$ is not piecewise continuous, but fortunately things are simple. Here is the result.

Proposition 2.4. *Given $\varepsilon > 0$, there is a $\tau_0 = \tau_0(\varepsilon)$ such that $\overline{\mathbb{D}} \subset \mathrm{sp}_\varepsilon W_\tau(\sigma)$ for all $\tau > \tau_0$.*

Proof. Pick $\lambda \in \overline{\mathbb{D}}$. The operator $W_\tau(\sigma - \lambda)$ and its adjoint $W_\tau(\overline{\sigma} - \overline{\lambda})$ converge strongly to $W(\sigma - \lambda)$ and this operator's adjoint $W(\overline{\sigma} - \overline{\lambda})$. Thus, were the norms $\|W_{\tau_n}(\sigma - \lambda)^{-1}\|_\infty$ uniformly bounded for some sequence $\tau_n \to \infty$, $W(\sigma - \lambda)$ would be invertible. As the latter is not the case due to Proposition 2.3, we conclude that $\|W_\tau(\sigma - \lambda)^{-1}\|_\infty \to \infty$ for each $\lambda \in \overline{\mathbb{D}}$. This together with the compactness of $\overline{\mathbb{D}}$ implies that for every $\varepsilon > 0$ there is a $\tau_0(\varepsilon)$ such that $\|W_\tau(\sigma - \lambda)^{-1}\|_\infty \ge 1/\varepsilon$ for all $\tau > \tau_0(\varepsilon)$ and all $\lambda \in \overline{\mathbb{D}}$. □

Proposition 2.4 is equivalent to saying that given $\varepsilon > 0$ and $\lambda \in \overline{\mathbb{D}}$, there exists a number $\tau_0 = \tau_0(\varepsilon)$ such that for every $\tau > \tau_0$ we can find $f \in L^2(0,\tau)$ satisfying $\|f\| = 1$ and $\|W_\tau(\sigma)f - \lambda f\| \le \varepsilon$. This is in the spirit of Landau's result [15]. He took λ from $\mathbb{T}$ only but showed much more, namely that for $\tau > \tau_0$ there are at least 1000τ orthonormal functions f in $L^2(0,\tau)$ such that $\|W_\tau(\sigma)f - \lambda f\| \le \varepsilon$.

3. An oscillatory multivariate integral

In this section we prove Theorem 1.1.

Lemma 3.1. *For every natural number $k \ge 1$, the operators F_ω^k as well as the operators $W_\tau^k(\sigma)$ and $W_\tau(\sigma^k)$ generated by $\sigma(\xi) := \mathrm{e}^{-\mathrm{i}\xi^2/4}$, are of trace class.*

Proof. Since F_ω is unitarily similar to $W_{2\sqrt{\omega}}(\sigma)$, it suffices to prove that $W_\tau(\sigma^k)$ is in the trace class. The operator $W_\tau(\sigma^k)$ acts by the rule

$$(W_\tau(\sigma^k)f)(x) = \int_0^\tau \ell_k(x-y)f(y)\,\mathrm{d}y, \quad x \in (0,\tau), \tag{10}$$

where

$$\ell_k(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sigma^k(\xi)\mathrm{e}^{-\mathrm{i}\xi t}\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}k\xi^2/4-\mathrm{i}\xi t}\,\mathrm{d}\xi = \frac{1}{\sqrt{\pi \mathrm{i} k}}\,\mathrm{e}^{\mathrm{i}t^2/k}.$$

Let $\ell_{k,\tau}$ be a C^2 function on $\mathbb{R}$ which coincides with ℓ_k on $(-\tau,\tau)$ and is identically zero outside $(-2\tau, 2\tau)$. As (10) does not depend on the values of ℓ_k outside $(-\tau,\tau)$, we have $W_\tau(\sigma^k) = W_\tau(\hat{\ell}_{k,\tau})$. The function $\hat{\ell}_{k,\tau}$ is in $L^\infty(\mathbb{R})$ because $\ell_{k,\tau} \in L^1(\mathbb{R})$, and twice integrating the integral

$$\hat{\ell}_{k,\tau}(\xi) = \int_{-2\tau}^{2\tau} \ell_{k,\tau}(t)\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{d}t$$

by parts, we obtain

$$\hat{\ell}_{k,\tau}(\xi) = \frac{1}{(\mathrm{i}\xi)^2}\int_{-2\tau}^{2\tau} \ell''_{k,\tau}(t)\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{d}t$$

for $\xi \neq 0$, which shows that $\hat{\ell}_{k,\tau} \in L^1(\mathbb{R})$. In the beginning of Section 2 we noticed that symbols in $L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ generate truncated Wiener–Hopf operators in the trace class. Hence $W_\tau(\hat{\ell}_{k,\tau}) \in \mathcal{C}_1$. $\square$

We have

$$(F_\omega^k f)(x) = \int_{-1}^{1} m_k(x,y) f(y)\,\mathrm{d}y, \quad x \in (-1,1),$$

where

$$m_1(x,y) = \sqrt{\frac{\omega}{\pi \mathrm{i}}}\,\mathrm{e}^{\mathrm{i}\omega(x-y)^2}, \quad m_2(x,y) = \left(\sqrt{\frac{\omega}{\pi \mathrm{i}}}\right)^2 \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega(x-z)^2}\mathrm{e}^{\mathrm{i}\omega(z-y)^2}\,\mathrm{d}z,$$

$$m_3(x,y) = \left(\sqrt{\frac{\omega}{\pi \mathrm{i}}}\right)^3 \int_{-1}^{1}\int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega(x-z)^2}\mathrm{e}^{\mathrm{i}\omega(z-w)^2}\mathrm{e}^{\mathrm{i}\omega(w-y)^2}\,\mathrm{d}z\,\mathrm{d}w,$$

and so on. Since F_ω^k is of trace class by Lemma 3.1 and m_k is continuous on $[-1,1]^2$, it follows that

$$\operatorname{tr} F_\omega^k = \int_{-1}^{1} m_k(x,x)\,\mathrm{d}x;$$

see [14, Corollary III.10.2]. Consequently,

$$\operatorname{tr} F_\omega^k = \left(\sqrt{\frac{\omega}{\pi \mathrm{i}}}\right)^k I_k \tag{11}$$

where

$$I_k := \int_{-1}^{1} \dots \int_{-1}^{1} \exp\left(\mathrm{i}\omega \sum_{j=1}^{k} (x_j - x_{j+1})^2\right)\,\mathrm{d}x_1 \dots \mathrm{d}x_k \tag{12}$$

with $x_{k+1} := x_1$. By virtue of some lucky circumstances, it is not difficult to compute I_k straightforwardly for $k \le 4$. Trivially, $I_1 = 2$. Letting

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-\zeta^2}\,\mathrm{d}\zeta, \quad z \in \mathbb{C},$$

one almost immediately gets

$$\begin{aligned} I_2 &= \frac{2}{\sqrt{\omega}}\sqrt{\frac{\pi\mathrm{i}}{2}}\,\mathrm{erf}\left(\sqrt{\frac{8\omega}{\mathrm{i}}}\right)+\frac{\mathrm{i}}{2}\,\frac{\mathrm{e}^{8\mathrm{i}\omega}}{\omega}-\frac{\mathrm{i}}{2\omega} \\ &= \frac{2}{\sqrt{2}}\sqrt{\frac{\pi\mathrm{i}}{\omega}}-\frac{\mathrm{i}}{2\omega}+O\left(\frac{1}{\omega^2}\right)=\frac{2}{\sqrt{2}}\sqrt{\frac{\pi\mathrm{i}}{\omega}}-\frac{1}{2\pi}\left(\sqrt{\frac{\pi\mathrm{i}}{\omega}}\right)^2+O\left(\frac{1}{\omega^2}\right), \end{aligned}$$

while with a little more labour, one obtains

$$\begin{aligned} I_3 &= \frac{1}{\omega^{3/2}}\,\frac{\pi\mathrm{i}}{\sqrt{3}}\int_0^{\sqrt{\omega}}\mathrm{erf}\left(\sqrt{\frac{6}{\mathrm{i}}}\,y\right)\left[\mathrm{erf}\left(\sqrt{\frac{2}{\mathrm{i}}}\,y\right)+\mathrm{erf}\left(\sqrt{\frac{2}{\mathrm{i}}}\,(2\sqrt{\omega}-y)\right)\right]\mathrm{d}y \\ &= \frac{2}{\sqrt{3}}\left(\sqrt{\frac{\pi\mathrm{i}}{\omega}}\right)^2-\frac{1}{\pi\sqrt{2}}\left(\sqrt{\frac{\pi\mathrm{i}}{\omega}}\right)^3+o\left(\frac{1}{\omega^{3/2}}\right) \end{aligned}$$

and

$$\begin{aligned} I_4 &= \frac{1}{\omega^2}\,\frac{(\pi\mathrm{i})^{3/2}}{4}\int_0^{\sqrt{\omega}}\mathrm{erf}\left(\sqrt{\frac{4}{\mathrm{i}}}\,y\right)\left[\mathrm{erf}\left(\sqrt{\frac{2}{\mathrm{i}}}\,y\right)+\mathrm{erf}\left(\sqrt{\frac{2}{\mathrm{i}}}\,(2\sqrt{\omega}-y)\right)\right]^2\mathrm{d}y \\ &= \frac{2}{\sqrt{4}}\left(\sqrt{\frac{\pi\mathrm{i}}{\omega}}\right)^3+O\left(\frac{1}{\omega^2}\right). \end{aligned}$$

However, to tackle the general case we have to proceed differently.

Theorem 3.2. *As* $\omega\to\infty$,

$$I_k=\frac{2}{\sqrt{k}}\left(\sqrt{\frac{\pi\mathrm{i}}{\omega}}\right)^{k-1}(1+o(1)).$$

Proof. To establish the pattern for general k, we first consider the case $k=3$. The oscillator function in (12) is

$$g(x_1,x_2,x_3):=(x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2,$$

and its stationary points are on the straight line $x_1=x_2=x_3$. We make the change of variables

$$t=x_1-x_2,\quad u=x_2-x_3,\quad v=x_1+x_2+x_3$$

in (12). The determinant of the Jacobian is $1/3$, hence

$$I_3=\frac{1}{3}\int_\Delta \exp[\mathrm{i}\omega(t^2+u^2+(t+u)^2)]\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v$$

where Δ is some polytope containing the origin in its interior. The new oscillator function

$$h(t,u,v)=t^2+u^2+(t+u)^2$$

is independent of v, and as a function of t and u only, it has the single stationary point $t = u = 0$. The Hessian for h, again thought of as a function of solely t and u, is

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

This is a positive definite matrix, and therefore h can be written as $r^2 + s^2$ in suitable coordinates r and s. To find the new coordinates, we try the ansatz

$$r = at + bu, \quad s = cu. \tag{13}$$

The equation

$$(at + bu)^2 + (cu)^2 = t^2 + u^2 + (t + u)^2$$

is satisfied for

$$a = \sqrt{2}, \quad b = \frac{1}{\sqrt{2}}, \quad c = \sqrt{\frac{3}{2}}.$$

The Jacobi determinant of the substitution (13) with these coefficients equals $1/(ac) = 1/\sqrt{3}$. Consequently,

$$I_3 = \frac{1}{3}\frac{1}{\sqrt{3}}\int_\Omega \exp[\mathrm{i}\omega(r^2 + s^2)]\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s$$

where Ω is again a polytope with the origin in its interior. Integrating over v we get

$$\begin{aligned} I_3 &= \frac{1}{3}\frac{1}{\sqrt{3}}\int_{\Omega_1}\left(\int_{v_1(r,s)}^{v_2(r,s)} \exp[\mathrm{i}\omega(r^2 + s^2)]\,\mathrm{d}v\right)\mathrm{d}r\,\mathrm{d}s \\ &= \frac{1}{3}\frac{1}{\sqrt{3}}\int_{\Omega_1} V(r,s)\exp[\mathrm{i}\omega(r^2 + s^2)]\,\mathrm{d}r\,\mathrm{d}s \end{aligned}$$

with $V(r,s) := v_2(r,s) - v_1(r,s)$ and some (planar) polytope Ω_1 with the origin in its interior. For $r = s = 0$, the variable v ranges from -3 to 3. Hence $V(0,0) = 6$. The stationary phase formula

$$\int_{-\alpha}^{\beta} f(x)\mathrm{e}^{\mathrm{i}\omega x^2}\,\mathrm{d}x = f(0)\sqrt{\frac{\pi\mathrm{i}}{\omega}}\,(1 + o(1))$$

can now be applied independently for r and s. The outcome is

$$I_3 = \frac{1}{3}\frac{1}{\sqrt{3}}\,6\left(\sqrt{\frac{\pi\mathrm{i}}{\omega}}\right)^2(1 + o(1)) = \frac{2}{\sqrt{3}}\left(\sqrt{\frac{\pi\mathrm{i}}{\omega}}\right)^2(1 + o(1)).$$

The pattern in the general case is now obvious. Substituting

$$t_j = x_j - x_{j+1} \quad (1 \le j \le k-1), \quad t_k = x_1 + x_2 + \ldots + x_k$$

in (12), we get

$$I_k = \frac{1}{k}\int_\Delta \exp[\mathrm{i}\omega h(t_1, \ldots, t_{k-1})]\,\mathrm{d}t_1 \ldots \mathrm{d}t_k$$

with

$$h(t_1,\ldots,t_{k-1}) = \sum_{j=1}^{k-1} t_j^2 + \left(\sum_{j=1}^{k-1} t_j\right)^2 .$$

The Hessian of this function is the $(k-1)\times(k-1)$ matrix

$$H := \begin{pmatrix} 4 & 2 & 2 & \ldots \\ 2 & 4 & 2 & \ldots \\ 2 & 2 & 4 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

The determinant of the $m\times m$ matrix constituted by the first m rows and columns of H is $2^m(m+1)$. Thus, by Sylvester's theorem, H is positive definite. We look for a change of variables

$$\begin{pmatrix} s_1 \\ s_2 \\ \ldots \\ s_{k-1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1,k-1} \\ 0 & a_{22} & \ldots & a_{2,k-1} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{k-1,k-1} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ \ldots \\ t_{k-1} \end{pmatrix}$$

such that

$$s_1^2 + s_2^2 + \ldots + s_{k-1}^2 = h(t_1, t_2, \ldots, t_{k-1}).$$

It is easily seen that such a change of variables can be found with

$$a_{11} = \sqrt{2}, \quad a_{22} = \sqrt{\frac{3}{2}}, \quad \ldots, \quad a_{k-1,k-1} = \sqrt{\frac{k}{k-1}}.$$

The Jacobi determinant equals $1/(a_{11}a_{22}\ldots a_{k-1,k-1}) = 1/\sqrt{k}$. We so arrive at the representation

$$\begin{aligned} I_k &= \frac{1}{k}\frac{1}{\sqrt{k}}\int_\Omega \exp[\mathrm{i}\omega(s_1^2+\ldots+s_{k-1}^2)]\,\mathrm{d}t_k\,\mathrm{d}s_1\ldots\mathrm{d}s_{k-1} \\ &= \frac{1}{k}\frac{1}{\sqrt{k}}\int_{\Omega_1} V(s_1,\ldots,s_{k-1})\exp[\mathrm{i}\omega(s_1^2+\ldots+s_{k-1}^2)]\,\mathrm{d}s_1\ldots\mathrm{d}s_{k-1}. \end{aligned}$$

As $V(0,\ldots,0) = 2k$, the usual stationary phase formula argument yields

$$I_k = \frac{1}{k}\frac{1}{\sqrt{k}}\,2k\left(\sqrt{\frac{\pi\mathrm{i}}{\omega}}\right)^{k-1}(1+o(1)) = \frac{2}{\sqrt{k}}\left(\sqrt{\frac{\pi\mathrm{i}}{\omega}}\right)^{k-1}(1+o(1)),$$

as desired. □

Lemma 2.1 in conjunction with (11) and Theorem 3.2 proves Theorem 1.1. As already said, (5) is equivalent to (1). Thus, also the proof of Theorem 1.3 is at this point complete.

4. The logarithmic spiral ansatz

We now prove Theorem 1.2. Letting b_ω be as in that theorem, we have

$$\int_0^\infty b_\omega^k(x)\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-[\alpha(\omega)+\mathrm{i}\beta(\omega)]kx^\nu}\,\mathrm{d}x,$$

and hence (3) is true for some k if and only if

$$\int_0^\infty \mathrm{e}^{-[\alpha(\omega)+\mathrm{i}\beta(\omega)]kx^\nu}\,\mathrm{d}x = 2k^{-1/2}\sqrt{\frac{\omega}{\pi\mathrm{i}}}\,(1+o(1)), \tag{14}$$

or equivalently, after substituting $kx^\nu \to x^\nu$,

$$k^{-1/\nu}\int_0^\infty \mathrm{e}^{-[\alpha(\omega)+\mathrm{i}\beta(\omega)]x^\nu}\,\mathrm{d}x = 2k^{-1/2}\sqrt{\frac{\omega}{\pi\mathrm{i}}}\,(1+o(1)). \tag{15}$$

Taking (14) for $k=1$, we obtain

$$\int_0^\infty \mathrm{e}^{-[\alpha(\omega)+\mathrm{i}\beta(\omega)]x^\nu}\,\mathrm{d}x = 2\sqrt{\frac{\omega}{\pi\mathrm{i}}}\,(1+o(1)),$$

whereas (15) for $k=2$ states that

$$\int_0^\infty \mathrm{e}^{-[\alpha(\omega)+\mathrm{i}\beta(\omega)]x^\nu}\,\mathrm{d}x = 2^{1+1/\nu-1/2}\sqrt{\frac{\omega}{\pi\mathrm{i}}}\,(1+o(1)).$$

Comparing the last two formulas, we arrive at the conclusion that if (14) holds for $k=1$ and $k=2$, then necessarily $\nu=2$.

Now consider (14) with $\nu=2$. Computing the integral, we obtain that (14) is equivalent to the statement that

$$\frac{1}{2}\sqrt{\frac{\pi}{\alpha(\omega)+\mathrm{i}\beta(\omega)}}\,\frac{1}{\sqrt{k}} = \frac{2}{\sqrt{k}}\sqrt{\frac{\omega}{\pi\mathrm{i}}}\,(1+o(1)),$$

which holds if and only if

$$\alpha(\omega)+\mathrm{i}\beta(\omega) = \frac{\pi^2\mathrm{i}}{16\omega}\,(1+o(1)). \tag{16}$$

Writing $o(1) = o(1) + \mathrm{i}o(1)$ with two real $o(1)$ on the right, we arrive at the conclusion that (16) is valid if and only if

$$\alpha(\omega) = o\left(\frac{1}{\omega}\right), \quad \beta(\omega) = \frac{\pi^2}{16\omega} + o\left(\frac{1}{\omega}\right).$$

This completes the proof of Theorem 1.2.

5. Symbols with a Fox–Li discontinuity

This section is devoted to the proof of Theorem 1.5.

Lemma 5.1. (a) *Let A_τ and B_τ be operators on $L^2(0,\tau)$. If $\|A_\tau\|_1 = o(\tau)$ and $\|B_\tau\|_\infty = O(1)$, then $\|A_\tau B_\tau\|_1 = o(\tau)$ and $\|B_\tau A_\tau\|_1 = o(\tau)$.*

(b) *If $b, d \in L^\infty(\mathbb{R})$ and $H(b), H(\widetilde{b}) \in \mathcal{C}_1$, then*

$$\|W_\tau(b)W_\tau(d) - W_\tau(bd)\|_1 = o(\tau), \quad \|W_\tau(d)W_\tau(b) - W_\tau(bd)\|_1 = o(\tau).$$

(c) *If $b \in L^\infty(\mathbb{R})$ and $H(b) \in \mathcal{C}_1$, then $H(b^k) \in \mathcal{C}_1$ for every natural number $k \geq 1$.*

Proof. Part (a) follows from the inequalities

$$\|A_\tau B_\tau\|_1 \leq \|A_\tau\|_1 \|B_\tau\|_\infty, \quad \|B_\tau A_\tau\|_1 \leq \|B_\tau\|_\infty \|A_\tau\|_1.$$

To prove (b) note that, by (8),

$$W_\tau(b)W_\tau(d) - W_\tau(bd) = -P_\tau H(b)H(\widetilde{d})P_\tau - R_\tau H(\widetilde{b})H(d)R_\tau$$

and that

$$\|P_\tau H(b)H(\widetilde{d})P_\tau\|_1 \leq \|P_\tau\|_\infty \|H(b)\|_1 \|H(\widetilde{d})P_\tau\|_\infty = O(1) = o(\tau),$$
$$\|R_\tau H(\widetilde{b})H(d)R_\tau\|_1 \leq \|R_\tau\|_\infty \|H(\widetilde{b})\|_1 \|H(d)R_\tau\|_\infty = O(1) = o(\tau).$$

Finally, part (c) is obviously true for $k = 1$. So suppose that $H(b^k) \in \mathcal{C}_1$ for some $k \geq 1$. The identity

$$H(b^{k+1}) = H(b^k)W(\widetilde{b}) + W(b^k)H(b),$$

which is the continuous analogue of formula (2.19) in [10], shows that then $H(b^{k+1})$ is also in $\mathcal{C}_1$. $\square$

Proposition 5.2. *Let $a(\xi) = c(\xi)\sigma(\xi)$ where $\sigma(\xi) := \mathrm{e}^{-\mathrm{i}\xi^2/4}$ and c is a function in $C(\dot{\mathbb{R}})$ such that $H((c - c(\infty))\sigma^\nu) \in \mathcal{C}_1$ and $H((\widetilde{c} - c(\infty))\sigma^\nu) \in \mathcal{C}_1$ for every integer $\nu \geq 0$. Then the operator $W_\tau^k(a)$ and $W_\tau(a^k)$ are of trace class for every natural number $k \geq 1$ and*

$$\operatorname{tr} W_\tau^k(a) = \operatorname{tr} W_\tau(a^k) + o(\tau) = \frac{\tau}{2\pi}\int_{-\infty}^{\infty} a^k(\xi)\,\mathrm{d}\xi + o(\tau). \tag{17}$$

Proof. Again by Widom's formula (8),

$$W_\tau(a^k) = W_\tau(c^k)W_\tau(\sigma^k) + P_\tau H(c^k)H(\sigma^k)P_\tau + R_\tau H(\widetilde{c}^k)H(\sigma^k)R_\tau;$$

notice that $\widetilde{\sigma}(\xi) := \sigma(-\xi) = \sigma(\xi)$. Lemma 3.1 tells us that $W_\tau(\sigma^k)$ is in $\mathcal{C}_1$. Let $u := c - c(\infty)$. The Hankel operator induced by a constant function is the zero operator. Hence $H(c) = H(u)$ and $H(\widetilde{c}) = H(\widetilde{u})$. By our assumption, $H(u)$ and $H(\widetilde{u})$ are in $\mathcal{C}_1$. From Lemma 5.1(c) we therefore deduce that the operators $H(c^k)$ and $H(\widetilde{c}^k)$ are also in $\mathcal{C}_1$. This shows that $W_\tau(a^k) \in \mathcal{C}_1$ and thus also that $W_\tau^k(a) = [W_\tau(a^1)]^k \in \mathcal{C}_1$.

In what follows we write $A_\tau \equiv B_\tau$ if $\|A_\tau - B_\tau\|_1 = o(\tau)$. Recall that $u(\xi)$ is defined as $c(\xi) - c(\infty)$. Thus, $a = u\sigma + c(\infty)\sigma$. We claim that for each natural number $k \geq 1$ it is true that

$$W_\tau^k(a) \equiv W_\tau[(u\sigma + c(\infty)\sigma)^k - c(\infty)^k\sigma^k] + c(\infty)^k W_\tau^k(\sigma). \tag{18}$$

This is trivial for $k = 1$. So assume the claim is true for some $k \geq 1$. Then

$$\begin{aligned} W_\tau^{k+1}(a) &\equiv \Big[W_\tau(u\sigma) + c(\infty)W_\tau(\sigma)\Big] \times \\ &\quad \times \left[\sum_{j=1}^{k} \binom{k}{j} c(\infty)^{k-j} W_\tau((u\sigma)^j\sigma^{k-j}) + c(\infty)^k W_\tau^k(\sigma)\right]. \end{aligned}$$

The operators $H(u\sigma^\nu)$ and $H(\widetilde{u}\sigma^\nu)$ are in $\mathcal{C}_1$ for all natural numbers $\nu \geq 1$ by our assumption. From Lemma 5.1(b) we therefore obtain that

$$W_\tau(u\sigma)W_\tau((u\sigma)^j\sigma^{k-j}) \equiv W_\tau((u\sigma)(u\sigma)^j\sigma^{k-j})$$

and using parts (a) and (b) of Lemma 5.1 we get

$$\begin{aligned} W_\tau(\sigma)W_\tau((u\sigma)^j\sigma^{k-j}) &\equiv W_\tau(\sigma)W_\tau(u\sigma)W_\tau((u\sigma)^{j-1}\sigma^{k-j}) \\ &\equiv W_\tau(u\sigma^2)W_\tau((u\sigma)^{j-1}\sigma^{k-j}) \equiv W_\tau(\sigma(u\sigma)^j\sigma^{k-j}) \end{aligned}$$

and

$$W_\tau(u\sigma)W_\tau^k(\sigma) \equiv W_\tau(u\sigma^2)W_\tau^{k-1}(\sigma) \equiv W_\tau(u\sigma^3)W_\tau^{k-2}(\sigma) \equiv \ldots \equiv W_\tau(u\sigma^{k+1}).$$

Consequently,

$$\begin{aligned} W_\tau^{k+1}(a) &\equiv W_\tau\left[(u\sigma + c(\infty)\sigma)\sum_{j=1}^{k}\binom{k}{j} c(\infty)^{k-j}(u\sigma)^j\sigma^{k-j}\right] \\ &\quad + W_\tau\Big[c(\infty)^k u\sigma^{k+1}\Big] + c(\infty)^{k+1}W_\tau^{k+1}(\sigma), \end{aligned}$$

and the sum of the symbols in the brackets on the right is

$$\begin{aligned} &(u\sigma + c(\infty)\sigma)[(u\sigma + c(\infty)\sigma)^k - c(\infty)^k\sigma^k] + c(\infty)^k u\sigma^{k+1} \\ &= (u\sigma + c(\infty)\sigma)^{k+1} - c(\infty)^{k+1}\sigma^{k+1}. \end{aligned}$$

This proves our claim (18) for $k + 1$.

If $A_\tau \equiv B_\tau$, then $\operatorname{tr} A_\tau = \operatorname{tr} B_\tau + o(\tau)$. Since $u\sigma + c(\infty)\sigma = a$, the trace of the first term on the right of (18) equals

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(a^k(\xi) - c(\infty)^k\sigma^k(\xi)\Big)\,\mathrm{d}\xi,$$

and from Theorem 1.3 we know that the trace of the second term on the right of (18) is

$$\operatorname{tr}(c(\infty)^k W_\tau^k(\sigma)) = c(\infty)^k \operatorname{tr} W_\tau(\sigma^k) + o(\tau) = \frac{c(\infty)^k}{2\pi}\int_{-\infty}^{\infty}\sigma^k(\xi)\,\mathrm{d}\xi + o(\tau).$$

Adding the two results we arrive at (17). □

The hypothesis of Proposition 5.2 stipulates that the Hankel operators $H(u\sigma^\nu)$ and $H(\widetilde{u}\sigma^\nu)$ are in $\mathcal{C}_1$ for every integer $\nu \geq 0$. Peller showed that the two Hankel operators $H(b)$ and $H(\widetilde{b})$ are of trace class if and only if b is in the Besov space $B_1^1(\mathbb{R})$; see [17, p. 277]. Here is simple sufficient condition for $H(b)$ and $H(\widetilde{b})$ to be in the trace class.

Lemma 5.3. *If $b \in C^3(\mathbb{R})$ and the functions $\xi^2 b(\xi)$, $\xi^2 b'(\xi)$, $\xi^2 b''(\xi)$, $b'''(\xi)$ belong to $L^1(\mathbb{R})$ and have zero limits as $\xi \to \pm\infty$, then $H(b)$ and $H(\widetilde{b})$ are trace class operators.*

Proof. Let

$$\ell(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(\xi)\mathrm{e}^{-\mathrm{i}\xi t}\,\mathrm{d}\xi, \quad t \in \mathbb{R}.$$

Since $\xi b(\xi)$ and $\xi^2 b(\xi)$ are in $L^1(\mathbb{R})$, we may twice differentiate the integral to see that ℓ is in $C^2(\mathbb{R})$ and

$$\ell''(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-\mathrm{i}\xi)^2 b(\xi) e^{-\mathrm{i}\xi t}\,\mathrm{d}\xi.$$

Using that $(\xi^2 b(\xi))' = 2\xi b(\xi) + \xi^2 b'(\xi)$ and $(\xi^2 b(\xi))'' = 2b(\xi) + 4\xi b'(\xi) + \xi^2 b''(\xi)$ are also in $L^1(\mathbb{R})$ and have zero limits at infinity, we may twice integrate by parts to obtain that

$$\ell''(t) = \frac{1}{2\pi}\frac{(-\mathrm{i})^2}{(\mathrm{i}t)^2}\int_{-\infty}^{\infty} (\xi^2 b(\xi))'' \mathrm{e}^{-\mathrm{i}\xi t}\,\mathrm{d}\xi,$$

which shows that

$$\int_{-\infty}^{\infty} |t|\,|\ell''(t)|^2\,\mathrm{d}t < \infty. \tag{19}$$

As b', b'', b''' are in L^1 and have zero limits at infinity, we have

$$\ell(t) = \frac{1}{2\pi}\frac{1}{(\mathrm{i}t)^3}\int_{-\infty}^{\infty} b'''(\xi)\mathrm{e}^{-\mathrm{i}\xi t}\,\mathrm{d}\xi$$

and hence

$$\int_{-\infty}^{\infty} |t|^4|\ell(t)|^2\,\mathrm{d}t < \infty. \tag{20}$$

Basor and Widom [1, p. 398] showed that $H(b)$ and $H(\widetilde{b})$ are of trace class if (19) and (20) hold. □

Corollary 5.4. *If c is as in Theorem 1.5, then the Hankel operators $H(u\sigma^\nu)$ and $H(\widetilde{u}\sigma^\nu)$ are in $\mathcal{C}_1$ for every real number ν.*

Proof. The function $b := u\sigma^\nu$ satisfies the hypothesis of Lemma 5.3. □

Combining Corollary 5.4 and Proposition 5.2, we arrive at Theorem 1.5.

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