Spectral theory of large Wiener–Hopf operators with complex-symmetric kernels and rational symbols

Albrecht Böttcher Sergei Grudsky* Arieh Iserles

November 12, 2010

Abstract

This paper is devoted to the asymptotic behaviour of individual eigenvalues of truncated Wiener–Hopf integral operators over increasing intervals. The kernel of the operators is complex-symmetric and has a rational Fourier transform. Under additional hypotheses, the main result describes the location of the eigenvalues and provides their asymptotic expansions in terms of the reciprocal of the length of the truncation interval. Also determined is the structure of the eigenfunctions.

Keywords: Wiener–Hopf operator, complex-symmetric kernel, rational symbol, eigenvalue, eigenfunction, Fox–Li operator

Mathematics Subject Classification (2010): Primary 47B35; Secondary 45C05, 45E10

1 Introduction

A truncated Wiener–Hopf operator is of the form

$$(K_\tau f)(t) := f(t) + \int_0^\tau \mathrm{k}(t-s) f(s)\,\mathrm{d}s, \qquad t \in (0,\tau). \tag{1.1}$$

We suppose that k is a function in $L^2(\mathbb{R})$, so that the integral operator in (1.1) is a Hilbert–Schmidt operator and thus compact on $L^2(0,\tau)$ for all $\tau > 0$. Let $\operatorname{sp} K_\tau$ be the spectrum of K_τ. Since $K_\tau - I$ is compact, all points in $\operatorname{sp} K_\tau \setminus \{1\}$ are eigenvalues. We are interested in the location and the asymptotic behaviour of these eigenvalues as τ tends to infinity.

The two basic assumptions stipulated in this paper are that the kernel $\mathrm{k}(t-s)$ is complex-symmetric, which means that k is a complex-valued function satisfying $\mathrm{k}(t) = \mathrm{k}(-t)$ for all $t \in \mathbb{R}$, and that the so-called *symbol* of the operator,

$$a(x) := 1 + \int_{-\infty}^{\infty} \mathrm{k}(t) \mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t, \quad x \in \mathbb{R},$$

is a rational function. These two assumptions are equivalent to the requirement that

$$\mathrm{k}(t) = \begin{cases} \displaystyle\sum_{\ell=1}^{m} p_\ell(t)\mathrm{e}^{-\lambda_\ell t} & \text{for} \quad t > 0, \\ \displaystyle\sum_{\ell=1}^{m} p_\ell(-t)\mathrm{e}^{\lambda_\ell t} & \text{for} \quad t < 0, \end{cases}$$

*This author acknowledges support of the present work by a grant of the DAAD and by CONACYT grants 80503 and 102800.

where λ_ℓ are complex numbers with $\operatorname{Re}\lambda_\ell > 0$ and $p_\ell(t)$ are polynomials with complex coefficients. As $\mathrm{k}(t) = \mathrm{k}(-t)$ for all $t \in \mathbb{R}$ if and only if $a(x) = a(-x)$ for all $x \in \mathbb{R}$, the Wiener–Hopf operators considered here are just those with even rational symbols. We may write

$$a(x) = \prod_{j=1}^{r} \frac{x^2 - \zeta_j^2}{x^2 + \mu_j^2}, \quad x \in \mathbb{R}, \tag{1.2}$$

where $\zeta_j \in \mathbb{C}$, $\mu_j \in \mathbb{C}$, $\operatorname{Re}\mu_j > 0$, and $-\zeta_j^2 \neq \mu_k^2$ for all j, k. To indicate the dependence of K_τ on the symbol a and in accordance with the literature, we henceforth denote K_τ by $W_\tau(a)$.

This paper was motivated by the recent papers [3], [4], [8], [9] dedicated to the Fox–Li operator and it was encouraged by the recent investigations in [1], [5] on individual eigenvalues of certain Hessenberg or Hermitian Toeplitz matrices. The Fox–Li operator is (1.1) with the kernel $\mathrm{k}(t) = \mathrm{e}^{\mathrm{i}t^2}$. Clearly, $\mathrm{k}(t-s)$ is complex-symmetric, and although the function k is not in $L^2(\mathbb{R})$, the Fox–Li operator can be shown to be the identity plus a trace class operator. Its symbol is

$$a(x) = 1 + \sqrt{\pi}\, \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\mathrm{i}x^2/4}. \tag{1.3}$$

Numerical computations and arguments from physics indicate that the eigenvalues of $W_\tau(a)$ line up along a spiral commencing near the point $1 + \sqrt{\pi/\tau}\, \mathrm{e}^{\mathrm{i}\pi/4}$ and rotating clockwise to the point 1. However mathematically rigorous and at the same time satisfactory results are very sparse. These include Henry Landau's analysis of the pseudo-eigenvalues [13] of the Fox–Li operator and Henry Landau and Harold Widom's paper [14], which, as shown in [3], provides deep insight into the singular values of the Fox–Li operator. Of course, (1.3) is far from being a rational function, but we think that exploring the case of even rational symbols might well be a first step towards gaining an understanding of the situation for the Fox–Li symbol (1.3).

We take the liberty to repeat the quote from Cochran's book [10, p. 279] made already in [8]: "The analysis of integral equations with general complex-symmetric kernels remains, at present, an art form in which each separate equation appears to necessitate treatment based almost solely on its own individual features and peculiarities." The features and peculiarities exploited here are the difference kernel and the rationality of the symbol.

We extend a from the real line $\mathbb{R}$ to $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ by defining $a(\pm\infty) := 1$. Let $\mathcal{R}(a) := a(\overline{\mathbb{R}})$. By our assumptions, $\mathcal{R}(a)$ is an analytic curve in the plane such that when x moves from $-\infty$ to 0, the symbol $a(x)$ traces out this curve from 1 to $a(0)$, and when x goes further from 0 to ∞ then $a(x)$ follows this curve back from $a(0)$ to 1. The winding number of the function a about any point outside $\mathcal{R}(a)$ is zero.

The Wiener–Hopf analogue of the Brown–Halmos theorem for Toeplitz matrices [6, Proposition 2.17] tells us that $\operatorname{sp} W_\tau(a)$ is contained in the convex hull of $\mathcal{R}(a)$ for every $\tau > 0$. Classical results on the finite section method for Wiener–Hopf operators [12, Theorem III.3.1] show that if a is even and continuous on $\mathbb{R} \cup \{\infty\}$ and $U \subset \mathbb{C}$ is any open neighbourhood of $\mathcal{R}(a)$, then $\operatorname{sp} W_\tau(a) \subset U$ for all sufficiently large τ. In [7], the limit of $\operatorname{sp} W_\tau(a)$ in the Hausdorff metric was determined for arbitrary rational symbols a. In the case of even rational symbols, the result of [7] implies that $\operatorname{sp} W_\tau(a)$ converges to $\mathcal{R}(a)$ in the Hausdorff metric. There are also results of the type of Szegő's limit theorems, which describe the limiting density of the points in $\operatorname{sp} W_\tau(a)$. One such result is Theorem 2.6 of [3]; see also [6, Example 5.39], [15], [18] for the Toeplitz case. Finally, if a is real-valued, theorems on the asymptotic behaviour of the extreme eigenvalues of $W_\tau(a)$ may already be found in Widom's papers [16], [17].

The results of this paper are different from those cited in the previous paragraph. We provide here asymptotic expansions for individual eigenvalues. Under additional hypotheses, which include that the set $\mathcal{R}(a)$ is a curve without self-intersections and that the roots of certain polynomials are all simple, we prove the following. We associate a number $\beta > 0$ with

a, consider the half-stripe

$$S_\tau := \{z \in \mathbb{C} : \operatorname{Re} z > 0, \quad |\operatorname{Im} z| \le \beta/\tau\},$$

and show that, for τ large enough, all the eigenvalues of $W_\tau(a)$ are contained in $a(S_\tau)$. Further, we construct a finite cover $I_1 \cup \ldots \cup I_N = (0, \infty)$ by open intervals, and for each $I \in \{I_n\}_{n=1}^N$ we consider, subject to certain circumstances, either the segments

$$I_{k,\tau} := \left[\left(k - \frac{1}{2}\right)\frac{\pi}{\tau}, \left(k + \frac{1}{2}\right)\frac{\pi}{\tau}\right]$$

contained in I or the segments

$$I_{k,\tau} := \left[k\frac{\pi}{\tau}, (k+1)\frac{\pi}{\tau}\right]$$

being subsets of I. In this way we obtain N families of rectangles

$$S_{k,\tau} := \{z \in S_\tau : \operatorname{Re} z \in I_{k,\tau}\}.$$

We prove that if τ is sufficiently large then each set $a(S_{k,\tau})$ contains exactly one eigenvalue, and the eigenvalue $\lambda_{k,\tau}$ in $a(S_{k,\tau})$ has an asymptotic expansion

$$\lambda_{k,\tau} \sim a(k\pi/\tau) + \frac{c_1(k\pi/\tau)}{2\mathrm{i}\tau} + \frac{c_2(k\pi/\tau)}{(2\mathrm{i}\tau)^2} + \ldots$$

with computable coefficients $c_1(k\pi/\tau), c_2(k\pi/\tau), \ldots$. We also show that the eigensubspaces are all one-dimensional and describe the structure of the eigenfunctions.

In the following Section 2 we illustrate the strategy of our approach by a simple example. Section 3 contains the precise statement of our main result, while Sections 4 to 6 are devoted to its proof. The big advantage of rational symbols is that we have an explicit formula for the Fredholm determinant

$$\det\left(\frac{1}{1-\lambda}\Big(W_\tau(a) - \lambda I\Big)\right). \tag{1.4}$$

Such a formula was established in [2]. See also [11] for an alternative approach. A point $\lambda \neq 1$ is an eigenvalue for $W_\tau(a)$ if and only if (1.4) is zero. Our assumption that a is an even function facilitates the analysis of (1.4) significantly, and after writing λ in the form $\lambda = a(z)$, we will arrive at a nonlinear equation for z. If z is restricted to one of the small rectangles $S_{k,\tau}$, this equation is of the form $z = \Phi_{k,\tau}(z)$ with a contractive map $\Phi_{k,\tau}$ of $S_{k,\tau}$ into itself. Banach's fixed point theorem then accomplishes the rest.

We remark that basic ideas of the present work can already be found in Harold Widom's 1958 paper [16]. He studied Hermitian kernels, but the equations he obtained and his analysis of these equations are very similar to what will be done in Section 4, and the method we will use in Section 6 to determine the eigenfunctions is also essentially already in Widom's paper [16]. However, the non-Hermitian case has several subtleties, and we see the main contribution of this paper in overcoming the obstacles caused by them. We also consider the application of Banach's fixed point theorem to the eigenvalue analysis of Wiener–Hopf operators as a certain novelty. Finally, we consider this paper as another piece of the art mentioned in the quote of Cochran and hope it will be useful for forthcoming research into complex-symmetric kernels.

2 A preparatory example

In order to gain an understanding for some aspects of the general setting, we first consider the case $r = 1$, therefore

$$a(x) = \frac{x^2 - \zeta^2}{x^2 + \mu^2}.$$

To avoid unnecessarily cumbersome expressions, let us take $\zeta = 0$ and $\mu = 1$. The resulting symbol $a(x) = x^2/(x^2+1)$ is real-valued and the corresponding kernel

$$\mathrm{k}(t) = \begin{cases} -\mathrm{e}^{-t}/2 & \text{for} \quad t > 0, \\ -\mathrm{e}^{t}/2 & \text{for} \quad t < 0, \end{cases} \tag{2.1}$$

is Hermitian, but the example nevertheless illustrates the full extent of phenomena associated with the general setting. For $\lambda \neq 1$, the two complex solutions of the equation $a(z) = \lambda$ are $z = \omega_1 = \omega$ and $z = \omega_2 = -\omega$, where ω is any choice of the square root $\sqrt{\lambda/(1-\lambda)}$. In the case at hand, the determinant (1.4) is

$$\det W_\tau\left(\frac{a-\lambda}{1-\lambda}\right) = \mathrm{e}^{\kappa\tau}\left(W_1\mathrm{e}^{\mathrm{i}\omega\tau} + W_2\mathrm{e}^{-\mathrm{i}\omega\tau}\right) \tag{2.2}$$

with

$$\kappa := \frac{1}{2}\frac{1}{1-\lambda} - 1, \quad W_1 := \frac{(1+\mathrm{i}\omega)^2}{4\mathrm{i}\omega}, \quad W_2 := \frac{(1-\mathrm{i}\omega)^2}{-4\mathrm{i}\omega}$$

if $\lambda \notin \{0,1\}$, while

$$\det W_\tau(a) = \mathrm{e}^{-\tau/2}(1+\tau/2)$$

when $\lambda = 0$. We see in particular that $\lambda = 0$ is not an eigenvalue. Setting (2.2) to zero, we arrive at the equation

$$\mathrm{e}^{2\mathrm{i}\tau\omega} = -\frac{W_2}{W_1} = \left(\frac{1-\mathrm{i}\omega}{1+\mathrm{i}\omega}\right)^2 =: b(\omega). \tag{2.3}$$

The eigenvalues of $W_\tau(a)$ are just the numbers $\lambda = a(z)$ resulting from the solutions of the equation $\mathrm{e}^{2\mathrm{i}\tau z} = b(z)$. Since $|b(z)| = 1$ for all z, all solutions of this equation are real numbers, and we therefore write x instead of z. Specifically, the eigenvalues are $\lambda = a(x)$, where x satisfies one of the equations

$$x = \frac{1}{2\tau}\arg b(x) + \frac{k\pi}{\tau}, \quad k \in \mathbb{Z}. \tag{2.4}$$

A moment's thought reveals that

$$\arg b(x) = 2m\pi - 4\arctan x.$$

Fix $m = m_0$ and insert the corresponding argument of b in (2.4). Denote the right-hand side of (2.4) by $\Psi_{k,\tau}(x)$. We then have

$$|\Psi_{k,\tau}(x) - \Psi_{k,\tau}(y)| = \frac{2}{\tau}|\arctan x - \arctan y| \leq \frac{2}{\tau}|x-y|$$

for all $x, y \in \mathbb{R}$, which shows that if $\tau > 2$, then $\Psi_{k,\tau}$ is a contractive map of $\mathbb{R}$ into itself. Hence, each of the equations (2.4) has a unique solution. The set of solutions may be denoted by $\{x_{k,\tau}\}_{k\in\mathbb{Z}}$. In fact, this set does not depend on m_0: see Figure 2.1. The set $\{x_{k,\tau}\}_{k\in\mathbb{Z}}$ is symmetric about the origin, and hence the eigenvalues of $W_\tau(a)$ are the (real) numbers $\lambda_{k,\tau} = a(x_{k,\tau})$ obtained from taking all $x_{k,\tau}$ with $x_{k,\tau} > 0$.

In the general case, things are more elaborate. We will again arrive at an equation $z = \Psi_{k,\tau}(z)$ with a map $\Psi_{k,\tau}$ that is contractive for sufficiently large τ. However, z is then located in a complex domain which, moreover, may not contain all of $\mathbb{R}$. The problem then is to find appropriate closed subsets of $\mathbb{C}$, the complete metric spaces required by Banach's fixed point theorem, so that $\Psi_{k,\tau}$ maps these subsets into themselves. We will construct small rectangles $S_{k,\tau}$ which do that job. In the special case at hand, these rectangles collapse to line segments on $\mathbb{R}$ and the general construction that will follow below amounts to the following.

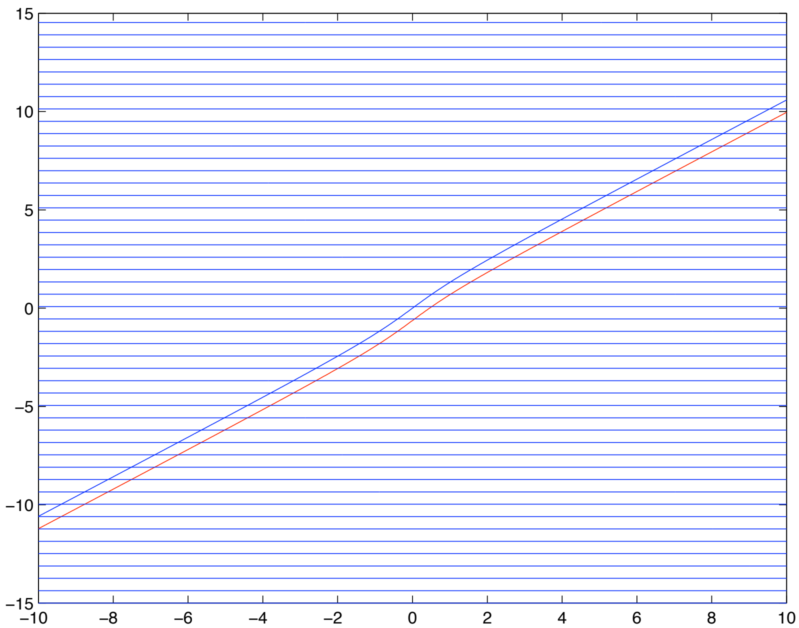

Figure 2.1: We took $\tau = 5$. The horizontal lines are $y = k\pi/\tau$, the upper curve is the graph of $y = x + (2/\tau)\arctan x$ (corresponding to $m = 0$), and the lower curve is the graph of $y = x + (2/\tau)\arctan x - \pi/\tau$ (corresponding to $m = 1$).

Let I_1 be the open interval $I_1 := (0, 2/3)$ and choose $\arg b(x) = -4\arctan x$ for $x \in I_1$. Note that then $\arg b(x)$ lies in $(-\pi, \pi)$ and is separated from the boundary points $\pm\pi$ as x ranges over I_1. Let $\mathcal{K}_\tau(I_1)$ denote the set of all integers k for which the closed line segments

$$I_{k,\tau} := \left[\left(k - \frac{1}{2}\right)\frac{\pi}{\tau}, \left(k + \frac{1}{2}\right)\frac{\pi}{\tau}\right] \tag{2.5}$$

are contained in I_1. For each $k \in \mathcal{K}_\tau(I_1)$, consider equation (2.4). Since $\arg b(x)$ is in $(-\pi, \pi)$, it follows that

$$\frac{1}{2\tau}\arg b(x) + \frac{k\pi}{\tau} \in \left(\left(k - \frac{1}{2}\right)\frac{\pi}{\tau}, \left(k + \frac{1}{2}\right)\frac{\pi}{\tau}\right).$$

Thus, $\Psi_{k,\tau}$ maps $I_{k,\tau}$ into itself. Banach's fixed point theorem now yields a set $X_1 := \{x_{k,\tau,1}\}$ of solutions of equation (2.4). Next, let $I_2 := (1/3, 3)$ and choose $\arg b(x) = 2\pi - 4\arctan x$ for $x \in I_2$. This time $\arg b(I_2)$ is contained in a closed subset of $(0, 2\pi)$, we denote by $\mathcal{K}_\tau(I_2)$ the integers k for which

$$I_{k,\tau} := \left[k\frac{\pi}{\tau}, (k+1)\frac{\pi}{\tau}\right]$$

is contained in I_2, and consider equation (2.4) for $k \in \mathcal{K}_\tau(I_2)$. Our choice of the argument guarantees that $\Psi_{k,\tau}(I_{k,\tau}) \subset I_{k,\tau}$ and we again have recourse to Banach's fixed point theorem. We obtain a second set $X_2 := \{x_{k,\tau,2}\}$ of solutions. Finally, for x in $I_3 := (2, \infty)$ we proceed as for $x \in I_1$, that is, take $\arg b(x) = -4\arctan x$, consider equation (2.4) for k in the set $\mathcal{K}_\tau(I_3)$ of all integers for which the segments (2.5) are subsets of I_3, and eventually arrive at a third set $X_3 := \{x_{k,\tau,3}\}$ of solutions of (2.4). Note that the sets X_i and X_j coincide on $I_i \cap I_j$,

$$X_1 \cap I_1 \cap I_2 = X_2 \cap I_1 \cap I_2, \qquad X_2 \cap I_2 \cap I_3 = X_3 \cap I_2 \cap I_3.$$

The set of eigenvalues of $W_\tau(a)$ is $a(X_1 \cup X_2 \cup X_3)$.

Let us now turn to the eigenfunctions. Suppose that $\lambda = a(\omega)$ is an eigenvalue of $W_\tau(a)$. We start with the initial guess $\varphi_\tau(t) = c_1 \mathrm{e}^{\mathrm{i}\omega t} + c_2 \mathrm{e}^{-\mathrm{i}\omega t}$ and consider the equation

$$(1-\lambda)\varphi_\tau(t) + \int_0^\tau \mathrm{k}(t-s)\varphi_\tau(s)\,\mathrm{d}s = 0 \tag{2.6}$$

with k given by (2.1). Evaluating the integrals $\int_0^\tau \mathrm{k}(t-s)\mathrm{e}^{\pm\omega s}\,\mathrm{d}s$ and writing

$$\lambda = \frac{\omega^2}{\omega^2+1} = 1 - \frac{1}{2}\frac{1}{1+\mathrm{i}\omega} - \frac{1}{2}\frac{1}{1-\mathrm{i}\omega},$$

equation (2.6) becomes

$$\left(\frac{1}{1+\mathrm{i}\omega}c_1 + \frac{1}{1-\mathrm{i}\omega}c_2\right)\mathrm{e}^{-t} + \left[\frac{\mathrm{e}^{(-1+\mathrm{i}\omega)\tau}}{1-\mathrm{i}\omega}c_1 + \frac{\mathrm{e}^{(-1-\mathrm{i}\omega)\tau}}{1+\mathrm{i}\omega}c_2\right]\mathrm{e}^{t} = 0.$$

The left-hand side is identically zero if and only if

$$\begin{pmatrix} \dfrac{1}{1+\mathrm{i}\omega} & \dfrac{1}{1-\mathrm{i}\omega} \\ \dfrac{\mathrm{e}^{\mathrm{i}\omega\tau}}{1-\mathrm{i}\omega} & \dfrac{\mathrm{e}^{-\mathrm{i}\omega\tau}}{1+\mathrm{i}\omega} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.7}$$

The determinant of the matrix is

$$\frac{\mathrm{e}^{-\mathrm{i}\omega\tau}}{(1+\mathrm{i}\omega)^2} - \frac{\mathrm{e}^{\mathrm{i}\omega\tau}}{(1-\mathrm{i}\omega)^2},$$

and this is zero exactly when (2.3) holds. As equation (2.3) is satisfied when $\lambda = a(\omega)$ is an eigenvalue, the set of solutions to (2.7) is a one-dimensional space. Taking the solution $c_1 = 1+\mathrm{i}\omega$, $c_2 = -(1-\mathrm{i}\omega)$ we get the eigenfunction

$$\varphi_\tau(t) = (1+\mathrm{i}\omega)\mathrm{e}^{\mathrm{i}\omega t} - (1-\mathrm{i}\omega)\mathrm{e}^{-\mathrm{i}\omega t}. \tag{2.8}$$

Now note that equation (2.3) is equivalent to the satisfaction of one of the two equations

$$\mathrm{e}^{\mathrm{i}\tau\omega} = \frac{1-\mathrm{i}\omega}{1+\mathrm{i}\omega}, \qquad \mathrm{e}^{\mathrm{i}\tau\omega} = -\frac{1-\mathrm{i}\omega}{1+\mathrm{i}\omega}. \tag{2.9}$$

Letting $\theta := -1$ if the first equation in (2.9) is satisfied and $\theta := 1$ if the second holds, we get

$$\frac{1-\mathrm{i}\omega}{1+\mathrm{i}\omega} = -\theta\mathrm{e}^{\mathrm{i}\omega\tau} \tag{2.10}$$

and may rewrite (2.8) in the form

$$\varphi_\tau(t) = (1+\mathrm{i}\omega)\Big[\mathrm{e}^{\mathrm{i}\omega t} + \theta\mathrm{e}^{\mathrm{i}\omega\tau}\mathrm{e}^{-\mathrm{i}\omega t}\Big].$$

Thus, ignoring scalar multiples, we finally may represent the eigenfunction (2.8) also in the form

$$\varphi_\tau(t) = \begin{cases} \cos\left(\omega\left(t-\frac{\tau}{2}\right)\right) & \text{for} \quad \theta = 1, \\ \sin\left(\omega\left(t-\frac{\tau}{2}\right)\right) & \text{for} \quad \theta = -1. \end{cases} \tag{2.11}$$

We remark that, after replacing ω by x, the first and second equations in (2.9) are in turn equivalent to equation (2.4) with odd and even k, respectively. In other terms, $\theta = 1$ corresponds to even k in (2.4) while $\theta = -1$ originates in an odd k in (2.4).

3 The main results

Consider the symbol a given by (1.2) with $\zeta_j \in \mathbb{C}$, $\mu_j \in \mathbb{C}$, $\operatorname{Re}\mu_j > 0$, and $-\zeta_j^2 \neq \mu_k^2$ for all j,k. Throughout what follows we assume that the set $\mathcal{R}(a)$ has no self-intersections (which means in particular that $a(0) \neq a(\infty) = 1$), that $a'(x) \neq 0$ for $0 < x < \infty$, and that each of the equations $a(z) = a(0)$ and $a(z) = 1$ has $2r - 2$ zeros in $\mathbb{C} \setminus \mathbb{R}$. The latter requirement is equivalent to the stipulation that the second derivatives of $a(x)$ and $a(1/x)$ are nonzero for $x = 0$.

Our assumptions imply that for each $\lambda \in \mathcal{R}(a) \setminus \{1\}$ the equation $a(z) = \lambda$ has exactly $2r$ complex roots

$$\omega_1(\lambda), \omega_2(\lambda), \ldots, \omega_r(\lambda), -\omega_1(\lambda), -\omega_2(\lambda), \ldots, -\omega_r(\lambda). \tag{3.1}$$

We may label the zeros so that $\omega_1(\lambda) \in \mathbb{R}$ and $\operatorname{Im}\omega_j(\lambda) > 0$ for $j \geq 2$. Furthermore, we denote by $\pm\omega_j(1)$ $(j \geq 2)$ the roots of the equation $a(z) = 1$, labelled again so that $\operatorname{Im}\omega_j(1) > 0$. Clearly, there is a compact set Ω in the open upper half-plane such that $\omega_j(\lambda) \in \Omega$ for all $\lambda \in \mathcal{R}(a)$ and all $j \geq 2$. We may also assume that the points $\mathrm{i}\mu_1, \ldots, \mathrm{i}\mu_r$ belong to Ω. Note there exist constants $\delta > 0$ and $C < \infty$ such that

$$\operatorname{Im}\omega_j(\lambda) > 4\delta, \quad |\omega_j(\lambda)| < C/2$$

for $\lambda \in \mathcal{R}(a)$ and $j \geq 2$.

Now take λ from an open neighbourhood $U \subset \mathbb{C}$ of $\mathcal{R}(a)$ and suppose that λ is not equal to 1. Then the equation $a(z) = \lambda$ has again the roots (3.1). Labelling the roots appropriately and choosing U small enough, we can guarantee that

$$\operatorname{Im}\omega_j(\lambda) > 3\delta, \quad |\omega_j(\lambda)| < C \tag{3.2}$$

for $\lambda \in U$ and $j \geq 2$ and that

$$|\operatorname{Im}\omega_1(\lambda)| < \delta \quad \text{or} \quad |\omega_1(\lambda)| > 2C \tag{3.3}$$

for $\lambda \in U \setminus \{1\}$. Note also that if $\omega_2(\lambda_0), \ldots, \omega_r(\lambda_0)$ are distinct, then, again under appropriate labelling, $\omega_2(\lambda), \ldots, \omega_r(\lambda)$ depend analytically on λ in an open neighbourhood of λ_0.

Proposition 3.1 *If the points $\omega_2(1), \ldots, \omega_r(1)$ are distinct then there exist an open neighbourhood $U_1 \subset \mathbb{C}$ of the point 1 and a number τ_1 such that $|\operatorname{Im}\omega_1(\lambda)| < \delta$ for all $\lambda \in (U_1 \setminus \{1\}) \cap \operatorname{sp} W_\tau(a)$ with $\tau > \tau_1$.*

Thus, once the hypothesis of Proposition 3.1 is satisfied, U may be chosen so small that instead of (3.3) we have

$$|\operatorname{Im}\omega_1(\lambda)| < \delta \tag{3.4}$$

for all $\lambda \neq 1$ in $U \cap \operatorname{sp} W_\tau(a)$ with $\tau > \tau_1$. Since $\operatorname{sp} W_\tau(a) \subset U$ for all $\tau > \tau'$, we conclude that, under the hypothesis of Proposition 3.1 and for $\tau > \tau'' := \max(\tau', \tau_1)$, all eigenvalues λ of $W_\tau(a)$ lie in U and are of the form $\lambda = a(z)$ with $|\operatorname{Im} z| < \delta$. To state it differently, the eigenvalues near the point 1 are not just close to $\mathcal{R}(a)$, they are *very* close to $\mathcal{R}(a)$.

Theorem 3.2 *If the points $\omega_2(a(0)), \ldots, \omega_r(a(0))$ are distinct, then there is a number τ_2 such that for $\tau > \tau_2$ the operator $W_\tau(a)$ has no eigenvalues $\lambda \in U$ with $|\operatorname{Re}\omega_1(\lambda)| \leq \pi/(2\tau)$.*

The hypothesis of Theorem 3.2 ensures in particular that $\lambda = a(0)$ is not an eigenvalue for all sufficiently large τ. Moreover, since we may assume *a priori* that all eigenvalues lie in U, it follows that all eigenvalues may be sought in the form $\lambda = a(z)$ with $\operatorname{Re} z > \pi/(2\tau)$. Note that $z = \omega_1(\lambda)$, provided that $\omega_1(\lambda)$ is chosen to have positive real part.

Fix an open neighborhood $U \subset \mathbb{C}$ of $\mathcal{R}(a)$. Then $\operatorname{sp} W_\tau(a) \subset U$ for all sufficiently large τ. Let $\Pi = \{z \in \mathbb{C} : |\operatorname{Im} z| < \delta, a(z) \in U\}$. For $z \in \Pi$ consider the two functions

$$Q(z) := \prod_{\ell=1}^{r} (z - \mathrm{i}\mu_\ell),$$
$$P(z) := \prod_{\ell=2}^{r} [z - \omega_\ell(a(z))]$$

and set

$$b(z) := \frac{Q(-z)^2}{Q(z)^2} \cdot \frac{P(z)^2}{P(-z)^2}.$$

Using (3.2) and (3.3) and taking into account that $\operatorname{Im}(\mathrm{i}\mu_j) > 4\delta$, it is easily seen that there exists a constant $\beta \in (0, \infty)$ such that

$$\mathrm{e}^{-\beta} \leq |b(z)| \leq \mathrm{e}^{\beta} \tag{3.5}$$

for all $z \in \Pi$.

Now let $I \subset (0, \infty)$ be an open interval and suppose there exists a $\sigma > 0$ such that either

$$b(I) \subset \{z \in \mathbb{C} : -\pi + 2\sigma < \arg z < \pi - 2\sigma\} \tag{3.6}$$

or

$$b(I) \subset \{z \in \mathbb{C} : 2\sigma < \arg z < 2\pi - 2\sigma\}. \tag{3.7}$$

Since $b(z) \to 1$ as $|z| \to \infty$, there is an $x_0 > 0$ such that (3.6) is satisfied for $I = (x_0, \infty)$. Analogously, because $b(0) = 1$, we have case (3.6) whenever $I = (0, y_0)$ for some sufficiently small $y_0 > 0$. Clearly, if the length of I is sufficiently small, then always one of (3.6) or (3.7) holds. We can therefore cover $(0, \infty)$ by finitely many open intervals I such that either (3.6) or (3.7) is in force for each I. In the case of (3.6) we let $\mathcal{K}_\tau(I)$ denote the set of all integers $k \geq 1$ for which

$$I_{k,\tau} := \left[\left(k - \frac{1}{2}\right)\frac{\pi}{\tau}, \left(k + \frac{1}{2}\right)\frac{\pi}{\tau}\right] \subset I,$$

while if (3.7) is valid then we denote by $\mathcal{K}_\tau(I)$ the integers $k \geq 1$ for which

$$I_{k,\tau} := \left[k\frac{\pi}{\tau}, (k+1)\frac{\pi}{\tau}\right] \subset I.$$

If both (3.6) and (3.7) are true then we make a choice in favour of either of the two possibilities. For $k \in \mathcal{K}_\tau(I)$, let $S_{k,\tau}$ be the rectangle

$$S_{k,\tau} := \{z \in \mathbb{C} : \operatorname{Re} z \in I_{k,\tau}, |\operatorname{Im} z| \leq \beta/\tau\}.$$

Clearly, there exists a τ''' such that if $\tau > \tau'''$ then

$$b(S_{k,\tau}) \subset \{z \in \mathbb{C} : -\pi + \sigma < \arg z < \pi - \sigma\}$$

for all $k \in \mathcal{K}_\tau(I)$, provided that (3.6) holds and

$$b(S_{k,\tau}) \subset \{z \in \mathbb{C} : \sigma < \arg z < 2\pi - \sigma\}$$

for all $k \in \mathcal{K}_\tau(I)$ if (3.7) is satisfied. Finally, for $z \in S_{k,\tau}$, define $\log b(z)$ as $\log|b(z)| + \mathrm{i}\arg b(z)$ with $\arg b(z)$ in $(-\pi + \sigma, \pi - \sigma)$ in the case (3.6) and in $(\sigma, 2\pi - \sigma)$ in the case (3.7).

Herewith our main result.

Theorem 3.3 *Let* $\operatorname{clos} I$ *be the closure of* I *in* $[0,\infty]$ *and suppose that for* λ *in* $a(\operatorname{clos} I)$ *the roots* $\omega_2(\lambda),\dots,\omega_r(\lambda)$ *are distinct. Then there exists a* τ_0 *such that the following is true for every* $\tau > \tau_0$.

(a) *If* $\lambda = a(z) \in U$ *is an eigenvalue of* $W_\tau(a)$ *such that* $\operatorname{Re} z \in I_{k,\tau}$ *for some* $k \in \mathcal{K}_\tau(I)$, *then* $z \in S_{k,\tau}$.

(b) *For each* $k \in \mathcal{K}_\tau(I)$, *the set* $a(S_{k,\tau})$ *contains exactly one eigenvalue* $\lambda_{k,\tau}$ *of the operator* $W_\tau(a)$. *The algebraic multiplicity of this eigenvalue is* 1.

(c) *The function*

$$\Phi_{k,\tau}(z) := \frac{k\pi}{\tau} + \frac{1}{2\mathrm{i}\tau}\log b(z)$$

is a contractive map of $S_{k,\tau}$ *into itself and, letting*

$$z^{(0)}_{k,\tau} := \frac{k\pi}{\tau}, \quad z^{(n)}_{k,\tau} := \Phi_{k,\tau}(z^{(n-1)}_{k,\tau}) \qquad (n \geq 1),$$

we have

$$\lambda_{k,\tau} = a(z^{(n)}_{k,\tau}) + O(1/\tau^{n+1}) \text{ as } \tau \to \infty$$

uniformly in $k \in \mathcal{K}_\tau(I)$, *that is, there exist constants* $C_n < \infty$ *independent of* k *and* τ *such that*

$$|\lambda_{k,\tau} - a(z^{(n)}_{k,\tau})| \leq C_n/\tau^{n+1}$$

for all $\tau > \tau_0$ *and all* $k \in \mathcal{K}_\tau(I)$.

We remark that we have to work with intervals I instead of $(0,\infty)$ for two reasons. The first is that restricting the function b to I allows us to choose an argument of b with properties that enable the application of Banach's fixed point theorem: we have encountered this issue already in Section 2. The second reason is our assumption that the roots $\omega_2(\lambda),\dots,\omega_r(\lambda)$ are distinct. We did not encounter this problem in the special case studied in Section 2. In general, however, there may occur multiple roots, but this will happen at isolated points only. Theorem 3.3 is then applicable to intervals I whose closure does not contain those isolated points.

Corollary 3.4 *If the points* $\omega_2(1),\dots,\omega_r(1)$ *are distinct then* $W_\tau(a)$ *has infinitely many eigenvalues for every sufficiently large* τ.

Indeed, in that case Theorem 3.3(b) is applicable with $I = (x_0,\infty)$ for some sufficiently large $x_0 > 0$. We just quote this corollary because we don't know any rigorous argument that would imply that the Fox–Li operator has infinitely many eigenvalues.

The first three iterations in Theorem 3.3(c) give

$$z_{k,\tau} = z_0 + \frac{c_1(z_0)}{2\mathrm{i}\tau} + \frac{c_2(z_0)}{(2\mathrm{i}\tau)^2} + \frac{c_3(z_0)}{(2\mathrm{i}\tau)^3} + O\left(\frac{1}{\tau^4}\right)$$

with $z_0 := z^{(0)}_{k,\tau} := k\pi/\tau$ and

$$c_1(z_0) = \log b(z_0), \qquad c_2(z_0) = \frac{b'(z_0)}{b(z_0)}\log b(z_0),$$

$$c_3(z_0) = \frac{b'(z_0)^2}{b(z_0)^2}\log b(z_0) + \frac{b''(z_0)b(z_0) - b'(z_0)^2}{2b(z_0)^2}(\log b(z_0))^2.$$

Accordingly, $\lambda_{k,\tau}$ equals

$$\begin{aligned} a(z_0) &+ \frac{1}{2\mathrm{i}\tau}a'(z_0)c_1(z_0) + \frac{1}{(2\mathrm{i}\tau)^2}\left[a'(z_0)c_2(z_0) + \frac{a''(z_0)}{2}c_1(z_0)^2\right] \\ &+ \frac{1}{(2\mathrm{i}\tau)^3}\left[a'(z_0)c_3(z_0) + a''(z_0)c_1(z_0)c_2(z_0) + \frac{a'''(z_0)}{6}c_1(z_0)^3\right] + O\left(\frac{1}{\tau^4}\right). \end{aligned} \tag{3.8}$$

If a is real valued, which occurs if and only if $\mathrm{k}(t) = \overline{\mathrm{k}(-t)}$ for all t, then $W_\tau(a)$ is a selfadjoint operator. In this case $|b(x)| = 1$ for $x \in \mathbb{R}$, hence the function $\Phi_{k,\tau}$ in Theorem 3.3(c) maps $I_{k,\tau}$ into itself and becomes

$$\Phi_{k,\tau}(x) = \frac{k\pi}{\tau} + \frac{1}{2\tau} \arg b(x)$$

for $x \in I_{k,\tau}$. It follows in particular that all eigenvalues are real, as they should be for a selfadjoint operator.

The first approximation in (3.8) gives

$$\begin{aligned}\lambda_{k,\tau} &= a(z_0) + \frac{1}{2\mathrm{i}\tau} a'(z_0) \log b(z_0) + O(1/\tau^2) \\ &= a(z_0) + \frac{1}{2\tau} a'(z_0) \arg b(z_0) - \frac{\mathrm{i}}{2\tau} a'(z_0) \log |b(z_0)| + O(1/\tau^2).\end{aligned}$$

The tangent to $\mathcal{R}(a)$ through $a(z_0)$ has the parametric representation $\lambda = a(z_0) + a'(z_0)t$, $t \in \mathbb{R}$, and increasing values of the parameter t provide the tangent with an orientation. The point $a(z_0) + (1/2\tau)a'(z_0) \arg b(z_0)$ lies on this tangent. It follows that, up to the $O(1/\tau^2)$ term, the eigenvalue $\lambda_{k,\tau}$ is located on the right of the tangent if $|b(z_0)| > 1$, while $\lambda_{k,\tau}$ is on the left of the tangent if $|b(z_0)| < 1$.

Figures 3.1 and 3.2 illustrate Theorem 3.3 by an example. The symbol a is given by (7.1). Note that b starts at the point 1 and is first inside the unit circle when tracing out the curve $\mathcal{R}(b)$ clockwise. The corresponding eigenvalues of $W_\tau(a)$ are accordingly on the left of the curve $\mathcal{R}(a)$. Eventually b enters the exterior of the unit circle and stays there, which implies that the eigenvalues of $W_\tau(a)$ move to the right of the curve $\mathcal{R}(a)$. As the latter eigenvalues are extremely close to $\mathcal{R}(a)$, this is almost not visible at the resolution of the plot.

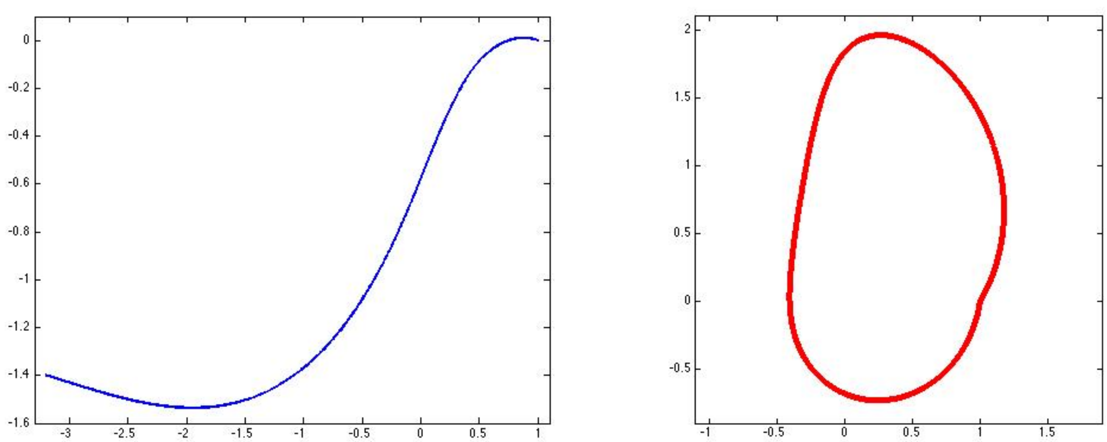

Figure 3.1: The range $\mathcal{R}(a)$ is indicated on the left, while the range of b on $(0, \infty)$ is indicated on the right. The latter is traced out clockwise, starting and terminating at 1.

The following result represents a version of Theorem 3.3 that avoids the use of the intervals $I_{k,\tau}$. In contrast to Theorem 3.3, we now let $\log b$ denote any logarithm of b which is continuous on Π; note that such logarithms exist because $b(0) = 1$ and $b(z) \to 1$ as $|z| \to \infty$.

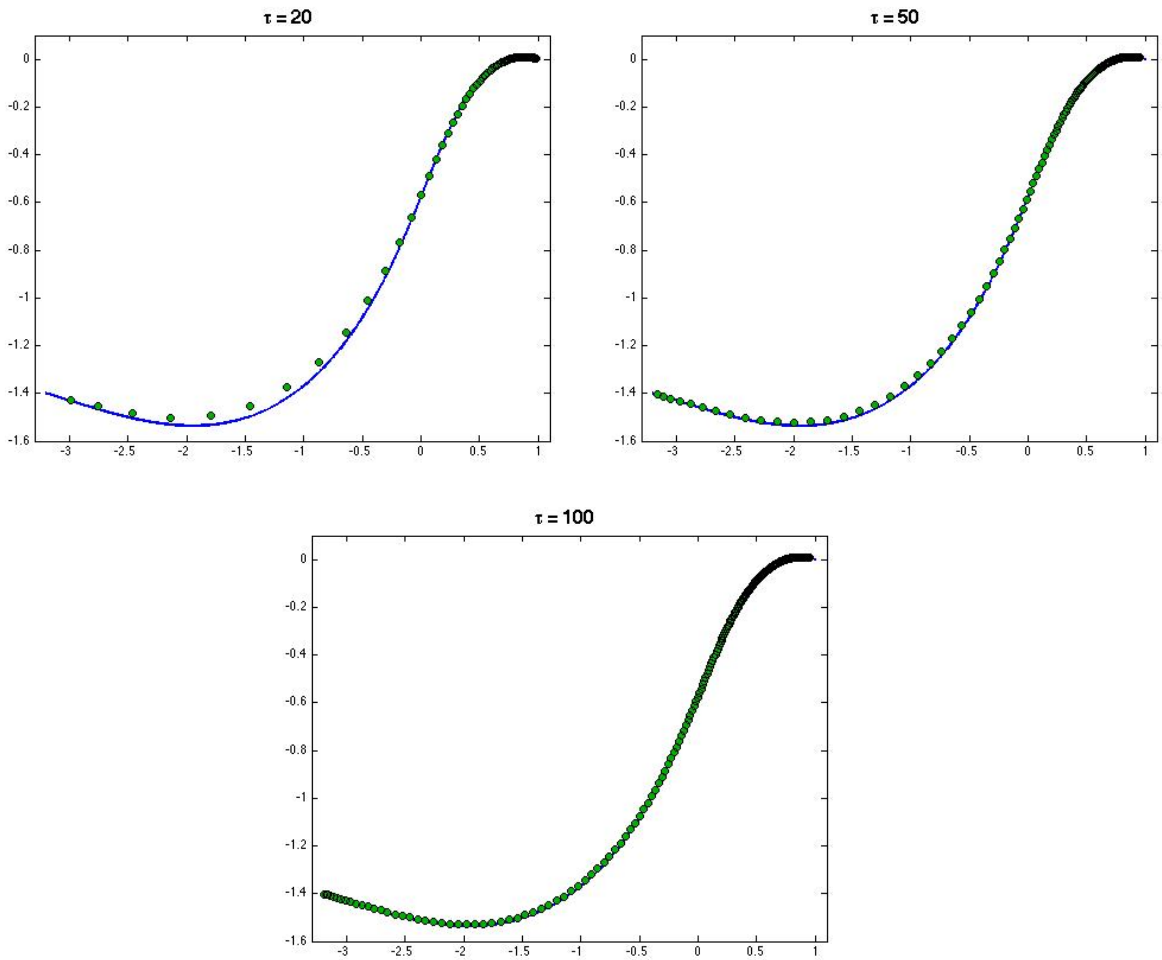

Figure 3.2: The eigenvalues, denoted by small discs and overlaid on $\mathcal{R}(a)$, for $\tau = 20, 50, 100$.

Theorem 3.5 *Let the hypothesis of Theorem 3.3 be satisfied and choose an open neighbourhood $V \subset U$ of $a(\mathrm{clos}\, I)$ such that the roots $\omega_2(\lambda), \dots, \omega_r(\lambda)$ are distinct for $\lambda \in V$. Then there exists a τ_0 such that the following is true for every $\tau > \tau_0$. The eigenvalues of $W_\tau(a)$ belonging to V are given by $\lambda_{k,\tau} = a(z_{k,\tau})$ where $k_1 \le k \le k_2$ and $k_1, k_2 \ge 1$ depend on τ and V. Moreover $|\mathrm{Im}\, z_{k,\tau}| < \beta/\tau$ and $\mathrm{Re}\, z_{k,\tau} < \mathrm{Re}\, z_{k+1,\tau}$ for all k. Defining $z_{k,\tau}^{(n)}$ as in Theorem 3.3(c), we have*

$$\lambda_{k,\tau} = a(z_{k,\tau}^{(n)}) + O(1/\tau^{n+1}) \;\; \text{as} \;\; \tau \to \infty$$

uniformly in k.

We remark that if the roots $\omega_1(\lambda), \dots, \omega_r(\lambda)$ are distinct for all $\lambda \in \mathcal{R}(a)$, so that Theorem 3.3 may be employed with $I = (0, \infty)$, then Theorem 3.5 holds with $k_1 \le k \le k_2$ replaced by $1 \le k < \infty$.

Finally, a result on eigenfunctions.

Theorem 3.6 *Suppose that the numbers $\mu_1, \dots, \mu_r$ are distinct. Let λ be an eigenvalue of $W_\tau(a)$ and assume that the roots $\omega_2(\lambda), \dots, \omega_r(\lambda)$ are distinct. Then every eigenfunction $\varphi_\tau \in$*

$L^2(0,\tau)$ *of* $W_\tau(a)$ *corresponding to* λ *is of the form*

$$\varphi_\tau(t) = \sum_{j=1}^{r} \left[c_j \mathrm{e}^{\mathrm{i}\omega_j(\lambda)t} + c_{r+j}\mathrm{e}^{-\mathrm{i}\omega_j(\lambda)t} \right], \tag{3.9}$$

satisfies $\varphi_\tau(\tau - t) = \theta\varphi_\tau(t)$ *for all* $t \in (0,\tau)$ *with* $\theta \in \{\pm 1\}$, *and can be rewritten in the form*

$$\varphi_\tau(t) = \begin{cases} \displaystyle\sum_{j=1}^{r} 2c_j \mathrm{e}^{\mathrm{i}\omega_j(\lambda)\tau/2} \cos\left(\omega_j(\lambda)\left(t - \frac{\tau}{2}\right)\right) & \text{for} \quad \theta = 1, \\ \displaystyle\sum_{j=1}^{r} 2\mathrm{i}c_j \mathrm{e}^{\mathrm{i}\omega_j(\lambda)\tau/2} \sin\left(\omega_j(\lambda)\left(t - \frac{\tau}{2}\right)\right) & \text{for} \quad \theta = -1. \end{cases}$$

The coefficients c_j can be computed from the linear algebraic system that arises after inserting ansatz (3.9) in equation (2.6). We demonstrated this in Section 2 for the kernel k given by (2.1).

4 The Wiener–Hopf determinant

Let $U \subset \mathbb{C}$ be a sufficiently small open neighbourhood of $\mathcal{R}(a)$ and take a point $\lambda \in U \setminus \{a(0), 1\}$ such that the roots $\omega_2(\lambda), \ldots, \omega_r(\lambda)$ are all distinct. We then have

$$\frac{a(x) - \lambda}{1 - \lambda} = \frac{(x - \xi_1(\lambda)) \ldots (x - \xi_{2r}(\lambda))}{(x^2 + \mu_1^2) \ldots (x^2 + \mu_r^2)} = \prod_{j=1}^{r} \frac{x^2 - \omega_j(\lambda)^2}{x^2 + \mu_j^2}.$$

Thus, $\xi_1(\lambda), \ldots, \xi_{2r}(\lambda)$ are simply the roots $\pm\omega_1(\lambda), \ldots, \pm\omega_r(\lambda)$ labelled in a different manner. In what follows we will frequently abbreviate $\xi_j(\lambda)$ and $\omega_j(\lambda)$ to ξ_j and ω_j. From (3.2) and (3.3) we see that $\omega_1 \neq 0$ and that $\pm\omega_1$ cannot coincide with any of the roots $\pm\omega_2, \ldots, \pm\omega_r$. This shows that the roots $\xi_1, \ldots, \xi_{2r}$ are distinct. Proposition 2.4 of [2] tells us that $W_\tau((a - \lambda)/(1 - \lambda))$ is the identity plus a trace class operator, and Theorem 5.1 of [2] shows that

$$\det W_\tau \left(\frac{a - \lambda}{1 - \lambda} \right) = \mathrm{e}^{\kappa\tau} \sum_M W_M \mathrm{e}^{w_M \tau} \tag{4.1}$$

where $\kappa = \kappa(\lambda)$ is some constant, the sum is over all subsets $M \subset \{\xi_1, \ldots, \xi_{2r}\}$ of cardinality r, and, with $M^c := \{\xi_1, \ldots, \xi_{2r}\} \setminus M$ and $R := \{\mu_1, \ldots, \mu_r\}$,

$$w_M := \sum_{\xi_j \in M^c} \mathrm{i}\xi_j,$$

$$W_M := \frac{\prod_{\xi_j \in M^c, \mu_m \in R} (\mathrm{i}\xi_j + \mu_m) \prod_{\mu_\ell \in R, \xi_k \in M} (\mu_\ell - \mathrm{i}\xi_k)}{\prod_{\mu_\ell \in R, \mu_m \in R} (\mu_\ell + \mu_m) \prod_{\xi_j \in M^c, \xi_k \in M} (\mathrm{i}\xi_j - \mathrm{i}\xi_k)}.$$

The point λ belongs to $\operatorname{sp} W_\tau(a)$ if and only if (4.1) is zero, whereby its algebraic multiplicity is its multiplicity as a zero of (4.1). The dominant terms in (4.1) are those for which

$$\operatorname{Im} w_M = \sum_{\xi_j \in M^c} \operatorname{Im} \xi_j \tag{4.2}$$

is minimal. Recall that (3.2) and (3.3) are valid and that $\operatorname{Re} \mu_j > 4\delta$, $|\mu_j| < C$ for $j \geq 2$.

Proof of Proposition 3.1. Subject to our assumption, there exists an open neighbourhood $U_1 \subset \mathbb{C}$ of 1 such that $\omega_2(\lambda), \ldots, \omega_r(\lambda)$ are distinct for $\lambda \in U_1$. Take $\lambda \in U_1 \setminus \{1\}$ and suppose that $\lambda \in \operatorname{sp} W_\tau(a)$. We may assume that $\lambda \neq a(0)$. Because of (3.3), it remains to show that the two inequalities $|\omega_1(\lambda)| > 2C$ and $|\operatorname{Im} \omega_1(\lambda)| \geq \delta$ cannot hold simultaneously if τ is large enough. So assume that $|\omega_1(\lambda)| > 2C$ and, for the sake of definiteness, $\operatorname{Im} \omega_1(\lambda) \geq \delta$. Note that (4.1) holds.

If $M_0^c = \{-\omega_1, -\omega_2, \ldots, -\omega_r\}$, then the number $\operatorname{Im} w_{M_0}$ given by (4.2) takes a certain negative value, and changing any of the minus signs in M_0^c to plus signs results in a value of (4.2) that is by at least 2δ larger than the value of $\operatorname{Im} w_{M_0}$. Suppose for a moment that $W_{M_0}^{-1} W_M = O(1)$ for $M \neq M_0$. Then (4.1) yields

$$\det W_\tau \left(\frac{a - \lambda}{1 - \lambda} \right) = \mathrm{e}^{\kappa\tau} W_{M_0} \mathrm{e}^{\mathrm{i} w_{M_0} \tau} \left(1 + O(\mathrm{e}^{-2\delta\tau})\right),$$

and hence λ cannot be in $\operatorname{sp} W_\tau(a)$ for all τ larger than some τ_1, which is the desired contradiction.

It remains to prove that $W_{M_0}^{-1} W_M = O(1)$ for $M \neq M_0$. We put

$$\nu := \prod_{\mu_\ell \in R, \mu_m \in R} (\mu_\ell + \mu_m).$$

Note that ν is a nonzero constant. In what follows we use abbreviations like

$$\prod_{\mu_m \in R} (\ldots + \mu_m) =: \prod (\ldots + \mu_m), \qquad \prod_{\mu_\ell \in R} (\mu_\ell - \ldots) =: \prod (\mu_\ell - \ldots)$$

whenever the range of the product is clear from the context. We have $W_{M_0} = A_{M_0}(\lambda) B_{M_0}(\lambda)$ where

$$A_{M_0}(\lambda) := \frac{\prod(-\mathrm{i}\omega_1 + \mu_m) \prod(\mu_\ell - \mathrm{i}\omega_1)}{-2\mathrm{i}\omega_1 \prod_{k \geq 2}(-\mathrm{i}\omega_1 - \mathrm{i}\omega_k) \prod_{j \geq 2}(-\mathrm{i}\omega_j - \mathrm{i}\omega_1)}, \tag{4.3}$$

$$B_{M_0}(\lambda) := \frac{\prod_{j \geq 2} \prod(-\mathrm{i}\omega_j + \mu_m) \prod_{k \geq 2} \prod(\mu_\ell - \mathrm{i}\omega_k)}{\nu \prod_{j,k \geq 2}(-\mathrm{i}\omega_j - \mathrm{i}\omega_k)}. \tag{4.4}$$

Since $|\omega_j + \omega_k| < 2C$, $\operatorname{Re}(-\mathrm{i}\omega_j + \mu_m) > 3\delta + 4\delta = 7\delta$, $\operatorname{Re}(\mu_\ell - \mathrm{i}\omega_k) > 7\delta$ for $j, k \geq 2$, there exists a constant $C_1 \in (0, \infty)$ such that

$$1/C_1 \leq |B_{M_0}(\lambda)|.$$

Taking into account the inequalities $|\mu_m| < C/2$, $|\mu_\ell| < C/2$, $|\omega_k| < C$ $(k \geq 2)$, $|\omega_j| < C$ $(j \geq 2)$, $|\omega_1| > 2C$ and that ω_1 occurs in the power $2r$ in the numerator and in the power $2r - 1$ in the denominator of $A_{M_0}(\lambda)$, we see that there exists a constant $C_2 \in (0, \infty)$ such that

$$|\omega_1|/C_2 \leq |A_{M_0}(\lambda)|.$$

We now turn to W_M. Suppose first that

$$M^c = \{-\omega_1, \xi_2, \ldots, \xi_r\}, \quad M = \{\omega_1, \eta_2, \ldots, \eta_r\},$$

where $\{\xi_2, \ldots, \xi_r, \eta_2, \ldots, \eta_r\} = \{\pm\omega_2, \ldots, \pm\omega_r\}$. Then $W_M = A_M(\lambda) B_M(\lambda)$ with

$$A_M(\lambda) := \frac{\prod(-\mathrm{i}\omega_1 + \mu_m) \prod(\mu_\ell - \mathrm{i}\omega_1)}{-2\mathrm{i}\omega_1 \prod(-\mathrm{i}\omega_1 - \mathrm{i}\eta_k) \prod(\mathrm{i}\xi_j - \mathrm{i}\omega_1)}, \tag{4.5}$$

$$B_M(\lambda) := \frac{\prod\prod(\mathrm{i}\xi_j + \mu_m) \prod\prod(\mu_\ell - \mathrm{i}\eta_k)}{\nu \prod\prod(\mathrm{i}\xi_j - \mathrm{i}\eta_k)}. \tag{4.6}$$

The number $\xi_j - \eta_k$ is of one of the six forms $2\omega_j$, $-2\omega_j$, $\omega_j + \omega_k$, $\omega_j - \omega_k$, $-\omega_j + \omega_k$, $-\omega_j - \omega_k$, where $j \neq k$. Clearly, $|\mathrm{Im}\,(2\omega_j)| > 6\delta$ and $|\mathrm{Im}\,(\omega_j + \omega_k)| > 6\delta$. Our assumption that the ω_j and ω_k are distinct implies that

$$\inf_{\lambda \in U_1} |\omega_j(\lambda) - \omega_k(\lambda)| > 0.$$

Consequently,

$$|B_M(\lambda)| \leq C_3$$

for some constant $C_3 < \infty$. Furthermore, as above when considering $A_{M_0}(\lambda)$, we deduce the existence of a constant $C_4 < \infty$ such that

$$|A_M(\lambda)| \leq C_4 |\omega_1|.$$

Combining the estimates we obtain

$$|W_{M_0}^{-1} W_M| \leq C_1 C_2 C_3 C_4.$$

Suppose next that

$$M^c = \{-\omega_1, \omega_1, \xi_3, \ldots, \xi_r\}, \quad M = \{\eta_1, \eta_2, \ldots, \eta_r\}$$

with $\{\xi_3, \ldots, \xi_r, \eta_1, \ldots, \eta_r\} = \{\pm\omega_2, \ldots, \pm\omega_r\}$. Then $W_M = A_M(\lambda) B_M(\lambda)$ where

$$A_M(\lambda) := \frac{\prod(-\mathrm{i}\omega_1 + \mu_m) \prod(\mathrm{i}\omega_1 + \mu_m)}{\prod(-\mathrm{i}\omega_1 - \mathrm{i}\eta_k) \prod(\mathrm{i}\omega_1 - \mathrm{i}\eta_k)},$$

$$B_M(\lambda) = \frac{\prod\prod(\mathrm{i}\xi_j + \mu_m) \prod\prod(\mu_\ell - \mathrm{i}\eta_k)}{\nu \prod\prod(\mathrm{i}\xi_j - \mathrm{i}\eta_k)},$$

and we conclude as above that

$$|B_M(\lambda)| \leq C_5, \quad |A_M(\lambda)| \leq C_6,$$

which results in the estimate

$$|W_{M_0}^{-1} W_M| \leq C_1 C_2 C_5 C_6 / |\omega_1| < C_1 C_2 C_5 C_6 / (2C).$$

The case in which $M^c = \{\xi_1, \xi_2, \ldots, \xi_r\}$ or $M^c = \{\omega_1, \xi_2, \ldots, \xi_r\}$ can be tackled similarly. $\square$

Let us now suppose that the hypothesis of Theorem 3.3 is satisfied. Then there exists an open neighbourhood $U \subset \mathbb{C}$ of $a(\mathrm{clos}\, I)$ such that $\omega_2(\lambda), \ldots, \omega_r(\lambda)$ are distinct for $\lambda \in U$. Take $\lambda \in U \setminus \{a(0), 1\}$. Then formula (4.1) is applicable. Clearly, (3.2) holds. If I is an infinite interval, then Proposition 3.1 shows that (3.4) is valid. In the case where I is a finite interval, we can obviously guarantee (3.4) by choosing U sufficiently small.

The two candidates for sets M with minimal values (4.2) are given by

$$M_1^c := \{-\omega_1, -\omega_2, \ldots, -\omega_r\}, \quad M_2^c := \{\omega_1, -\omega_2, \ldots, -\omega_r\},$$

because changing one $-\omega_j$ with $j \geq 2$ to ω_j yields an increase by at least 6δ. Formula (4.1) gives

$$\mathrm{e}^{-\kappa\tau} \det W_\tau \left(\frac{a - \lambda}{1 - \lambda}\right) = W_{M_1} \mathrm{e}^{w_{M_1}\tau} + W_{M_2} \mathrm{e}^{w_{M_2}\tau} + \sum_{M \neq M_1, M_2} W_M \mathrm{e}^{w_M \tau}, \tag{4.7}$$

and since $\mathrm{e}^{(w_{M_2}-w_{M_1})\tau} = \mathrm{e}^{2\mathrm{i}\omega_1\tau}$, it follows that (4.7) equals

$$\begin{aligned} &W_{M_2}\mathrm{e}^{w_{M_1}\tau}\left[\mathrm{e}^{2\mathrm{i}\omega_1\tau} + W_{M_2}^{-1}W_{M_1} + \sum_{M\neq M_1,M_2} W_{M_2}^{-1}W_M\mathrm{e}^{(w_M-w_{M_1})\tau}\right] \\ &= W_{M_2}\mathrm{e}^{w_{M_1}\tau}\left[\mathrm{e}^{2\mathrm{i}\omega_1\tau} + W_{M_2}^{-1}W_{M_1}\left(1 + \sum_{M\neq M_1,M_2} W_{M_1}^{-1}W_M\mathrm{e}^{(w_M-w_{M_1})\tau}\right)\right]. \end{aligned} \tag{4.8}$$

We have $M_1 = \{\omega_1, \omega_2, \dots, \omega_r\}$ and thus

$$W_{M_1} = \frac{\prod(-\mathrm{i}\omega_1+\mu_m)\prod_{j\geq 2}\prod(-\mathrm{i}\omega_j+\mu_m)\prod(\mu_\ell-\mathrm{i}\omega_1)\prod_{k\geq 2}\prod(\mu_\ell-\mathrm{i}\omega_k)}{\nu(-2\mathrm{i}\omega_1)\prod_{k\geq 2}(-\mathrm{i}\omega_1-\mathrm{i}\omega_k)\prod_{j\geq 2}(-\mathrm{i}\omega_j-\mathrm{i}\omega_1)\prod_{j,k\geq 2}(-\mathrm{i}\omega_1-\mathrm{i}\omega_k)},$$

and since $M_2 = \{-\omega_1, \omega_2, \dots, \omega_r\}$, we obtain

$$W_{M_2} = \frac{\prod(\mathrm{i}\omega_1+\mu_m)\prod_{j\geq 2}\prod(-\mathrm{i}\omega_j+\mu_m)\prod(\mu_\ell+\mathrm{i}\omega_1)\prod_{k\geq 2}\prod(\mu_\ell-\mathrm{i}\omega_k)}{\nu(2\mathrm{i}\omega_1)\prod_{k\geq 2}(\mathrm{i}\omega_1-\mathrm{i}\omega_k)\prod_{j\geq 2}(-\mathrm{i}\omega_j+\mathrm{i}\omega_1)\prod_{j,k\geq 2}(-\mathrm{i}\omega_1-\mathrm{i}\omega_k)}.$$

Consequently, $W_{M_2}^{-1}W_{M_1}$ is equal to

$$\begin{aligned} &-\frac{\prod(-\mathrm{i}\omega_1+\mu_m)\prod(\mu_\ell-\mathrm{i}\omega_1)\prod_{k\geq 2}(\mathrm{i}\omega_1-\mathrm{i}\omega_k)\prod_{j\geq 2}(-\mathrm{i}\omega_j+\mathrm{i}\omega_1)}{\prod(\mathrm{i}\omega_1+\mu_m)\prod(\mu_\ell+\mathrm{i}\omega_1)\prod_{k\geq 2}(-\mathrm{i}\omega_1-\mathrm{i}\omega_k)\prod_{j\geq 2}(-\mathrm{i}\omega_j-\mathrm{i}\omega_1)} \\ &= -\frac{\prod(-\omega_1+\mathrm{i}\mu_m)\prod(-\mathrm{i}\mu_\ell-\omega_1)\prod_{k\geq 2}(\omega_1-\omega_k)\prod_{j\geq 2}(\omega_1-\omega_j)}{\prod(\omega_1-\mathrm{i}\mu_m)\prod(-\mathrm{i}\mu_\ell+\omega_1)\prod_{k\geq 2}(-\omega_1-\omega_k)\prod_{j\geq 2}(-\omega_1-\omega_j)} \\ &= -\frac{Q(-\omega_1)^2}{Q(\omega_1)^2}\,\frac{P(\omega_1)^2}{P(-\omega_1)^2} = -b(\omega_1). \end{aligned}$$

Recall next that we have $\lambda = a(z)$ with $z = \omega_1 = \omega_1(\lambda)$ in Π. Thus, taking into account (4.8), we see that the equation $\det W_\tau((a-\lambda)/(1-\lambda)) = 0$ may be written in the form

$$\mathrm{e}^{2\mathrm{i}\tau z} = b(z)(1+\varphi_\tau(z)) \tag{4.9}$$

where

$$\varphi_\tau(z) = \sum_{M\neq M_1,M_2} W_{M_1}^{-1}W_M\mathrm{e}^{(w_M-w_{M_1})\tau}.$$

As we have assumed that $\lambda \neq a(0)$, we may so far use (4.9) only for $z \in \Pi\setminus\{0\}$. The following lemma will justify the equation for $z = 0$ too.

Lemma 4.1 *Suppose that $\omega_2(a(0)), \dots, \omega_r(a(0))$ are distinct. Then, as $z \to 0$,*

$$\det W_\tau((a-a(z))/(1-a(z)))$$

equals

$$\mathrm{e}^{(\kappa_0+o(1))\tau} f(z)\mathrm{e}^{-\mathrm{i}\tau(\omega_2(a(z))+\ldots+\omega_r(a(z)))}\left[\frac{\mathrm{e}^{\mathrm{i}\tau z}/b(z)-\mathrm{e}^{-\mathrm{i}\tau z}}{z} + g_\tau(z)\right]$$

where $\kappa_0 \in \mathbb{R}$, $f(z) = f(0) + O(z)$ with $f(0) \neq 0$, and $|g_\tau(z)| \leq C\mathrm{e}^{-3\delta\tau}$ with some constant $C < \infty$ independent of z and τ.

Proof. We write $\omega_1 = z$ and $\lambda = a(z)$, and may suppose that $|\mathrm{Im}\, z| < \delta$. From (4.1) we obtain

$$\mathrm{e}^{-\kappa\tau} \det W_\tau \left(\frac{a - a(z)}{1 - a(z)}\right) = W_{M_1} \left(\mathrm{e}^{w_{M_1}\tau} + \sum_{M \neq M_1} W_{M_1}^{-1} W_M \mathrm{e}^{w_M \tau} \right)$$

where $M_1^c := \{-z, -\omega_2, \ldots, -\omega_r\}$. In the proof of Proposition 3.1, the set M_1^c was denoted by M_0^c. Formulæ (4.3), (4.4) give $W_{M_1} = A_{M_1}(\lambda) B_{M_1}(\lambda)$ with

$$A_{M_1}(\lambda) := \frac{\prod(-\mathrm{i}z + \mu_m) \prod(\mu_\ell - \mathrm{i}z)}{-2\mathrm{i}z \prod(-\mathrm{i}z - \mathrm{i}\omega_k) \prod(-\mathrm{i}\omega_j - \mathrm{i}z)},$$

$$B_{M_1}(\lambda) := \frac{\prod\prod(-\mathrm{i}\omega_j + \mu_m) \prod\prod(\mu_\ell - \mathrm{i}\omega_k)}{\nu \prod\prod(-\mathrm{i}\omega_j - \mathrm{i}\omega_k)},$$

which shows that

$$W_{M_1} =: -\frac{f(z)}{z} = -\frac{1}{z}(f(0) + O(z))$$

with $f(0) \neq 0$.

Let $M_2^c := \{-z, -\omega_2, \ldots, -\omega_r\}$. Above we saw that $W_{M_1}^{-1} W_{M_2} = -1/b(z)$. Consequently,

$$\begin{aligned} &W_{M_1} \left(\mathrm{e}^{w_{M_1}\tau} + W_{M_1}^{-1} W_{M_2} \mathrm{e}^{w_{M_2}\tau}\right) \\ &= -\frac{f(z)}{z} \left[\mathrm{e}^{\mathrm{i}\tau(-z-\omega_2-\ldots-\omega_r)} - \frac{1}{b(z)} \mathrm{e}^{\mathrm{i}\tau(z-\omega_2-\ldots-\omega_r)} \right] \\ &= f(z) \frac{\mathrm{e}^{\mathrm{i}\tau z}/b(z) - \mathrm{e}^{-\mathrm{i}\tau z}}{z} \mathrm{e}^{-\mathrm{i}\tau(\omega_2+\ldots+\omega_r)}. \end{aligned}$$

Now consider

$$\begin{aligned} M_-^c &:= \{-z, \xi_2, \ldots, \xi_r\}, \quad M_- = \{z, \eta_1, \ldots, \eta_r\}, \\ M_+^c &:= \{z, \xi_2, \ldots, \xi_r\}, \quad M_+ = \{-z, \eta_1, \ldots, \eta_r\} \end{aligned}$$

with $\{\xi_2, \ldots, \xi_r, \eta_1, \ldots, \eta_r\} = \{\pm\omega_2, \ldots, \pm\omega_r\}$. Then

$$W_{M_-} = \frac{\prod(-\mathrm{i}z + \mu_m) \prod\prod(\mathrm{i}\xi_j + \mu_m) \prod(\mu_\ell - \mathrm{i}z) \prod\prod(\mu_\ell - \mathrm{i}\eta_k)}{\nu(-2\mathrm{i}z) \prod(-\mathrm{i}z - \mathrm{i}\eta_k) \prod(\mathrm{i}\xi_j - \mathrm{i}z) \prod\prod(\mathrm{i}\xi_j - \mathrm{i}\eta_k)}$$

and hence $W_{M_1}^{-1} W_{M_-} = D_- E_-$ with

$$D_- = \frac{\prod(-\mathrm{i}z + \mathrm{i}\omega_k) \prod(-\mathrm{i}\omega_j - \mathrm{i}z)}{\prod(-\mathrm{i}z - \mathrm{i}\eta_k) \prod(\mathrm{i}\xi_j - \mathrm{i}z)},$$

$$E_- = \frac{\prod\prod(-\mathrm{i}\omega_j - \mathrm{i}\omega_k) \prod\prod(\mathrm{i}\xi_j + \mu_m) \prod\prod(\mu_\ell - \mathrm{i}\eta_k)}{\prod\prod(-\mathrm{i}\omega_j + \mu_m) \prod\prod(\mu_\ell - \mathrm{i}\omega_k) \prod\prod(\mathrm{i}\xi_j - \mathrm{i}\eta_k)}.$$

Thus, $D_- E_-$ converges to a finite limit G as $z \to 0$. Analogously,

$$W_{M_+} = \frac{\prod(\mathrm{i}z + \mu_m) \prod\prod(\mathrm{i}\xi_j + \mu_m) \prod(\mu_\ell + \mathrm{i}z) \prod\prod(\mu_\ell - \mathrm{i}\eta_k)}{\nu(2\mathrm{i}z) \prod(\mathrm{i}z - \mathrm{i}\eta_k) \prod(\mathrm{i}\xi_j + \mathrm{i}z) \prod\prod(\mathrm{i}\xi_j - \mathrm{i}\eta_k)},$$

which yields $W_{M_1}^{-1} W_{M_+} = D_+ E_+$ with

$$D_+ = \frac{\prod(-\mathrm{i}z - \mathrm{i}\omega_k) \prod(-\mathrm{i}\omega_j - \mathrm{i}z) \prod(\mathrm{i}z + \mu_m) \prod(\mu_\ell + \mathrm{i}z)}{\prod(-\mathrm{i}z + \mu_m) \prod(\mu_\ell - \mathrm{i}z) \prod(\mathrm{i}z - \mathrm{i}\eta_k) \prod(\mathrm{i}\xi_j + \mathrm{i}z)},$$

$$E_+ = \frac{\prod\prod(-\mathrm{i}\omega_j - \mathrm{i}\omega_k) \prod\prod(\mathrm{i}\xi_j + \mu_m) \prod\prod(\mu_\ell - \mathrm{i}\eta_k)}{\prod\prod(-\mathrm{i}\omega_j + \mu_m) \prod\prod(\mu_\ell - \mathrm{i}\omega_k) \prod\prod(\mathrm{i}\xi_j - \mathrm{i}\eta_k)}.$$

It follows that D_+E_+ tends to $-G$ as $z \to 0$. We arrive at the conclusion that

$$W_{M_1}\left[W_{M_1}^{-1}W_{M_-}\mathrm{e}^{w_{M_-}\tau} + W_{M_1}^{-1}W_{M_+}\mathrm{e}^{w_{M_+}\tau}\right]$$
$$= -\frac{f(z)}{z}\left[(G+O(z))\mathrm{e}^{\mathrm{i}\tau(-z+\xi_2+\ldots+\xi_r)} - (G+O(z))\mathrm{e}^{\mathrm{i}\tau(z+\xi_2+\ldots+\xi_r)}\right]$$
$$= f(z)\left[G\frac{\mathrm{e}^{\mathrm{i}\tau z}-\mathrm{e}^{-\mathrm{i}\tau z}}{\tau z}\tau\mathrm{e}^{\mathrm{i}\tau(\xi_2+\ldots+\xi_r)} + O(\mathrm{e}^{\tau\delta})\mathrm{e}^{\tau(\xi_2+\ldots+\xi_r)}\right].$$

Because $\mathrm{Im}\,(\xi_2+\ldots+\xi_r) \ge -\mathrm{Im}\,(\omega_2+\ldots+\omega_r)+6\delta$, the term in the brackets is

$$O(\tau)\mathrm{e}^{-\mathrm{i}\tau(\omega_2+\ldots+\omega_r)}O(\mathrm{e}^{-6\delta\tau}) + O(\mathrm{e}^{\delta\tau})\mathrm{e}^{-\mathrm{i}\tau(\omega_2+\ldots+\omega_r)}O(\mathrm{e}^{-6\delta\tau}),$$

and this is $O(\mathrm{e}^{-5\delta\tau}\mathrm{e}^{-\mathrm{i}\tau(\omega_2+\ldots+\omega_r)})$. If

$$M^C := \{z, -z, \xi_3, \ldots, \xi_r\} \quad \text{or} \quad M^c := \{\xi_1, \xi_2, \ldots, \xi_r\},$$

then $W_M = O(1)$ as $z \to 0$ and therefore $W_{M_1}^{-1}W_M = O(z)$ as $z \to 0$. In that case

$$\mathrm{Im}\sum \xi_j \ge -\mathrm{Im}\,(\omega_2+\ldots+\omega_r)+3\delta.$$

Thus,

$$W_{M_1}W_{M_1}^{-1}W_M\mathrm{e}^{w_M\tau} = f(z)O(1)\mathrm{e}^{-\mathrm{i}\tau(\omega_2+\ldots+\omega_r)}O(\mathrm{e}^{-3\delta\tau}).$$

Finally, the constant κ equals $-\mathrm{k}_\lambda(0)-(\mu_1+\ldots+\mu_r)$ (cf. Theorem 5.1 of [2]) with

$$\mathrm{k}_\lambda(0) := \int_{-\infty}^{\infty}\left(\prod_{j=1}^{r}\frac{x^2-\omega_j(\lambda)^2}{x^2+\mu_j^2}-1\right)\frac{\mathrm{d}x}{2\pi}.$$

Clearly, κ converges to a finite limit κ_0 as $\lambda \to a(0)$. □

By the last lemma, in a punctured neighbourhood of $z=0$ the equation

$$\det W_\tau((a-a(z))/(1-a(z))) = 0$$

is equivalent to

$$\frac{\mathrm{e}^{\mathrm{i}\tau z}/b(z)-\mathrm{e}^{-\mathrm{i}\tau z}}{z} + g_\tau(z) = 0,$$

which may be written in the form

$$\mathrm{e}^{2\mathrm{i}\tau z} = b(z)(1-z\mathrm{e}^{\mathrm{i}\tau z}g_\tau(z)) =: b(z)(1+\varphi_\tau(z)).$$

The lemma shows that $\varphi_\tau(z) \to 0$ as $z \to 0$. Let us define $\varphi_\tau(0) := 0$. Since obviously $b(0)=1$, it then follows that (4.9) may also be employed for $z=0$ and thus for all $z \in \Pi$ provided the points $\omega_2(a(0)), \ldots, \omega_r(a(0))$ are distinct.

Lemma 4.2 *There exists a constant $\gamma \in (0,\infty)$ such that $|b'(z)| \le \gamma$ for all $z \in \Pi$.*

Proof. Clearly,

$$\frac{1}{2}\frac{b'(z)}{b(z)} = -\frac{Q'(-z)}{Q(-z)} - \frac{Q'(z)}{Q(z)} + \frac{P'(z)}{P(z)} + \frac{P'(-z)}{P(-z)}.$$

The first two quotients on the right are bounded because $\mathrm{Im}\,(\mathrm{i}\mu_j) > 4\delta$. Furthermore,

$$\frac{P'(z)}{P(z)} = \sum_{j=2}^{r}\frac{1-\omega_j'(a(z))a'(z)}{z-\omega_j(a(z))} = \sum_{j=2}^{r}\frac{1-\omega_j'(\lambda)a'(z)}{z-\omega_j(\lambda)}$$

with $\lambda = a(z)$. We know that $|\mathrm{Im}\,(z-\omega_j(\lambda))| > 2\delta$ and that $a'(z)$ is bounded for $z \in \Pi$. Since $a(\omega_j(\lambda)) = \lambda$, it follows that $\omega_j'(\lambda) = 1/a'(\omega_j(\lambda))$, and as $\omega_j(\lambda)$ is a simple root of a, we conclude that the infimum of $|a'(\omega_j(\lambda))|$ over $\lambda \in U$ is strictly positive. This shows that $P'(z)/P(z)$ is bounded for $z \in \Pi$, and the same is of course also true for $P'(-z)/P(-z)$. □

Lemma 4.3 *If* $0 \notin \Pi$ *or if* $0 \in \Pi$ *but the roots* $\omega_2(a(0)), \dots, \omega_r(a(0))$ *are distinct, then*

$$\varphi_\tau(z) = O(\mathrm{e}^{-2\delta\tau}) \quad \text{and} \quad \varphi'_\tau(z) = O(\tau \mathrm{e}^{-2\delta\tau})$$

uniformly in $z \in \Pi$.

Proof. Suppose first that 0 is not in Π. We begin by considering the case where

$$M^c = \{-\omega_1, \xi_2, \dots, \xi_r\}, \quad M = \{\omega_1, \eta_2, \dots, \eta_r\}.$$

Then

$$\mathrm{e}^{(w_M - w_{M_1})\tau} = \mathrm{e}^{\mathrm{i}\tau \sum_{j\geq 2}(\xi_j + \omega_j)}.$$

Since $M \neq M_1$, there is a $j \geq 2$ such that $\xi_j = \omega_j$, in which case $\mathrm{Im}\,(\xi_j + \omega_j) > 6\delta$. Thus,

$$|\mathrm{e}^{(w_M - w_{M_1})\tau}| = O(\mathrm{e}^{-6\delta\tau}).$$

We have $W_M = A_M(\lambda)B_M(\lambda)$ with $\lambda = a(z)$ and $A_M(\lambda)$, $B_M(\lambda)$ given by (4.5), (4.6). Moreover, $W_{M_1} = A_{M_0}(\lambda)B_{M_0}(\lambda)$ where $A_{M_0}(\lambda)$, $B_{M_0}(\lambda)$ are as in (4.3), (4.4). Hence

$$\varphi_{M,\tau}(z) := W_{M_1}^{-1} W_M \mathrm{e}^{(w_M - w_{M_1})\tau} = D(\lambda)E(\lambda)\mathrm{e}^{\mathrm{i}\tau\sum(\xi_j(\lambda)+\omega_j(\lambda))}$$

with

$$D(\lambda) := \frac{A_M(\lambda)}{A_{M_0}(\lambda)} = \frac{\prod_{k\geq 2}(-\mathrm{i}\omega_1 - \mathrm{i}\omega_k)\prod_{j\geq 2}(-\mathrm{i}\omega_j - \mathrm{i}\omega_1)}{\prod(-\mathrm{i}\omega_1 - \mathrm{i}\eta_k)\prod(\mathrm{i}\xi_j - \mathrm{i}\omega_1)},$$

$$E(\lambda) := \frac{B_M(\lambda)}{B_{M_0}(\lambda)} = \frac{\prod\prod(\mathrm{i}\xi_j + \mu_m)\prod\prod(\mu_\ell - \mathrm{i}\eta_k)\prod_{j,k\geq 2}(-\mathrm{i}\omega_j - \mathrm{i}\omega_k)}{\prod_{j\geq 2}\prod(-\mathrm{i}\omega_j + \mu_m)\prod_{k\geq 2}\prod(\mu_\ell - \mathrm{i}\omega_k)\prod\prod(\mathrm{i}\xi_j - \mathrm{i}\eta_k)}.$$

As in the proof of Proposition 3.1 we see that $D(\lambda) = O(1)$ and $E(\lambda) = O(1)$. This shows that $\varphi_{M,\tau}(z) = O(\mathrm{e}^{-6\delta\tau})$. We further have $\varphi'_{M,\tau}(z) = T_1 + T_2 + T_3$ with

$$T_1 := E'(\lambda)a'(z)D(\lambda)\mathrm{e}^{\mathrm{i}\tau\sum(\xi_j(\lambda)+\omega_j(\lambda))},$$
$$T_2 := D'(\lambda)a'(z)E(\lambda)\mathrm{e}^{\mathrm{i}\tau\sum(\xi_j(\lambda)+\omega_j(\lambda))},$$
$$T_3 := D(\lambda)E(\lambda)\mathrm{e}^{\mathrm{i}\tau\sum(\xi_j(\lambda)+\omega_j(\lambda))}\mathrm{i}\tau\left[\sum(\xi'_j(\lambda) + \omega'_j(\lambda)\right] a'(z).$$

Since $a'(z) = O(1)$ and $E'(\lambda) = O(1)$, we get $T_1 = O(\mathrm{e}^{-6\delta\tau})$.

Next, because $\omega_1 = \omega_1(\lambda) = z$, we may write

$$D'(\lambda)a'(z) = \frac{\mathrm{d}}{\mathrm{d}z}\frac{\prod_{k\geq 2}[-\mathrm{i}z - \mathrm{i}\omega_k(a(z))]\prod_{j\geq 2}[-\mathrm{i}\omega_j(a(z)) - \mathrm{i}z]}{\prod[-\mathrm{i}z - \mathrm{i}\eta_k(a(z))]\prod[\mathrm{i}\xi_j(a(z)) - \mathrm{i}z]}.$$

Using the fact that $[-\mathrm{i}z + \omega_k(a(z))]' = -\mathrm{i} + \omega'_k(a(z))a'(z)$ and that $\omega_k(a(z))$ and $a'(z)$ are bounded and doing this also for the remaining factors in the expression on the right, we obtain

$$D'(\lambda)a'(z) = \frac{(-\mathrm{i}z)^{4r-5} + f_{4r-6}(z)(-\mathrm{i}z)^{4r-6} + \ldots + f_0(z)}{\prod_{k\geq 2}(-\mathrm{i}z - \mathrm{i}\eta_k(a(z)))^2\prod_{j\geq 2}(\mathrm{i}\xi_j(a(z)) - \mathrm{i}z)^2}$$

with bounded coefficients $f_{4r-6}(z), \dots, f_0(z)$. The denominator is of the form

$$(-\mathrm{i}z)^{4r-4} + g_{4r-5}(z)(-\mathrm{i}z)^{4r-5} + \ldots + g_0(z)$$

with bounded coefficients, and since

$$|\mathrm{Im}\, z| < \delta, \qquad |\mathrm{Im}\,\xi_j(a(z))| > 3\delta, \qquad |\mathrm{Im}\,\eta_k(a(z))| > 3\delta,$$

the denominator is bounded away from zero. This proves that $D'(\lambda)a'(z) = O(1)$ and hence $T_2 = O(\mathrm{e}^{-6\delta\tau})$.

Finally, taking into account that $a(\omega_j(\lambda)) = \lambda$ and $a(\xi_j(\lambda)) = \lambda$, we get

$$\omega_j'(\lambda) = 1/a'(\omega_j(\lambda)), \quad \xi_j'(\lambda) = 1/a'(\xi_j(\lambda)).$$

Our assumption that the roots $\omega_2(\lambda), \dots, \omega_r(\lambda)$ are all simple implies that

$$\inf_{\lambda \in U} |a'(\omega_j(\lambda))| > 0, \quad \inf_{\lambda \in U} |a'(\xi_j(\lambda))| > 0.$$

Hence $\omega_j'(\lambda) = O(1)$ and $\xi_j'(\lambda) = O(1)$, which shows that $T_3 = O(\tau \mathrm{e}^{-6\delta\tau})$.

The proof is analogous in the remaining cases. We remark that if

$$M^c = \{-\omega_1, \omega_1, \xi_3, \dots, \xi_r\}, \quad M = \{\xi_2, -\xi_2, -\xi_3, \dots, -\xi_r\},$$

then

$$\mathrm{e}^{(w_M - w_{M_1})\tau} = \mathrm{e}^{\mathrm{i}\tau(\xi_3 + \dots + \xi_r + \omega_1 + \omega_2 + \dots + \omega_r)},$$

and in the case $\xi_3 = -\omega_3, \dots, \xi_r = -\omega_r$ we have

$$\left| \mathrm{e}^{(w_M - w_{M_1})\tau} \right| = \mathrm{e}^{-\mathrm{Im}\,(\omega_1 + \omega_2)\tau}$$

with $-\mathrm{Im}\,\omega_1 - \mathrm{Im}\,\omega_2 < \delta - 3\delta = -2\delta$. This is the explanation for the $e^{-2\delta\tau}$ in the statement of the lemma. The proof in the case where $0 \notin \Pi$ is complete.

Let finally $0 \in \Pi$ but suppose that the roots $\omega_2(a(0)), \dots, \omega_r(a(0))$ are distinct. We may then combine the previous argument with Lemma 4.1 to deduce that $\varphi_\tau(z) = O(\mathrm{e}^{-2\delta\tau})$. By a more elaborate but still straightforward analysis, one can show that the function g_τ in Lemma 4.1 satisfies $|g_\tau'(z)| \le C\tau\mathrm{e}^{-3\delta\tau}$ as $|z| \to 0$ with some constant C independent of z and τ. After defining $\varphi_\tau'(0) := 0$, this yields the estimate $\varphi_\tau'(z) = O(\tau\mathrm{e}^{-2\delta\tau})$. □

5 The nonlinear equation

In this section we prove Theorems 3.2 and 3.3. Suppose that the hypothesis of Theorem 3.3 is satisfied. We may also assume that the neighbourhood U is replaced by a an open neighbourhood of $a(\mathrm{clos}\, I)$ such that $\omega_2(\lambda), \dots, \omega_r(\lambda)$ are separated away from each other for $\lambda \in U$. For the sake of definiteness, we assume that (3.6) is in force. We already know that if τ is large enough then $\lambda = a(z) \in U \setminus \{a(0), 1\}$ is an eigenvalue of $W_\tau(a)$ if and only if (4.9) holds. Note that we may assume that $0 \notin \Pi$ if $\mathrm{clos}\, I$ does not contain the origin. On the other hand, if $0 \in \mathrm{clos}\, I$ and thus $0 \in \Pi$, then the hypothesis of Theorem 3.3 guarantees that the points $\omega_2(a(0)), \dots, \omega_r(a(0))$ are distinct. Furthermore, taking τ sufficiently large, we may, by virtue of Lemma 4.3, guarantee that $\log|1 + \varphi_\tau(z)| \in (-\beta, \beta)$ and $\arg(1 + \varphi_\tau(z)) \in (-\sigma, \sigma)$.

Proof of Theorem 3.3(a). Let $z = x + \mathrm{i}\varepsilon$ with $x \in I_{k,\tau}$. If z satisfies (4.9), then

$$x = \frac{1}{2\tau}\Big[\arg b(z) + \arg(1 + \varphi_\tau(z))\Big] + \frac{\ell\pi}{\tau}, \tag{5.1}$$

$$\varepsilon = -\frac{1}{2\tau}\log|b(z)| - \frac{1}{2\tau}\log|1 + \varphi_\tau(z)|, \tag{5.2}$$

where the argument of $b(z)$ is chosen in $(-\pi + \sigma, \pi - \sigma)$ and $\ell \in \mathbb{Z}$. From (3.5) we see that $\log|b(z)| \in [-\beta, \beta]$, while we have ensured that $\log|1 + \varphi_\tau(z)| \in (-\beta, \beta)$. This proves that $\varepsilon \in (-2\beta/(2\tau), 2\beta/(2\tau))$. □

Proof of Theorem 3.3(b). The existence of exactly one eigenvalue in $a(S_{k,\tau})$ will follow from (4.9) in conjunction with Banach's fixed point theorem, once we have proved that the map

$$\Psi_{k,\tau}(z) := \frac{1}{2\mathrm{i}\tau}\Big(\log b(z) + \log(1+\varphi_\tau(z))\Big) + \frac{k\pi}{\tau} \tag{5.3}$$

is a contraction of $S_{k,\tau}$ into itself. Note that the right-hand sides of (5.1) and (5.2) with $\ell = k$ are just $\operatorname{Re}\Psi_{k,\tau}(z)$ and $\operatorname{Im}\Psi_{k,\tau}(z)$. Let $z \in S_{k,\tau}$. Then the right-hand of (5.1) with $\ell = k$ is in

$$\frac{1}{2\tau}(-\pi+\sigma, \pi-\sigma) + \frac{1}{2\tau}(-\sigma,\sigma) + \frac{k\pi}{\tau} = \left(\left(k-\frac{1}{2}\right)\frac{\pi}{\tau}, \left(k+\frac{1}{2}\right)\frac{\pi}{\tau}\right),$$

while the right-hand side of (5.2) lies in

$$-\frac{1}{2\tau}[-\beta,\beta] - \frac{1}{2\tau}(-\beta,\beta) = \left(-\frac{\beta}{\tau}, \frac{\beta}{\tau}\right).$$

Thus, $\Psi_{k,\tau}$ maps $S_{k,\tau}$ into itself. If f is an analytic function satisfying $|f'(z)| \le M$ for z in some convex domain G, then

$$|f(z_1) - f(z_2)| \le 2\sqrt{2}\, M |z_1 - z_2|$$

for $z_1, z_2 \in G$. Since

$$\Psi'_{k,\tau}(z) = \frac{1}{2\mathrm{i}\tau}\left(\frac{b'(z)}{b(z)} + \frac{\varphi'_\tau(z)}{1+\varphi_\tau(z)}\right)$$

and

$$\left|\frac{b'(z)}{b(z)}\right| < \gamma \mathrm{e}^{\beta}, \quad \frac{\varphi'_\tau(z)}{1+\varphi_\tau(z)} = O(\tau \mathrm{e}^{-2\delta\tau})$$

by virtue of (3.5) and Lemmas 4.2 and 4.3, we conclude that $\Psi_{k,\tau}$ is contractive whenever τ is sufficiently large.

The algebraic multiplicity of $\lambda_{k,\tau} = a(z_{k,\tau})$ is the multiplicity of $z := z_{k,\tau}$ as a zero of the equation (4.9). Assume that the multiplicity is at least 2. Differentiating equation (4.9) we get

$$2\mathrm{i}\tau \mathrm{e}^{2\mathrm{i}\tau z} = b'(z)(1+\varphi_\tau(z)) + b(z)\varphi'_\tau(z).$$

But this is impossible for large τ, since the absolute value of the left-hand side is $2\tau \mathrm{e}^{-2\tau \operatorname{Im} z} \ge 2\tau \mathrm{e}^{-2\beta}$ whereas that of the right-hand side is bounded due to (3.5) and Lemma 4.3. $\square$

Proof of Theorem 3.3(c). According to the proof of part (b), $\lambda_{k,\tau} = a(z_{k,\tau})$ where $z_{k,\tau} \in S_{k,\tau}$ is the unique solution of the equation $z = \Psi_{k,\tau}(z)$. The same argument as in the proof of part (b) shows that $\Phi_{k,\tau}$ is a contractive map of $S_{k,\tau}$ into itself. We now suppress the subscripts k, τ.

Put $z^{(0)} = w^{(0)} := k\pi/\tau$ and define $z^{(n)}$ and $w^{(n)}$ by

$$z^{(n)} := \Phi(z^{(n-1)}), \quad w^{(n)} := \Psi(w^{(n-1)}) \quad (n \ge 1).$$

From the proof of part (b) we know that there exists a constant $K < \infty$ such that

$$|\Phi(z_1) - \Phi(z_2)| \le \frac{K}{\tau}|z_1 - z_2|, \quad |\Psi(w_1) - \Psi(w_2)| \le \frac{K}{\tau}|w_1 - w_2|$$

for all $z_1, z_2, w_1, w_2 \in S_{k,\tau}$. We therefore deduce from Banach's fixed point theorem that $w^{(n)} \to z_{k,\tau}$ and that

$$|w^{(n)} - z_{k,\tau}| \le \left(\frac{K}{\tau}\right)^n |z^{(0)} - z_{k,\tau}| = O\left(\frac{1}{\tau^{n+1}}\right). \tag{5.4}$$

We claim that

$$z^{(n)} = w^{(n)} + O(\mathrm{e}^{-2\delta\tau}). \tag{5.5}$$

By Lemma 4.3,

$$\psi_\tau(z) := -\frac{1}{2\mathrm{i}\tau}\log(1+\varphi_\tau(z)) = O(\mathrm{e}^{-2\delta\tau}).$$

Thus, we have $\Phi(z) = \Psi(z) + \psi_\tau(z)$ and

$$\begin{aligned} z^{(1)} &= \Phi(z^{(0)}) = \Psi(z^{(0)}) + \psi_\tau(z^{(0)}) \\ &= \Psi(w^{(0)}) + O(\mathrm{e}^{-2\delta\tau}) = w^{(1)} + O(\mathrm{e}^{-2\delta\tau}), \end{aligned}$$

which is (5.5) for $n = 1$. But if (5.5) is true for some n, then

$$\begin{aligned} z^{(n+1)} &= \Phi(z^{(n)}) = \Phi\Big(w^{(n)} + O(\mathrm{e}^{-2\delta\tau})\Big) \\ &= \Phi(w^{(n)}) + \frac{K}{\tau}O(\mathrm{e}^{-2\delta\tau}) \\ &= \Psi(w^{(n)}) + \psi_\tau(w^{(n)}) + \frac{K}{\tau}O(\mathrm{e}^{-2\delta\tau}) \\ &= w^{(n+1)} + O(\mathrm{e}^{-2\delta\tau}), \end{aligned}$$

which is (5.5) for $n+1$. Consequently, (5.5) is true for all $n \ge 1$. Combining (5.4) and (5.5) we get

$$|z^{(n)} - z_{k,\tau}| = O\left(\frac{1}{\tau^{n+1}}\right),$$

whence

$$\lambda_{k,\tau} = a(z_{k,\tau}) = a\left(z^{(n)} + O\left(\frac{1}{\tau^{n+1}}\right)\right) = a(z^{(n)}) + O\left(\frac{1}{\tau^{n+1}}\right).$$

All estimates are uniform in k. □

Proof of Theorem 3.2. There exists an open neighbourhood $V \subset \mathbb{C}$ of the point $z = 0$ such that $\omega_2(a(z)), \ldots, \omega_r(a(z))$ are distinct for $z \in V$. The function

$$h_\tau(z) := \det W_\tau\left(\frac{a - a(z)}{1 - a(z)}\right)$$

is analytic in V. Since

$$\lim_{z\to 0}\frac{\mathrm{e}^{\mathrm{i}\tau z}/b(z) - \mathrm{e}^{-\mathrm{i}\tau z}}{z} = 2\mathrm{i}\tau - b'(0),$$

Lemma 4.1 implies that $h_\tau(0) \neq 0$ and that therefore $\lambda = a(0)$ is not an eigenvalue of $W_\tau(a)$ whenever τ is large enough.

Now assume that $W_\tau(a)$ has an eigenvalue $\lambda = a(z) \in U$ with $z \neq 0$ and $|\operatorname{Re} z| \le \pi/(2\tau)$. Then $z = x + \mathrm{i}\varepsilon$ with $x \in I_{0,\tau}$. The reasoning of the proof of Theorem 3.3(a) shows that $|\varepsilon| < \beta/\tau$. Hence $z \in S_{0,\tau}$. The map $\Psi_{k,\tau}$ introduced in the proof of Theorem 3.3(b) makes also sense for $k = 0$, and one can verify as in the proof of Theorem 3.3(b) that $\Psi_{k,\tau}$ is a contractive map of $S_{0,\tau}$ into itself. As z satisfies equation (4.9), it is a solution of the equation $z = \Psi_{0,\tau}(z)$. Now we can repeat the argument of the proof of Theorem 3.3(c) for $k = 0$ to conclude that $\Phi_{0,k} : S_{0,k} \to S_{0,k}$ is contractive and that z must also satisfy the equation $z = \Phi_{0,k}(z)$. However, by Banach's fixed point theorem, the unique solution of this equation is $z = 0$. This contradiction completes the proof. □

Proof of Theorem 3.5. We know that $\lambda \in V$ is an eigenvalue if and only if $\lambda = a(z)$ and z satisfies equation (4.9). This equation is in turn equivalent to the equation $z = \Psi_{k,\tau}(z)$ for some $k \ge 1$, where $\Psi_{k,\tau}$ is given by (5.3). Let m_0 be the maximum of $|\arg b(z)| + |\arg(1+\varphi_\tau(z))|$ as

z ranges over Π. If $z = x + \mathrm{i}\varepsilon$ and $z = \Psi_{k,\tau}(z)$, then x and ε satisfy (5.1) with $\ell = k$ and (5.2). We have $\log|b(z)| \in [-\beta, \beta]$, and can guarantee that $\log|1+\varphi_\tau(z)|$ is in $(-\beta, \beta)$. Consequently,

$$z \in S^*_{k,\tau} := \left\{ z \in \mathbb{C} : \operatorname{Re} z \in \left[\frac{k\pi}{\tau} - \frac{m_0}{2\tau}, \frac{k\pi}{\tau} + \frac{m_0}{2\tau} \right], |\operatorname{Im} z| \le \frac{\beta}{\tau} \right\}.$$

Note that $S^*_{k,\tau}$ is automatically a subset of Π for all $k \ge 1$ if only τ is large enough. This proves that the solutions of the equation $z = \Psi_{k,\tau}(z)$ necessarily belong to $S^*_{k,\tau}$. The existence of exactly one solution will follow from Banach's fixed point theorem once we have proved that the map $\Psi_{k,\tau}$ is a contraction of $S^*_{k,\tau}$ into itself. But it is obvious that $\Psi_{k,\tau}$ maps $S^*_{k,\tau}$ into itself, and exactly as in the proof of Theorem 3.3(b) we see that $\Psi_{k,\tau}$ is contractive on $S^*_{k,\tau}$. Let us denote the unique solution of the equation $z = \Psi_{k,\tau}(z)$ by $z_{k,\tau}$.

We now prove that $\operatorname{Re} z_{k,\tau} < \operatorname{Re} z_{k+1,\tau}$. Abbreviating $b(1+\varphi_\tau)$ to c, we have

$$\begin{aligned} &\operatorname{Re} z_{k+1,\tau} - \operatorname{Re} z_{k,\tau} = \operatorname{Re} \Psi_{k+1,\tau}(z_{k+1,\tau}) - \operatorname{Re} \Psi_{k,\tau}(z_{k,\tau}) \\ &= \frac{\pi}{\tau} + \frac{1}{2\tau}\Big(\arg c(z_{k+1,\tau}) - \arg c(z_{k,\tau}) \Big). \end{aligned} \tag{5.6}$$

In the proof of Theorem 3.3(b) we have observed that

$$\frac{1}{2\tau} |\arg c(z_{k+1,\tau}) - \arg c(z_{k,\tau})| \le \frac{K}{\tau} |z_{k+1,\tau} - z_{k,\tau}|$$

with some constant $K < \infty$, and since

$$|z_{k+1,\tau} - z_{k,\tau}| \le \frac{(k+1)\pi}{\tau} + \frac{m_0}{2\tau} - \frac{k}{\tau} + \frac{m_0}{2\tau} = \frac{m_0 + \pi}{\tau},$$

it follows that (5.6) is positive whenever $\tau > (m_0 + \pi)K/\pi$.

Finally, the argument of the proof of Theorem 3.3(c) remains literally true with $S_{k,\tau}$ replaced by $S^*_{k,\tau}$. This yields the remaining part of Theorem 3.5. □

6 The eigenfunctions

In this section we prove Theorem 3.6. The basic idea of the proof goes back to Widom [16], who considered Hermitian kernels.

Let $F : L^2(-\infty, \infty) \to L^2(\mathbb{R})$ be the Fourier transform,

$$(Ff)(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \mathrm{e}^{\mathrm{i}xt}\, \mathrm{d}t, \quad x \in \mathbb{R}.$$

We may think of $L^2(0,\tau)$ as a subspace of $L^2(-\infty,\infty)$ and denote by $FL^2(0,\tau)$ the image of $L^2(0,\tau)$ under the Fourier transform. Functions in $FL^2(0,\tau)$ may be continued to entire functions on $\mathbb{C}$. We let $\chi_{(0,\tau)}$ stand for the orthogonal projection of $L^2(-\infty,\infty)$ to $L^2(0,\tau)$. Note that $\chi_{(0,\tau)}$ is nothing but restriction to $(0,\tau)$ or, in other terms, multiplication by the characteristic function of the interval $(0,\tau)$. Finally, let $P_\tau := F\chi_{(0,\tau)}F^{-1}$. Thus, P_τ is the orthogonal projection of $L^2(\mathbb{R})$ to $FL^2(0,\tau)$.

Proof of Theorem 3.6. Let λ be an eigenvalue of the operator $W_\tau(a)$ and let $\varphi_\tau \in L^2(0,\tau)$ be an eigenfunction corresponding to λ. The equation $W_\tau(a)\varphi_\tau = \lambda\varphi_\tau$ is equivalent to

$$P_\tau((a(x) - \lambda)E_\tau(x)) = 0, \tag{6.1}$$

where $E_\tau := F\varphi_\tau$. Since $\mu_1, \ldots, \mu_r$ are distinct, the function a has the partial fraction decomposition

$$a(x) = 1 + \sum_{j=1}^{r} \frac{A_j}{x^2 + \mu_j^2} = 1 + \sum_{j=1}^{r} \left(\frac{\alpha_j}{x - \mathrm{i}\mu_j} - \frac{\alpha_j}{x + \mathrm{i}\mu_j} \right).$$

Inserting this in (6.1) we get

$$(1-\lambda)E_\tau(x)+\sum_{j=1}^{r}\alpha_j P_\tau\left(\frac{E_\tau(x)}{x-\mathrm{i}\mu_j}\right)-\sum_{j=1}^{r}\alpha_j P_\tau\left(\frac{E_\tau(x)}{x+\mathrm{i}\mu_j}\right)=0. \tag{6.2}$$

It is well known that if $\operatorname{Re}\mu>0$, then

$$P_\tau\left(\frac{E(x)}{x-\mathrm{i}\mu_j}\right)=\frac{E(x)-E(\mathrm{i}\mu)}{x-\mathrm{i}\mu},\qquad P_\tau\left(\frac{E(x)}{x+\mathrm{i}\mu_j}\right)=\frac{E(x)-\mathrm{e}^{\mathrm{i}\tau(x+\mathrm{i}\mu)}E(-\mathrm{i}\mu)}{x+\mathrm{i}\mu}$$

for every $E\in FL^2(0,\tau)$. Thus, (6.2) may be written as

$$(1-\lambda)E_\tau(x)+\sum_{j=1}^{r}\alpha_j\frac{E_\tau(x)-E_\tau(\mathrm{i}\mu_j)}{x-\mathrm{i}\mu_j}-\sum_{j=1}^{r}\alpha_j\frac{E_\tau(x)-\mathrm{e}^{\mathrm{i}\tau(x+\mathrm{i}\mu_j)}E_\tau(-\mathrm{i}\mu_j)}{x+\mathrm{i}\mu_j}=0,$$

or equivalently,

$$(a(x)-\lambda)E_\tau(x)=\sum_{j=1}^{r}\frac{y_j}{x-\mathrm{i}\mu_j}+\sum_{j=1}^{r}\frac{y_{r+j}\mathrm{e}^{\mathrm{i}\tau x}}{x+\mathrm{i}\mu_j} \tag{6.3}$$

with

$$y_j:=\alpha_j E_\tau(\mathrm{i}\mu_j),\quad y_{r+j}=-\alpha_j\mathrm{e}^{-\tau\mu_j}E_\tau(-\mathrm{i}\mu_j).$$

It follows that

$$\begin{aligned}E_\tau(x)&=\frac{1}{a(x)-\lambda}\left(\sum_{j=1}^{r}\frac{y_j}{x-\mathrm{i}\mu_j}+\sum_{j=1}^{r}\frac{y_{r+j}\mathrm{e}^{\mathrm{i}\tau x}}{x+\mathrm{i}\mu_j}\right)\\&=\frac{1}{a(x)-\lambda}\,\frac{p_{2r-1}(x)+\mathrm{e}^{\mathrm{i}\tau x}q_{2r-1}(x)}{(x^2+\mu_1^2)\dots(x^2+\mu_r^2)},\end{aligned}$$

where $p_{2r-1}(x)$ and $q_{2r-1}(x)$ are polynomials of degree at most $2r-1$. Now note that

$$\frac{1-\lambda}{a(x)-\lambda}=\frac{(x^2+\mu_1^2)\dots(x^2+\mu_r^2)}{(x-\xi_1)\dots(x-\xi_{2r})}$$

with $\xi_j:=\xi_j(\lambda)$ and that we may ignore the scalar multiple $1-\lambda$ when considering eigenfunctions. We therefore obtain

$$E_\tau(x)=\frac{p_{2r-1}(x)+\mathrm{e}^{\mathrm{i}\tau x}q_{2r-1}(x)}{(x-\xi_1)\dots(x-\xi_{2r})}=\sum_{j=1}^{2r}\frac{b_j+d_j\mathrm{e}^{\mathrm{i}\tau x}}{x-\xi_j}.$$

The entire function E_τ has no poles. Hence $b_j+d_j\mathrm{e}^{\mathrm{i}\tau\xi_j}=0$, which yields

$$E_\tau(x)=\sum_{j=1}^{2r}d_j\frac{\mathrm{e}^{\mathrm{i}\tau x}-\mathrm{e}^{\mathrm{i}\tau\xi_j}}{x-\xi_j}=\sum_{j=1}^{2r}c_j\frac{\mathrm{e}^{\mathrm{i}\tau(x-\xi_j)}-1}{x-\xi_j}.$$

Since

$$\int_0^\tau \mathrm{e}^{-\mathrm{i}\xi t}\mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}t=\frac{\mathrm{e}^{\mathrm{i}\tau(x-\xi)}-1}{x-\xi},$$

we arrive at the desired representation

$$\varphi_\tau(t)=\sum_{j=1}^{2r}c_j\mathrm{e}^{-\mathrm{i}\xi t}. \tag{6.4}$$

If

$$\varphi_\tau(t) + \int_0^\tau \mathrm{k}(t-s)\varphi_\tau(s)\,\mathrm{d}s = \lambda\varphi_\tau(t)$$

and $\mathrm{k}(t-s) = \mathrm{k}(s-t)$ then

$$\varphi_\tau(\tau-t) + \int_0^\tau \mathrm{k}(t-s)\varphi_\tau(\tau-s)\,\mathrm{d}s = \lambda\varphi_\tau(\tau-t).$$

Thus, if $\varphi_\tau(t)$ is an eigenfunction, then so is also $\varphi_\tau(\tau-t)$. As the eigensubspace is one-dimensional due to Theorem 3.3(b), we conclude that $\varphi_\tau(\tau-t) = \theta\varphi_\tau(t)$ with some scalar $\theta \neq 0$. Taking the Fourier transform we get

$$\mathrm{e}^{\mathrm{i}\tau x}E_\tau(-x) = \theta E_\tau(x). \tag{6.5}$$

If $E_\tau(0) \neq 0$, this implies that $\theta = 1$. In case $E_\tau(0) = 0$ and $E_\tau'(0) \neq 0$, we obtain after differentiation of (6.5) that $\theta = -1$. Continuing in this way we see that if

$$E_\tau(0) = \ldots = E_\tau^{(n-1)}(0) = 0, \quad E_\tau^{(n)}(0) \neq 0,$$

then $\theta = (-1)^n$.

Now recall that $\{\xi_1, \ldots, \xi_{2r}\} = \{\pm\omega_1, \ldots, \pm\omega_r\}$. We may therefore rewrite equality (6.4) in the form

$$\varphi_\tau(t) = \sum_{j=1}^{r} \left(c_j \mathrm{e}^{\mathrm{i}\omega_j t} + c_{r+j}\mathrm{e}^{-\mathrm{i}\omega_j t}\right),$$

whence

$$\varphi_\tau(\tau-t) = \sum_{j=1}^{r} \left(c_{r+j}\mathrm{e}^{-\mathrm{i}\omega_j \tau}\mathrm{e}^{\mathrm{i}\omega_j t} + c_j \mathrm{e}^{\mathrm{i}\omega_j \tau}\mathrm{e}^{-\mathrm{i}\omega_j t}\right).$$

Using the equality $\varphi_\tau(\tau-t) = \theta\varphi_\tau(t)$ and taking into consideration that $\mathrm{e}^{\pm\mathrm{i}\omega_j t}$ are linearly independent, we get $c_{r+j} = \theta\mathrm{e}^{\mathrm{i}\omega_j\tau}c_j$ and thus,

$$\varphi_\tau(t) = \sum_{j=1}^{r} c_j \left(\mathrm{e}^{\mathrm{i}\omega_j t} + \theta\mathrm{e}^{i\omega_j(\tau-t)}\right).$$

Clearly, this can be written in the form asserted in the theorem. □

As the left-hand side of (6.3) vanishes for $x = \xi_j$, so also must the right-hand side. Consequently,

$$\sum_{j=1}^{r} \frac{y_j}{\xi_k - \mathrm{i}\mu_j} + \sum_{j=1}^{r} \frac{y_{r+j}\mathrm{e}^{\mathrm{i}\tau\xi_k}}{\xi_k + \mathrm{i}\mu_j} = 0, \quad k = 1, \ldots, 2r. \tag{6.6}$$

This is a linear system with $2r$ equations and $2r$ variables $y_1, \ldots, y_{2r}$. It has a nontrivial solution if and only if its determinant is zero, and computing the determinant one gets an expression similar to the right-hand side of (4.1) and eventually arrives at equation (4.9). This is the way in which Widom proceeded in [16] to tackle the case of Hermitian kernels.

Let

$$a(x) = \frac{x^2}{x^2+1}$$

be as in Section 2. Then $r = 1$, $\mu = 1$, $\xi_1 = \omega$, $\xi_2 = -\omega$, formula (6.3) reads

$$(1-\lambda)\frac{x^2-\omega^2}{x^2+\mu^2}E_\tau(x) = \frac{y_1}{x-\mathrm{i}} + \frac{y_2\mathrm{e}^{\mathrm{i}\tau x}}{x+\mathrm{i}}, \tag{6.7}$$

and system (6.6) becomes

$$\begin{pmatrix} \dfrac{1}{\omega-\mathrm{i}} & \dfrac{\mathrm{e}^{\mathrm{i}\tau\omega}}{\omega+\mathrm{i}} \\ \dfrac{1}{-\omega-\mathrm{i}} & \dfrac{\mathrm{e}^{-\mathrm{i}\tau\omega}}{-\omega+\mathrm{i}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the matrix equals

$$\frac{\mathrm{e}^{\mathrm{i}\tau\omega}}{(\omega+\mathrm{i})^2} - \frac{\mathrm{e}^{-\mathrm{i}\tau\omega}}{(\omega-\mathrm{i})^2},$$

and this is zero if and only if (2.3) satisfied. Let y_1, y_2 be any nontrivial solution. From (6.7) we get

$$(1-\lambda)E_\tau(x) = y_1 \frac{x+\mathrm{i}}{x^2-\omega^2} + y_2 \frac{x-\mathrm{i}}{x^2-\omega^2}\,\mathrm{e}^{\mathrm{i}\tau x}, \tag{6.8}$$

and using

$$\frac{x+\mathrm{i}}{x^2-\omega^2} = \frac{\omega+\mathrm{i}}{2\omega(x-\omega)} + \frac{\omega-\mathrm{i}}{2\omega(x+\omega)}, \qquad \frac{x-\mathrm{i}}{x^2-\omega^2} = \frac{\omega-\mathrm{i}}{2\omega(x-\omega)} + \frac{\omega+\mathrm{i}}{2\omega(x+\omega)},$$

we deduce that (6.8) is equal to

$$\frac{y_1(\omega+\mathrm{i}) + y_2(\omega-\mathrm{i})\mathrm{e}^{\mathrm{i}\tau x}}{2\omega(x-\omega)} + \frac{y_1(\omega-\mathrm{i}) + y_2(\omega+\mathrm{i})\mathrm{e}^{\mathrm{i}\tau x}}{2\omega(x+\omega)}. \tag{6.9}$$

Since (6.8), hence also (6.9), cannot have poles, it follows that

$$\begin{aligned} y_1(\omega+\mathrm{i}) + y_2(\omega-\mathrm{i})\mathrm{e}^{\mathrm{i}\tau\omega} &= 0, \\ y_1(\omega-\mathrm{i}) + y_2(\omega+\mathrm{i})\mathrm{e}^{-\mathrm{i}\tau\omega} &= 0. \end{aligned}$$

Hence, (6.9) can be written in the form

$$\begin{aligned} &\frac{y_2(\omega-\mathrm{i})(\mathrm{e}^{\mathrm{i}\tau x} - \mathrm{e}^{\mathrm{i}\tau\omega})}{2\omega(x-\omega)} + \frac{y_2(\omega+\mathrm{i})(\mathrm{e}^{\mathrm{i}\tau x} - \mathrm{e}^{-\mathrm{i}\tau\omega})}{2\omega(x+\omega)} \\ &= \frac{y_2(\omega-\mathrm{i})}{2\omega}\mathrm{e}^{\mathrm{i}\tau\omega}\,\frac{\mathrm{e}^{\mathrm{i}\tau(x-\omega)}-1}{x-\omega} + \frac{y_2(\omega+\mathrm{i})}{2\omega}\mathrm{e}^{-\mathrm{i}\tau\omega}\,\frac{\mathrm{e}^{\mathrm{i}\tau(x+\omega)}-1}{x+\omega}. \end{aligned}$$

Taking the inverse Fourier transform we obtain

$$(1-\lambda)\varphi_\tau(t) = \frac{y_2(\omega-\mathrm{i})}{2\omega}\mathrm{e}^{\mathrm{i}\tau\omega}\mathrm{e}^{-\mathrm{i}\omega t} + \frac{y_2(\omega+\mathrm{i})}{2\omega}\mathrm{e}^{-\mathrm{i}\tau\omega}\mathrm{e}^{i\omega t}$$

and, ignoring scalar multiples, we finally get

$$\varphi_\tau(t) = c_1\mathrm{e}^{\mathrm{i}\omega t} + c_2\mathrm{e}^{-\mathrm{i}\omega t}$$

with $c_1 = (\omega+\mathrm{i})\mathrm{e}^{-\mathrm{i}\tau\omega}$ and $c_2 = (\omega-\mathrm{i})\mathrm{e}^{\mathrm{i}\tau\omega}$. By virtue of (2.10),

$$\frac{c_2\mathrm{e}^{-\mathrm{i}\tau\omega}}{c_1} = \frac{\omega-\mathrm{i}}{\omega+\mathrm{i}}\mathrm{e}^{\mathrm{i}\tau\omega} = \frac{-\mathrm{i}(1+\mathrm{i}\omega)}{\mathrm{i}(1-\mathrm{i}\omega)}\mathrm{e}^{\mathrm{i}\tau\omega} = \theta.$$

Thus, $\varphi_\tau(t) = c_1\Big(\mathrm{e}^{\mathrm{i}\omega t} + \theta\mathrm{e}^{\mathrm{i}\tau\omega}\mathrm{e}^{-\mathrm{i}\omega t}\Big)$, which results in (2.11).

7 Numerical examples

In this section we illustrate our narrative with two numerical examples. The first corresponds to the operator with the symbol

$$a(x) = \frac{-(16+68\mathrm{i}) - (10+30\mathrm{i})x^2 - (3+2\mathrm{i})x^2 + x^6}{(12+16\mathrm{i}) + (20+12\mathrm{i})x^2 + (9-4\mathrm{i})x^4 + x^6} = 1 + 2\sum_{k=1}^{3} \frac{\alpha_k \mu_k}{x^2 + \mu_k^2} \tag{7.1}$$

where $\boldsymbol{\alpha} = [-1, -\mathrm{i}, -2]$ and $\boldsymbol{\mu} = [1, 1+\mathrm{i}, 3-\mathrm{i}]$. The curve $\mathcal{R}(a)$ and the range of b on $(0, \infty)$ are plotted in Fig. 3.1. In Fig. 3.2 we display the eigenvalues of $W_\tau(a)$ for different values of τ, overlaid on $\mathcal{R}(a)$. Note that we have computed the eigenvalues both using Theorem 3.3(c) and by the finite section algorithm from [8]: both results match to very high accuracy. Note that the eigenvalues congregate near $\mathcal{R}(a)$, are nearly equispaced and their density grows linearly with τ: all this is in complete conformity with Theorem 3.3. The rapid speed of convergence of the iterative scheme from Theorem 3.3(c) is illustrated by Fig. 7.1, where we have zoomed into a small portion of $\mathcal{R}(a)$. Note that the first iteration $z_{k,\tau}^{(1)}$ is already difficult to distinguish from the exact eigenvalues at the resolution of the plot!

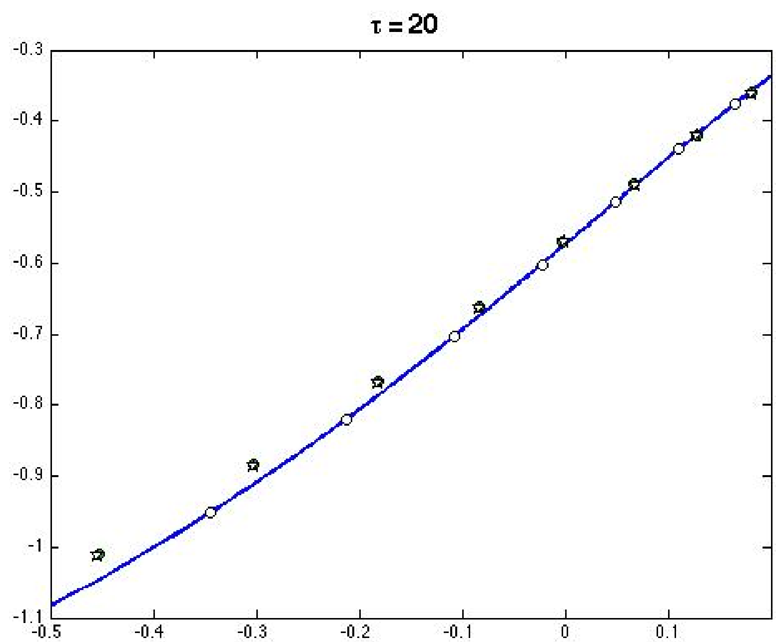

Figure 7.1: The speed of convergence of $a_{k,20}^{(n)}$ for growing n. The equispaced points $z_{k,20}^{(0)}$ are denoted by white circles, the first iteration $z_{k,20}^{(1)}$ by filled-in discs and the eigenvalues $\lambda_{k,20}^{(n)}$ by white stars.

Rapid convergence is further emphasized in Table 1, where we have displayed the error of the iterates visible in Fig. 7.1, as well as in Table 2, where similar information, corresponding to the same portion of $\mathcal{R}(a)$, has been displayed for $\tau = 100$. The decay of the error is consistent with the $O(1/\tau^{n+1})$ estimate from Theorem 3.3(c).

Table 1: The error $|z_{k,\tau}^{(n)} - \lambda_{k,\tau}|$ for $\tau = 20$, $k = 10, \ldots, 17$ and the iterations $n = 0, 1, 2, 3, 4$.

iteration	10	11	12	13	14	15	16	17
0	9.84_{-02}	7.69_{-02}	6.02_{-02}	4.75_{-02}	3.80_{-02}	3.10_{-02}	2.58_{-02}	2.19_{-02}
1	3.69_{-03}	2.68_{-03}	1.95_{-03}	1.42_{-03}	1.03_{-03}	7.55_{-04}	5.55_{-04}	4.11_{-04}
2	1.40_{-04}	9.47_{-05}	6.38_{-05}	4.26_{-05}	2.82_{-05}	1.84_{-05}	1.20_{-05}	7.70_{-06}
3	5.33_{-06}	3.55_{-06}	2.09_{-06}	1.28_{-06}	7.70_{-07}	4.51_{-07}	2.58_{-07}	1.44_{-07}
4	2.03_{-07}	1.18_{-07}	6.85_{-08}	3.86_{-08}	2.10_{-08}	1.10_{-08}	5.56_{-09}	2.71_{-09}

Table 2: The error $|z_{k,\tau}^{(n)} - \lambda_{k,\tau}|$ for $\tau = 100$, $k = 50, 55, \ldots, 85$ and the iterations $n = 0, 1, 2, 3, 4$.

iteration	50	55	60	65	70	75	80	85
0	2.09_{-02}	1.62_{-02}	1.26_{-02}	9.87_{-03}	7.86_{-03}	6.38_{-03}	5.29_{-03}	4.48_{-03}
1	1.66_{-04}	1.19_{-04}	8.50_{-05}	6.11_{-05}	4.42_{-05}	3.21_{-05}	2.35_{-05}	1.73_{-05}
2	1.32_{-06}	8.69_{-07}	5.75_{-07}	3.79_{-07}	2.48_{-07}	1.62_{-07}	1.05_{-07}	6.72_{-08}
3	1.05_{-08}	6.37_{-09}	3.89_{-09}	2.35_{-09}	1.40_{-09}	8.14_{-10}	4.65_{-10}	2.60_{-10}
4	8.32_{-11}	4.67_{-11}	2.63_{-11}	1.46_{-11}	7.86_{-12}	4.10_{-12}	2.07_{-12}	1.01_{-12}

Our second example is $a(x) = 1 + 2\sum_{k=1}^{4} \alpha_k \mu_k/(x^2 + \mu_k^2)$, where $\boldsymbol{\alpha} = [-1, 2+\mathrm{i}, 3\mathrm{i}, 1+\mathrm{i}]$ and $\boldsymbol{\mu} = [1+2\mathrm{i}, 2, 2-3\mathrm{i}, 3+4\mathrm{i}]$. The set $\mathcal{R}(a)$ and the range of b are displayed in Fig. 7.2. The transition of the eigenvalues from one side of $\mathcal{R}(a)$ to the other is illustrated in Fig. 7.3 for different values of τ. Fig. 7.4 recapitulates (to different scale) the right-hand side of Fig. 7.2, except that the portions corresponding to (3.6) and (3.7) are denoted differently: the first with circles and the second with pluses.

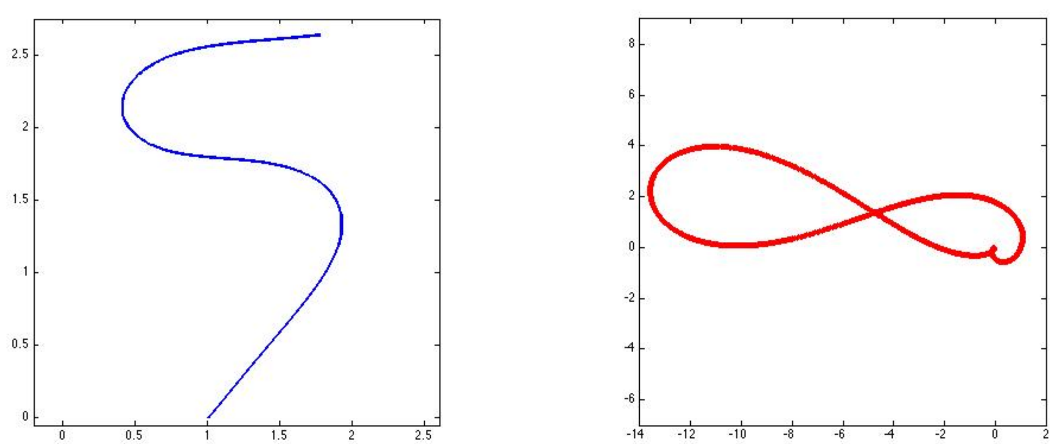

Figure 7.2: The range $\mathcal{R}(a)$ is indicated on the left, while the range of b on $(0, \infty)$ is indicated on the right.

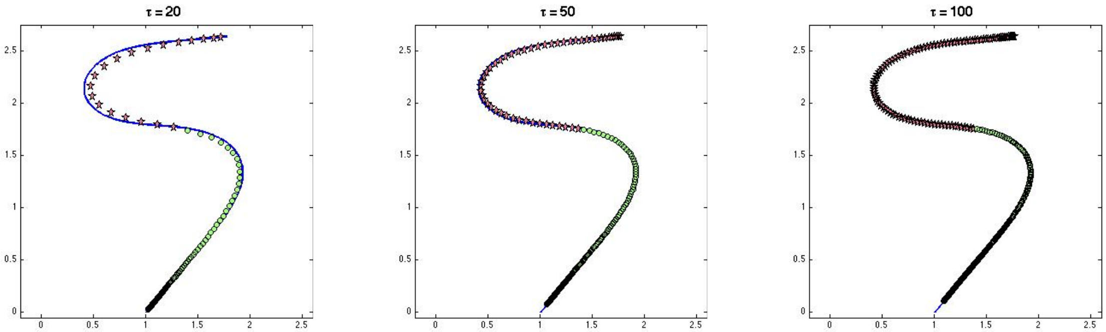

Figure 7.3: The eigenvalues of $W_\tau(a)$ for different values of τ. The eigenvalues consistent with (3.6) are denoted by a star and those conforming with (3.7) by a disc.

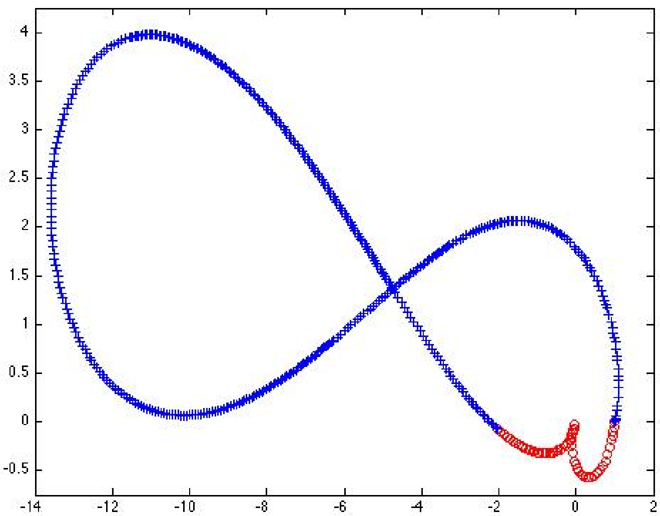

Figure 7.4: The range of b on $(0, \infty)$ with the points corresponding to (3.6) and to (3.7) denoted by circles and pluses, respectively.

References

[1] M. Bogoya, A. Böttcher, and S. Grudsky, *Asymptotics of individual eigenvalues of a class of large Hessenberg Toeplitz matrices.* Operator Theory: Advances and Applications, to appear.

[2] A. Böttcher, *Wiener–Hopf determinants with rational symbols.* Math. Nachr. **144** (1989), 39–64.

[3] A. Böttcher, H. Brunner, A. Iserles, and S. Nørsett, *On the singular values and eigenvalues of the Fox–Li and related operators.* New York J. Math., to appear.

[4] A. Böttcher, S. Grudsky, D. Huybrechs, and A. Iserles, *First-order trace formulas for the iterates of the Fox–Li operator.* To appear.

[5] A. Böttcher, S. Grudsky, and E. A. Maksimenko, *Inside the eigenvalues of certain Hermitian Toeplitz band matrices.* J. Comput. Appl. Math. **233** (2010), 2245–2264.

[6] A. Böttcher and B. Silbermann, *Introduction to Large Truncated Toeplitz Matrices.* Springer-Verlag, New York, 1999.

[7] A. Böttcher and H. Widom, *Two remarks on spectral approximations for Wiener–Hopf operators.* J. Integral Equations Appl. **6** (1994), 31–36.

[8] H. Brunner, A. Iserles, and S. P. Nørsett, *The spectral problem for a class of highly oscillatory Fredholm integral operators.* IMA J. Numer. Analysis **30** (2010), 108-130.

[9] H. Brunner, A. Iserles, and S. P. Nørsett, *The computation of the spectra of highly oscillatory Fredholm integral operators.* J. Integral Equations Appl., to appear.

[10] J. A. Cochran, *Analysis of Linear Integral Equations.* McGraw–Hill, New York, 1972.

[11] F. Gesztesy and K. A. Makarov, *(Modified) Fredholm determinants for operators with matrix–valued semi–separable integral kernels revisited.* Integral Equations Operator Theory **47** (2003), 457–497 and **48** (2004), 425–426.

[12] I. Gohberg and I. A. Feldman, *Convolution Equations and Projection Methods for Their Solution.* Amer. Math. Soc., Providence, RI, 1974.

[13] H. Landau, *The notion of approximate eigenvalues applied to an integral equation of laser theory.* Quart. Appl. Math. **35** (1977/78), 165–172.

[14] H. Landau and H. Widom, *Eigenvalue distribution of time and frequency limiting.* J. Math. Analysis Appl. **77** (1980), 469–481.

[15] P. Tilli, *Some results on complex Toeplitz eigenvalues.* J. Math. Anal. Appl. **239** (1999), 390–401.

[16] H. Widom, *On the eigenvalues of certain Hermitian forms.* Trans. Amer. Math. Soc. **88** (1958), 491–522.

[17] H. Widom, *Extreme eigenvalues of translation kernels.* Trans. Amer. Math. Soc. **100** (1961), 252–262.

[18] H. Widom, *Eigenvalue distribution of nonselfadjoint Toeplitz matrices and the asymptotics of Toeplitz determinants in the case of nonvanishing index.* Operator Theory: Adv. Appl. **48** (1990), 387–421.

Albrecht Böttcher, Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany

`aboettch@mathematik.tu-chemnitz.de`

Sergei M. Grudsky, Departamento de Matemáticas, CINVESTAV del I.P.N., Apartado Postal 14-740, 07000 México, D.F., México

`grudsky@math.cinvestav.mx`

Arieh Iserles, Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Cambridge CB3 0WA, United Kingdom

`A.Iserles@damtp.cam.ac.uk`