# Orthogonal polynomials on the unit circle and functional-differential equations

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**Abstract**

Classical orthogonal polynomials on the real line share the feature that they all obey a linear second-order differential equation. This is not the case with regard to orthogonal polynomials on the unit circle: such polynomials or for that matter their generating functions, are not known to satisfy a differential equation.

In this paper we study families of polynomials orthogonal on the unit circle, generalizing the familiar Geronimus and Rogers–Szegő polynomials, whose generating function obeys a second-order linear functional-differential equation, a special case of a so-called *pantograph equation.* This leads to a raft of new results, expressing such polynomials in terms of $q$-hypergeometric functions.

## 1 Introduction

Let $\{\phi_n\}_{n\in\mathbb{Z}_+}$ be the set of monic polynomials orthogonal on the unit circle with respect to some measure,

$$\int_{\mathbb{T}} \phi_n(z)\bar{\phi}_m(z)\,\mathrm{d}\mu(z) = 0, \qquad m \neq n.$$

We let $a_n = \phi_n(0)$, $n \in \mathbb{Z}_+$ – it is elementary that $|a_n| < 1$, $n \in \mathbb{N}$. It is known that the $\phi_n$s obey the three-term recurrence relation

$$\phi_n(z) = z\phi_{n-1}(z) + a_n\phi^*_{n-1}(z), \qquad n \in \mathbb{N},$$

where $p^*(z) = z^n \bar{p}(z^{-1})$, $p \in \mathbb{P}_n$, as well as the difference equation

$$(a_{n+1} + a_n z)\phi_n(z) = a_n \phi_{n+1}(z) + (1 - |a_n|^2) a_{n+1} z \phi_{n-1}(z), \qquad n \in \mathbb{N}, \tag{1.1}$$

with the initial conditions

$$\phi_0(z) \equiv 1, \qquad \phi_1(z) = z + a_1$$

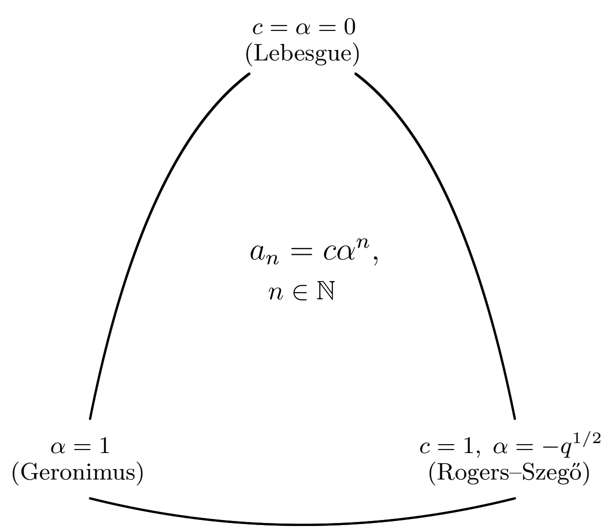
(Simon 2005). Note therefore that any sequence $\{a_n\}_{n \in \mathbb{Z}_+}$ of this kind uniquely defines a set of OPUC (orthogonal polynomials on the unit circle) and it is known as sequence of *Schur parameters.* Let $\alpha, c \in \mathbb{C}$, where $0 < |c|, |\alpha| < 1$. We are interested in the orthogonal polynomials corresponding to the sequence

$$a_n = \begin{cases} 1, & n = 0, \\ c\alpha^n, & n \in \mathbb{N}. \end{cases} \tag{1.2}$$

In that case (1.1) becomes

$$(\alpha + z)\phi_n(z) = \phi_{n+1}(z) + \alpha(1 - |c|^2 |\alpha|^{2n}) z \phi_{n-1}(z). \tag{1.3}$$

Note that $\alpha = 1$ in (1.2) corresponds to *Geronimus polynomials,* while $c = 1$ and $\alpha = -q^{1/2}$, $|q| < 1$, to *Rogers–Szegő polynomials.* Moreover, $\alpha = c = 0$ correspond to the standard Lebesgue measure on the unit circle, with $\phi_n(z) = z^n$ (in the context of this paper, for brevity, we call these "Lebesgue polynomials"). Thus, our concern here is to investigate a far-reaching generalization of these three important families of orthogonal polynomials on the unit circle: pictorially, this can be represented in the scheme




Note that our approach to OPUC in this paper is purely formal: the polynomials are defined, using the recurrence (1.1), directly from a set of Schur parameters. Indeed, the measure $\mathrm{d}\mu = \mathrm{d}\mu_{c,\alpha}$ that renders $\{\phi_n\}_{n\in\mathbb{Z}_+}$ is at present unknown, except for few special cases, and is subject to current investigation.

In Section 2 we demonstrate that a generating function of the OPUC with respect to the Schur parameters (1.2) obeys a functional-differential equation of a pantograph type. Such equations are known to possess an expansion in Dirichlet series (Iserles 1993) and this line of reasoning is pursued in Section 3 and leads to an explicit expression for the underlying OPUC. The main currency of this representation is a $q$-hypergeometric function which we explore in Section 4. A consequence of our analysis, presented in Section 5, is a representation of our OPUC in terms of $q$-Bessel functions: such representations in terms of hypergeometric or $q$-hypergeometric functions are ubiquitous in the theory of orthogonal polynomials on the real line but novel in the context of orthogonal polynomials on the unit circle. Finally, in Section 6 we explore a limiting behaviour of our representation once the parameters $\alpha$ and $c$ are allowed to approach values associated with Lebesgue, Rogers–Szegő and Geronimus polynomials.

## 2 Orthogonal polynomials on the unit circle and the pantograph equation

We consider the generating function

$$\Phi(t,z) = \sum_{n=0}^{\infty} \frac{\phi_n(z)}{n!} t^n.$$

We will often suppress the dependence of $\Phi$ upon $z$, in which case we write $\Phi = \Phi(t)$.

Multiplying (1.3) by $t^n/n!$ and summing up for $n = 1, 2, \ldots$ results in

$$(\alpha+z)\sum_{n=1}^{\infty} \frac{\phi_n}{n!} t^n = \sum_{n=1}^{\infty} \frac{\phi_{n+1}}{n!} t^n + \alpha z \sum_{n=1}^{\infty} \frac{\phi_{n-1}}{n!} t^n - \alpha |c|^2 z \sum_{n=1}^{\infty} \frac{\phi_{n-1}}{n!} |\alpha|^{2n} t^n.$$

However,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\phi_n}{n!} t^n &= \Phi(t) - \Phi(0),\\
\sum_{n=1}^{\infty} \frac{\phi_{n+1}}{n!} t^n &= \frac{\partial}{\partial t} \sum_{n=2}^{\infty} \frac{\phi_n}{n!} t^n = \frac{\partial}{\partial t}[\Phi(t) - \Phi(0) - \Phi'(0)t] = \Phi'(t) - \Phi'(0),\\
\sum_{n=1}^{\infty} \frac{\phi_{n-1}}{n!} t^n &= \sum_{n=0}^{\infty} \frac{\phi_n}{(n+1)!} t^{n+1} = \int_0^t \Phi(x)\,\mathrm{d}x,\\
\sum_{n=1}^{\infty} \frac{\phi_{n-1}}{n!} |\alpha|^{2n} t^n &= \sum_{n=0}^{\infty} \frac{\phi_n}{(n+1)!} (|\alpha|^2 t)^{n+1} = \int_0^{|\alpha|^2 t} \Phi(x)\,\mathrm{d}x.
\end{aligned}$$

Putting all this together, we have

$$(\alpha+z)[\Phi(t) - \Phi(0)] = \Phi'(t) - \Phi'(0) + \alpha z \int_0^t \Phi(x)\,\mathrm{d}x - \alpha z |c|^2 \int_0^{|\alpha|^2 t} \Phi(x)\,\mathrm{d}x.$$

Finally, we differentiate this expression with respect to $t$, whence

$$(\alpha+z)\Phi'(t)=\Phi''(t)+\alpha z\Phi(t)-\alpha|\alpha|^2|c|^2 z\Phi(|\alpha|^2 t).$$

We rewrite this in the form

$$\Phi''(t)=(\alpha+z)\Phi'(t)-\alpha z\Phi(t)+\alpha\tau z\Phi(qt), \tag{2.1}$$

where $q=|\alpha|^2$, $\tau=q|c|^2$ are both in $(0,1)$, with the initial conditions $\Phi(0)=\phi_0(z)\equiv 1$, $\Phi'(0)=\phi_1(z)=z+c\alpha$.

The equation (2.1) is a special instance of the *pantograph equation*

$$\boldsymbol{y}'(t)=A\boldsymbol{y}(t)+B\boldsymbol{y}(qt),\quad t\geq 0,\qquad \boldsymbol{y}(0)=\boldsymbol{y}_0\in\mathbb{C}^d, \tag{2.2}$$

where $A$ and $B$ are $d\times d$ complex matrices and $q\in(0,1)$ (Iserles 1993). It is known that (2.2) has a unique solution for all $t\in[0,\infty)$ and that, as long as the eigenvalues of $A$ reside in the open left complex half-plane and the eigenvalues of $A^{-1}B$ in the open complex unit disc, it is true that $\lim_{t\to\infty}\boldsymbol{y}(t)=\boldsymbol{0}$. Moreover, as long as $A$ is nonsingular and the spectral radius of $A^{-1}B$ is less than one, the solution of (2.2) can be expanded into Dirichlet series (Iserles 1993). This has profound implications to our study of the generating function $\Phi$.

Just to verify that we are on the right track, we note that for $z=0$ the pantograph reduces to the linear differential equation

$$\Phi''(t)=\alpha\Phi'(t),\quad t\geq 0,\qquad \Phi(0)=1,\quad \Phi'(0)=c\alpha,$$

with the solution $\Phi(t)=c\mathrm{e}^{\alpha t}+1-c$. Therefore

$$a_n=\Phi^{(n)}(0)=\begin{cases}1, & n=0,\\ c\alpha^n, & n\in\mathbb{N},\end{cases}$$

as required.

It is instructive to examine (2.1) in the three important special cases. For the Lebesgue case $\alpha=c=0$ the equation reduces to $\Phi''(t)=z\Phi'(t)$, with the initial conditions $\Phi(0)=1$, $\Phi'(0)=z$, therefore $\Phi(t,z)=\mathrm{e}^{tz}$ and we recover $\phi_n(z)=z^n$, $n\in\mathbb{Z}_+$. For Geronimus polynomials $\alpha=1$, hence $q=1$ and $\tau=|c|^2$, (2.1) reduces to an ordinary differential equation,

$$\Phi''-(1+z)\Phi'+(1-|c|^2)z\Phi=0,\qquad t\geq 0,$$

whose general solution is $\Phi(t)=\beta_+\mathrm{e}^{t\varrho_+}+\beta_-\mathrm{e}^{t\varrho_-}$, where

$$\varrho_\pm=\frac{1+z\pm\sqrt{(1-z)^2+4|c|^2 z}}{2}$$

are the roots of the quadratic $\varrho^2-(1+z)\varrho+(1-|c|^2)z=0$. Fitting the initial conditions $\Phi(0)=1$, $\Phi'(0)=z+c$, we have

$$\beta_\pm=\frac{1}{2}\mp\frac{(1-z)-2c}{\sqrt{(1-z)^2+4|c|^2 z}}.$$

This results in the known representation of Geronimus polynomials, namely

$$\phi_n(z) = \left[\frac{1}{2} - \frac{(1-z)-2c}{\sqrt{(1-z)^2+4|c|^2 z}}\right]\left[\frac{1+z+\sqrt{(1-z)^2+4|c|^2 z}}{2}\right]^n \\ + \left[\frac{1}{2} + \frac{(1-z)-2c}{\sqrt{(1-z)^2+4|c|^2 z}}\right]\left[\frac{1+z-\sqrt{(1-z)^2+4|c|^2 z}}{2}\right]^n, \qquad n \in \mathbb{Z}_+ \tag{2.3}$$

(Simon 2005, p. 87). This representation of an orthogonal polynomial system using 'non-polynomial' building blocks is similar in this sense to the familiar formula for Chebyshev polynomials of first and second kind (Rainville 1960, p. 301).

Finally, in the Rogers–Szegő case $c = 1$, $\alpha = -q^{1/2}$, we stay with a pantograph equation, specifically

$$\Phi''(t) = (z - q^{1/2})\Phi'(t) + q^{1/2} z\Phi(t) - q^{3/2} z\Phi(qt), \qquad t \ge 0, \tag{2.4}$$

with $\Phi(0) = 1$, $\Phi'(0) = z - q^{1/2}$.

Our next step is to study the solution of (2.1) in order to obtain a general expression for the $\phi_n$s – this is the subject matter of the next section. In the last section we apply our results and we analyse the special cases of Geronimus and Rogers–Szegő polynomials from the point of view of this paper, i.e. commencing from the pantograph equation (2.1).

# 3 Study of the solutions through Dirichlet series

Provided that the pantograph equation (2.2) is in a stable regime, its solution can be expanded in Dirichlet series,

$$\boldsymbol{y}(t) = \sum_{m=0}^{\infty} \boldsymbol{e}^{tq^n A}\boldsymbol{v}_n, \qquad t \ge 0 \tag{3.1}$$

(Iserles 1993).

The general pantograph equation

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + B\boldsymbol{y}(qt), \quad t \ge 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0,$$

has a Dirichlet solution provided that $A$ is invertible and $\|A^{-1}B\|_2 < 1$. In our case, we have

$$\boldsymbol{y} = \begin{bmatrix} \Phi \\ \Phi' \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ -\alpha z & \alpha + z \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ \alpha z \tau & 0 \end{bmatrix},$$

therefore

$$A^{-1}B = \tau \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of $A$ are $\alpha$ and $z$, both nonzero, hence the matrix is nonsingular, while the spectral radius of $A^{-1}B$ is $\tau \in [0, 1)$. Consequently, the solution of (2.1) can be expanded into a Dirichlet series of the form (2.1). Specifically, $\Phi$ possesses the expansion

$$\Phi(t)=\sum_{m=0}^{\infty} v_m \mathrm{e}^{\lambda q^m t}, \qquad v_0\neq 0,$$

where $\lambda$ and $\{v_m\}_{m\in\mathbb{Z}_+}$ are constant with respect to $t$ (the variable $z$ is treated as a parameter). Substituting this into (2.1), we have

$$\begin{aligned}\lambda^2\sum_{m=0}^{\infty} v_m q^{2m}\mathrm{e}^{\lambda q^m t} = (\alpha+z)\lambda\sum_{m=0}^{\infty} v_m q^{m}\mathrm{e}^{\lambda q^m t}-\alpha z\sum_{m=0}^{\infty} v_m \mathrm{e}^{\lambda q^m t}\\ +\alpha\tau z\sum_{m=1}^{\infty} v_{m-1}\mathrm{e}^{\lambda q^m t}.\end{aligned}$$

Assuming $\lambda\neq 0$, the functions $\mathrm{e}^{\lambda q^m t}$ are linearly independent for all $m\in\mathbb{Z}$, therefore it follows that

$$[\lambda^2 q^{2m}-(\alpha+z)q^m+\alpha z]v_m=\begin{cases}0, & m=0,\\ \alpha z\tau v_{m-1}, & m\in\mathbb{N}.\end{cases} \tag{3.2}$$

The immediate consequence of (3.2) is that, letting $m=0$, $v_0\neq 0$ implies

$$\lambda^2-(\alpha+z)\lambda+\alpha z=(\lambda-\alpha)(\lambda-z)=0$$

and we deduce that there exist two admissible values of $\lambda$, namely $\lambda=\alpha$ and $\lambda=z$.

Next, we consider the case $m\in\mathbb{Z}$. Now

$$(\alpha-q^m\lambda)(z-q^m\lambda)v_m=\alpha z\tau v_{m-1},$$

therefore,

$$\begin{aligned}\lambda=\alpha: &\qquad (1-q^m)\left(1-q^m\frac{\alpha}{z}\right)v_m=\tau v_{m-1},\\ \lambda=z: &\qquad (1-q^m)\left(1-q^m\frac{z}{\alpha}\right)v_m=\tau v_{m-1}.\end{aligned}$$

Using easy induction, we have

$$v_m=\frac{\tau^m}{(q,q)_m(\alpha/z,q)_m}v_0 \qquad \text{and} \qquad v_m=\frac{\tau^m}{(q,q)_m(z/\alpha,q)_m}v_0$$

respectively, where $(\kappa,q)_m$ is the Gauss–Heine symbol,

$$(\kappa,q)_m=\prod_{j=0}^{m-1}(1-\kappa q^j)$$

(Gasper & Rahman 2004). This argument has led us to a Dirichlet-series representation of $\Phi$.

**Theorem 1** *The generating function* $\Phi(t,z) = \sum_{m=0}^{\infty} \phi_m(z) t^m/m!$ *of OPUC with respect to the Schur parameters (1.2) can be expressed explicitly in the form*

$$\Phi(t,z) = \beta_1(z) \sum_{m=0}^{\infty} \frac{\tau^m}{(q,q)_m(\alpha/z,q)_m} \mathrm{e}^{\alpha q^m t} + \beta_2(z) \sum_{m=0}^{\infty} \frac{\tau^m}{(q,q)_m(z/\alpha,q)_m} \mathrm{e}^{z q^m t}. \quad (3.3)$$

*where* $\beta_1$ *and* $\beta_2$ *are determined by the conditions* $\Phi(0,z) \equiv 1$, $\partial\Phi(0,z)/\partial t = z + c\alpha$.

**Corollary 1** *The monic OPUC with respect to the Schur parameters (1.2) is*

$$\phi_m(z) = \alpha^m \beta_1(z) F(\alpha z^{-1}, q^m\tau, q) + z^m \beta_2(z) F(\alpha^{-1} z, q^m \tau, q), \qquad m \in \mathbb{Z}_+, \quad (3.4)$$

*where*

$$F(\zeta,\tau,q) = \sum_{m=0}^{\infty} \frac{\tau^m}{(q,q)_m(\zeta,q)_m}.$$

*Proof* Repeatedly differentiating the Dirichlet series (3.3) term-by-term, a procedure which is justified by its absolute convergence. □

Note that $F(0,\tau,q)$ is the so-called "little $q$-exponential function",

$$F(0,\tau,q) = \sum_{m=0}^{\infty} \frac{\tau^m}{(q,q)_m} = \mathrm{e}_q(\tau) = \frac{1}{(\tau,q)_\infty}$$

(Gasper & Rahman 2004, p. 236), while

$$\lim_{|\zeta|\to\infty} F(\zeta, q^m\tau, q) \equiv 1, \qquad \lim_{\zeta\to 0} \zeta^m F(\zeta; q^m\tau, q) = \begin{cases} 1, & m = 0, \\ 0, & m \in \mathbb{N}, \end{cases}$$

where $|\zeta| \to \infty$ does so in a sector of the form $|\arg\zeta| > \delta$ for some $\delta > 0$. Therefore

$$\phi_0(0) = \beta_1(0) + \beta_2(0), \qquad \phi_n(0) = \beta_1(0)\alpha^n, \quad n \in \mathbb{N},$$

where we recall that $\beta_1$ and $\beta_2$ are determined by the initial conditions,

$$\phi_0(z) \equiv 1, \qquad \phi_1(z) = z + c\alpha.$$

Thus, $\beta_1(0) = c$, $\beta_2(0) = 1 - c$, and we verify from (3.4) the explicit form of Schur parameters,

$$\phi_n(0) = c\alpha^n, \qquad n \in \mathbb{Z}_+.$$

It is convenient to reformulate (3.4) somewhat. Thus, we let

$$\eta_1(z) = \beta_1(z) F(\alpha z^{-1}, \tau, q), \qquad \eta_2(z) = \beta_2(z) F(\alpha^{-1} z, \tau, q)$$

and

$$H_m(\zeta,\tau,q) = \frac{F(\zeta, q^m\tau, q)}{F(\zeta,\tau,q)}, \qquad m \in \mathbb{Z}_+.$$

Then (3.4) can be rewritten in the form

$$\phi_m(z) = \alpha^m \eta_1(z) H_m(\alpha z^{-1}, \tau, q) + z^m \eta_2(z) H_m(\alpha^{-1} z, \tau, q), \qquad m \in \mathbb{Z}_+. \quad (3.5)$$

The initial conditions being

$$\begin{aligned} \eta_1 + \eta_2 &= 1, \\ \alpha H_1(\alpha z^{-1}, \tau, q)\eta_1 + z H_1(\alpha^{-1} z, \tau, q)\eta_2 &= z + c\alpha, \end{aligned}$$

we obtain

$$\begin{aligned} \eta_1(z) &= \frac{z + c\alpha - z H_1(\alpha^{-1} z, \tau, q)}{\alpha H_1(\alpha z^{-1}, \tau, q) - z H_1(\alpha^{-1} z, \tau, q)}, \\ \eta_2(z) &= \frac{\alpha H_1(\alpha z^{-1}, \tau, q) - z - c\alpha}{\alpha H_1(\alpha z^{-1}, \tau, q) - z H_1(\alpha^{-1} z, \tau, q)}. \end{aligned} \tag{3.6}$$

The representation (3.5) is not the final form in which we can cast the OPUC $\{\phi_n\}_{n\in\mathbb{Z}_+}$.

**Theorem 2** *The OPUC with respect to the Schur parameters (1.2) is*

$$\phi_m(z) = \alpha^m \eta_1(z) \prod_{\ell=1}^{m} H_1(\alpha z^{-1}, q^\ell \tau, q) + z^m \eta_2(z) \prod_{\ell=1}^{m} H_1(\alpha^{-1} z, q^\ell \tau, q), \qquad m \in \mathbb{Z}_+, \tag{3.7}$$

*where $\eta_1$ and $\eta_2$ have been given in (3.6).*

*Proof* Follows at once from (3.5), noting that

$$H_m(\zeta, \tau, q) = \frac{F(\zeta, q\tau, q)}{F(\zeta, \tau, q)} \times \frac{F(\zeta, q^2\tau, q)}{F(\zeta, q\tau, q)} \times \cdots \times \frac{F(\zeta, q^m\tau, q)}{F(\zeta, q^{m-1}\tau, q)} = \prod_{\ell=1}^{m} H_1(\zeta, q^\ell \tau, q).$$

□

The representation (3.7) has an important advantage in comparison with the seemingly simpler form (3.5): we need to deal with just a single function $H_1$, rather than with $H_m$ for all $m \in \mathbb{N}$. It is also reminiscent of the representation (2.3) of Geroniumus polynomials and indeed we will prove in the sequel that (3.7) becomes (2.3) as $\alpha \to 1$.

# 4 The function $H_1$

## 4.1 Analyticity

It will be proved later in this section that the function

$$F(\zeta, \tau, q) = \sum_{m=0}^{\infty} \frac{\tau^m}{(q,q)_m (\zeta, q)_m}, \qquad \tau \in (0, q),$$

is meromorphic in $\zeta \in \mathbb{C}$. Specifically, it is analytic except for simple polar singularities at $q^{-\ell}$ for all $\ell \in \mathbb{Z}_+$, because $(q^{-\ell}, q)_m = 0$ for $m \geq \ell + 1$. Our interest is, however, not in the function $F$ *per se* but in the ratios $H_m(\alpha z^{-1}, \tau, q)$ and $H_m(\alpha^{-1} z, \tau, q)$ for $m \in \mathbb{N}$.

Let $\zeta = q^{-\ell} + \varepsilon$ for some $\ell \in \mathbb{Z}_+$ and $0 < |\varepsilon| \ll 1$. It is an easy calculation that

$$(\zeta, q)_m = (-1)^m q^{(m-1-\ell)m} \frac{(q,q)_\ell}{(q,q)_{\ell-m}} + \mathcal{O}(\varepsilon), \qquad m \le \ell,$$
$$(\zeta, q)_m = (-1)^{\ell+1} \varepsilon q^{-\frac{1}{2}(\ell-1)\ell} (q,q)_\ell (q,q)_{m-\ell-1} + \mathcal{O}(\varepsilon^2), \qquad m \ge \ell+1.$$

Therefore, after further algebra,

$$F(q^{-\ell}+\varepsilon, \tau, q) = \frac{1}{\varepsilon} \frac{(-1)^{\ell+1} q^{\frac{1}{2}(\ell-1)\ell} \tau^{\ell+1}}{(q,q)_\ell (q,q)_{\ell+1}} F(q^{\ell+2}, \tau, q) + \mathcal{O}(1).$$

We deduce that

$$H_1(q^{-\ell}+\varepsilon, \tau, q) = q^{\ell+1} H_1(q^{\ell+2}, \tau, q) + \mathcal{O}(\varepsilon).$$

Therefore the singularity at $q^{-\ell}$ is removable. This, however, does not mean that $H_1$, unlike $F$, is an entire function, because it has polar singularities at the zeros of $F(\,\cdot\,, \tau, q)$. Indeed, we demonstrate in the sequel that $H_1$ is meromorphic, with a countable number of isolated poles accumulating at infinity.

Note that, according to Section 3, $\lim_{|\zeta|\to\infty} F(\zeta, \tau, q) = 1$, *as long as $\zeta$ is restricted to a sector of the form $|\arg \zeta| > \delta > 0$.* Hence, subject to this restriction, $\lim_{|\zeta|\to\infty} H_1(\zeta, \tau, q) = 1$, while $\lim_{M\to\infty} H_1(q^{-M}, \tau, q) = 0$.

## 4.2 An expansion in $\zeta$

The function $F$ has been given as a power series in $\tau$. However, and given its analyticity in $\zeta$, it is instructive to expand it in power series in the latter variable. We commence by observing that

$$\begin{aligned} F(\zeta, \tau, q) - F(q\zeta, \tau, q) &= \sum_{m=1}^{\infty} \frac{\tau^m}{(q,q)_m} \left[ \frac{1}{(\zeta,q)_m} - \frac{1}{(q\zeta,q)_m} \right] \\ &= \frac{\zeta}{1-\zeta} \sum_{m=0}^{\infty} \frac{(1-q^m)\tau^m}{(q,q)_m (q\zeta,q)_m} \\ &= \frac{\zeta}{1-\zeta} [F(q\zeta, \tau, q) - F(q\zeta, q\tau, q)]. \end{aligned}$$

This yields the recurrence relation

$$F(\zeta, \tau, q) = \frac{1}{1-\zeta} [F(q\zeta, \tau, q) - \zeta F(q\zeta, q\tau, q)]. \tag{4.1}$$

Before we advance any further, it is useful to recall the definition of an ${}_r\phi_s$ *basic hypergeometric function:* given $r, s \in \mathbb{Z}_+$ and $q, a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{C}$, $|q| < 1$,

$${}_r\phi_s \left[ \begin{matrix} a_1, \ldots, a_r; \\ b_1, \ldots, b_s; \end{matrix} \; q, z \right] = \sum_{m=0}^{\infty} \frac{(a_1,q)_m (a_2,q)_m \cdots (a_r,q)_m}{(q,q)_m (b_1,q)_m (b_2,q)_m \cdots (b_s,q)_m} \left[ (-1)^m q^{\binom{m}{2}} \right]^{1+s-r} z^m$$

(Gasper & Rahman 2004, p. 4).

**Proposition 3** *The function $F$ can be expressed in the form*

$$F(\zeta,\tau,q) = \frac{1}{(\zeta,q)_\infty(\tau,q)_\infty}\sum_{m=0}^{\infty}\frac{(\tau,q)_m}{(q,q)_m}q^{\frac{1}{2}(m-1)m}(-\zeta)^m \qquad (4.2)$$
$$= \frac{1}{(\zeta,q)_\infty(\tau,q)_\infty}\,{}_1\phi_1\left[\begin{matrix}\tau;\\ 0;\end{matrix}\, q,\zeta\right].$$

*Proof* We commence by proving that, for any $r \in \mathbb{Z}_+$,

$$F(\zeta,\tau,q) = \frac{1}{(\zeta,q)_r}\sum_{m=0}^{r}\left[\begin{matrix} r\\ m\end{matrix}\right]_q q^{\frac{1}{2}(m-1)m}(-\zeta)^m F(q^r\zeta, q^m\tau, q),$$

where

$$\left[\begin{matrix} n\\ m\end{matrix}\right]_q = \frac{(q,q)_n}{(q,q)_m(q,q)_{n-m}}, \qquad 0 \le m \le n,$$

is the *$q$-binomial symbol* (Gasper & Rahman 2004, p. 235).

This is certainly true for $r = 0$ and, because of (4.1), for $r = 1$. Moreover, using induction on $r$ and applying (4.1) on the right-hand side,

$$\begin{aligned}
&F(\zeta,\tau,q)\\
&= \frac{1}{(\zeta,q)_r}\sum_{m=0}^{r}\left[\begin{matrix} r\\ m\end{matrix}\right]_q q^{\frac{1}{2}(m-1)m}\frac{(-\zeta)^m}{1-q^r\zeta}[F(q^{r+1}\zeta, q^m\tau, q) - q^r\zeta F(q^{r+1}\zeta, q^{m+1}\tau, q)]\\
&= \frac{1}{(\zeta,q)_{r+1}}\left\{\sum_{m=0}^{r}\left[\begin{matrix} r\\ m\end{matrix}\right]_q q^{\frac{1}{2}(m-1)m}(-\zeta)^m F(q^{r+1}\zeta, q^m\tau, q)\right.\\
&\qquad\left. + \sum_{m=1}^{r+1}\left[\begin{matrix} r\\ m-1\end{matrix}\right]_q q^{\frac{1}{2}(m-2)(m-1)+r}(-\zeta)^m F(q^{r+1}\zeta, q^m\tau, q)\right\}
\end{aligned}$$

and the desired expression follows from identity

$$\left[\begin{matrix} r\\ m\end{matrix}\right]_q + q^{r-m+1}\left[\begin{matrix} r\\ m-1\end{matrix}\right]_q = \left[\begin{matrix} r+1\\ m\end{matrix}\right]_q$$

(Gasper & Rahman 2004, p. 235).

To prove (4.2), we let $r \to \infty$, noting that for every fixed $m$

$$\lim_{r\to\infty}\left[\begin{matrix} r\\ m\end{matrix}\right]_q = \frac{1}{(q,q)_m}$$

and that

$$\lim_{r\to\infty} F(q^r\zeta,\tau,q) = F(0,\tau,q) = \sum_{m=0}^{\infty}\frac{\tau^m}{(q,q)_m} = \mathrm{e}_q(\tau) = \frac{1}{(\tau,q)_\infty}$$

(cf. Section 2). □

Using (4.2), we investigate the analycity of $H_1$. Our point of departure is the observation that

$$F(\zeta,\tau,q) = \frac{G(\zeta,\tau,q)}{(\tau,q)_\infty(\zeta,q)_\infty},$$

where

$$G(\zeta,\tau,q) = \sum_{m=0}^{\infty}(-1)^m \frac{(\tau,q)_m}{(q,q)_m} q^{\frac{1}{2}(m-1)m}\zeta^m.$$

It is obvious that $G$ is an entire function of $\zeta$.

Given an entire function $f(\zeta) = \sum_{m=0}^{\infty} f_m\zeta^m$, its *order* is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log\max_{-\pi\le\theta\le\pi}|f(r\mathrm{e}^{\mathrm{i}\theta})|}{\log r}$$

(Hille 1962, p. 182) and, while at the first instance it describes the behaviour near the singularity at $\infty$, it can be used to reveal many other interesting features. An alternative expression for $\rho(f)$ is

$$\rho(f) = \limsup_{n\to\infty} \frac{m\log m}{\log|f_m|^{-1}}$$

(Hille 1962, p. 186). Therefore

$$\rho(G) = \limsup_{m\to\infty} \frac{m\log m}{\log(q,q)_m - \log(\tau,q)_m + \frac{1}{2}(m-1)m|\log q|} = 0.$$

Since $\rho(G)=0$, $G(0,\tau,q)=1$ and $G$ is clearly not a polynomial in $\zeta$ (recall that $\tau<1$), we use the *Hadamard factorization theorem* to argue that it can be represented in the form

$$G(\zeta,\tau,q) = \prod_{n=1}^{\infty}\left(1-\frac{\zeta}{\sigma_n}\right), \qquad \zeta\in\mathbb{C},$$

where $\sigma_n = \sigma_n(\tau,q)\in\mathbb{C}$ accumulate at $\infty$. We deduce that

$$H_1(\zeta,\tau,q) = \frac{(\tau,q)_\infty}{(q\tau,q)_\infty}\frac{G(\zeta,q\tau,q)}{G(\zeta,\tau,q)} = (1-\tau)\prod_{n=1}^{\infty}\frac{1-\zeta/\sigma_n(q\tau,q)}{1-\zeta/\sigma_n(\tau,q)}. \tag{4.3}$$

In particular, this indeed proves that $H_1$ is meromorphic.

The only possible impediment to the analyticity of $H_1$ are poles, i.e. the zeros of $G(\,\cdot\,,\tau,q)$. However,

$$G(\zeta,0,q) = \sum_{m=0}^{\infty}(-1)^m\frac{q^{\frac{1}{2}(m-1)m}}{(q,q)_m}\zeta^m = \mathrm{E}_q(-\zeta) = (\zeta,q)_\infty$$

– $\mathrm{E}_q$ is the 'big $q$-exponential function' (Gasper & Rahman 2004, p. 236). Therefore, for $\tau=0$ the only zeros of $G(\,\cdot\,,0,q)$ are $q^{-\ell}$, $\ell\in\mathbb{Z}_+$, all positive, distinct and cancelling each other in the quotient $H_1$.

Next, we compute $G(q^{-\ell},\tau,q)$ for $\ell\in\mathbb{Z}_+$ and $\tau>0$. To this end we utilize the identity

$$(\tau,q)_m=\sum_{k=0}^{m}(-1)^k\left[\begin{array}{c}m\\ k\end{array}\right]_q q^{\frac{1}{2}(k-1)k}\tau^k,\qquad m\in\mathbb{Z}_+,$$

whose inductive proof is trivial and left to the reader. Thus,

$$\begin{aligned}
G(q^{-\ell},\tau,q)&=\sum_{m=0}^{\infty}(-1)^m\frac{q^{\frac{1}{2}(m-1)m}}{(q,q)_m}\sum_{k=0}^{m}(-1)^k\begin{bmatrix}m\\ k\end{bmatrix}_q q^{\frac{1}{2}(k-1)k-m\ell}\tau^k\\
&=\sum_{k=0}^{\infty}\frac{(-1)^k}{(q,q)_k}q^{\frac{1}{2}(k-1)k}\tau^k\sum_{m=k}^{\infty}(-1)^m\frac{q^{\frac{1}{2}(m-1)m-m\ell}}{(q,q)_{m-k}}\\
&=\sum_{k=0}^{\infty}\frac{q^{(k-1)k-k\ell}}{(q,q)_k}\tau^k\sum_{m=0}^{\infty}(-1)^m\frac{q^{\frac{1}{2}(m-1)m+(k-\ell)m}}{(q,q)_m}\\
&=\sum_{k=0}^{\infty}\frac{q^{(k-1)k-k\ell}}{(q,q)_k}\tau^k\mathrm{E}_q(-q^{k-\ell})=\sum_{k=0}^{\infty}\frac{q^{(k-1)k-k\ell}}{(q,q)_k}\tau^k(q^{k-\ell},q)_\infty\\
&=(q,q)_\infty\sum_{k=\ell+1}^{\infty}\frac{q^{(k-1)k-k\ell}}{(q,q)_k(q,q)_{k-\ell-1}}\tau^k>0
\end{aligned}$$

(note that all the series above converge).

Likewise, given $0<|\delta|\ll 1$, an identical algebra yields

$$G(q^{-\ell+\delta},\tau,q)=\sum_{k=0}^{\infty}q^{(k-1)k-k(\ell-\delta)}(q^{k-\ell+\delta},q)_\infty.$$

Now, while for $k\geq\ell+1$ it is true that

$$(q^{k-\ell+\delta},q)_\infty=\frac{(q,q)_\infty}{(q,q)_{k-\ell-1}},$$

for $k=0,1,\ldots,\ell$ we obtain

$$(q^{k-\ell+\delta},q)_\infty=(1-q^\delta)(-1)^{\ell-k}q^{-\frac{1}{2}(\ell-k-1)(\ell-k)}(q,q)_{\ell-k}(q,q)_\infty[1+\mathcal{O}(\delta)].$$

Therefore

$$\begin{aligned}
&(q^{k-\ell+\delta},q)_\infty\\
&=(-1)^{\ell-k}(1-q^\delta)q^{-\frac{1}{2}(\ell-1)\ell}(q,q)_\infty\sum_{k=0}^{\ell}(-1)^k\tau^k\frac{(q,q)_{\ell-k}}{(q,q)_k}q^{\frac{1}{2}(k-1)k+\delta k}[1+O(\delta)]\\
&\quad+G(q^{-\ell},\tau,q)[1+O(\delta)].
\end{aligned}$$

While $G(q^{-\ell},\tau,q)>0$, we can render the first sum negative by choose $\delta>0$ when $\ell$ is even, $\delta<0$ otherwise. Moreover, $G(q^{-\ell},\tau,q)=O(\tau^\ell)$, hence small ($\tau\in(0,q)$), while the first sum is $O(1)$ in $\tau$. We deduce that for every sufficiently small $\tau>0$ and $\ell\in\mathbb{N}$

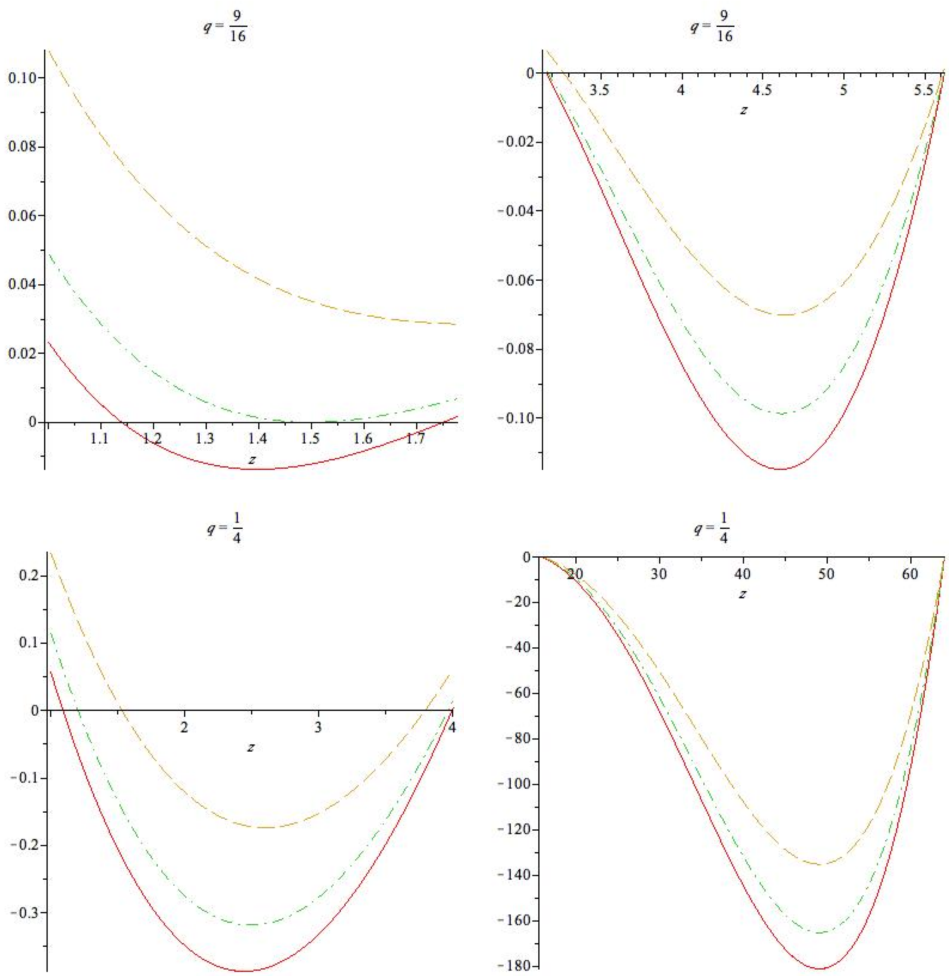



Figure 4.1: The function $G(\zeta,\tau,q)$ for two different values of $q$, sketched for $\zeta \in [1,q^{-1}]$ and $\zeta \in [q^{-2},q^{-3}]$ for $\tau = \frac{1}{20},\frac{1}{10},\frac{1}{5}$ (for $q = \frac{9}{16}$) and $\tau = \frac{1}{16},\frac{1}{8},\frac{1}{4}$ (for $q = \frac{1}{4}$) denotes by solid, dash-dot and dash line styles.

it is true that $\sigma_{2\ell-1}(\tau) < \sigma_{2\ell}(\tau)$ lie in the interval $(q^{-2\ell+1}, q^{-2\ell})$, the first very near the left endpoint and the second very near the right endpoint.

Moreover, while $q^{-\frac{1}{2}(\ell-1)\ell}$ increases very rapidly with $\ell$, the other terms depend on $\ell$ only in a fairly weak manner. Therefore we can expect $|\sigma_\ell - q^{-\ell}|$ to decrease very rapidly as $\ell$ grows, and this is confirmed by numerical computations. On the other hand, the interval $(1, q^{-1})$ is the obvious place where things are more interesting. For $0 < \tau \ll 1$ two zeros emerge from the endpoints, 'sliding' inwards: numerical calculations confirm that after a short while they may coalesce into a double zero, which subsequently bifurcates into the complex plane as a conjugate pair of zeros.

Fig 4.1 displays $G$ for different values of $q$ and $\tau$ in the first two intervals of the form $[q^{-2\ell}, q^{-2\ell-1}]$. In the first interval in the case $q = \frac{9}{16}$ there are two zeros for $\tau = \frac{1}{20}$, which have emerged for $\tau = 0$ from the endpoints and which coalesce very near $\tau = \frac{1}{10}$ (actually, at $\tau \approx 0.09992063019$) and, having moved to complex plane, $G$ is positive throughput the interval for increasing $\tau$. For $q = \frac{1}{4}$, however, double zeros persist in the first interval for all $\tau \in (0, q]$. Shifting our attention to the second interval (in the right column), we observe the presence of zeros *very* near the endpoints: we are already in the asymptotic regime.

Although the analysis of the function $G$ is valuable in understanding the behaviour of $H_1$, it is of interest to convert $F$ into a 'proper' power series in $\zeta$, thereby representing $H_1$ as a quotient of two power series. To this end we replace $\mathrm{e}_q(\zeta) = 1/(\zeta, q)_\infty$ by its expansion $\sum_{n=0}^\infty \zeta^n/(q,q)_n$. It then follows from (4.2) that

$$\begin{aligned}
(\tau, q)_\infty F(\zeta, \tau, q) &= \sum_{n=0}^\infty \frac{\zeta^n}{(q,q)_n} \sum_{m=0}^\infty \frac{(\tau, q)_m}{(q,q)_m} q^{\frac{1}{2}(m-1)m} (-1)^m \zeta^m \\
&= \sum_{m=0}^\infty \frac{(\tau, q)_m}{(q,q)_m} (-1)^m q^{\frac{1}{2}(m-1)m} \sum_{n=m}^\infty \frac{\zeta^n}{(q,q)_{n-m}} \\
&= \sum_{n=0}^\infty \frac{\zeta^n}{(q,q)_n} \sum_{m=0}^n (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{1}{2}(m-1)m} (\tau, q)_m .
\end{aligned}$$

The outcome is a power series representation of $F$,

$$F(\zeta, \tau, q) = \frac{1}{(\tau, q)_\infty} \sum_{n=0}^\infty d_n(\tau) \zeta^n,$$

where

$$d_n(\tau) = \frac{1}{(q,q)_n} \sum_{m=0}^n (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{1}{2}(m-1)m} (\tau, q)_m, \qquad n \in \mathbb{Z}_+. \tag{4.4}$$

**Proposition 4** *The above coefficients $d_n(\tau)$ satisfy $d_0 \equiv 1$ and*

$$d_n(\tau) = \sum_{m=1}^n \frac{q^{(m-1)m}}{(q,q)_m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q \tau^m, \qquad n \in \mathbb{N}. \tag{4.5}$$

*Proof* The expressions (4.4) and (4.5) match for $n = 0, 1$. We continue by induction on $n \geq 1$. Firstly, using identity I.45 from (Gasper & Rahman 2004, p. 235), we deduce from (4.4) that

$$\begin{aligned}
d_n(\tau) &= \frac{1}{(q,q)_n} \sum_{m=0}^n (-1)^m \left\{ \begin{bmatrix} n-1 \\ m \end{bmatrix}_q + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q \right\} q^{\frac{1}{2}(m-1)m} (\tau, q)_m \\
&= \frac{d_{n-1}(\tau)}{1-q^n} - q^{n-1} \frac{1-\tau}{(q,q)_n} \sum_{m=0}^{n-1} (-1)^m \begin{bmatrix} n-1 \\ m \end{bmatrix}_q q^{\frac{1}{2}(m-1)m} (q\tau, q)_m \\
&= \frac{1}{1-q^n} [d_{n-1}(\tau) - q^{n-1}(1-\tau) d_{n-1}(q\tau)], \qquad n \in \mathbb{N}.
\end{aligned}$$

Likewise, for $n \geq 2$ (4.5) yields

$$
\begin{aligned}
&\frac{1}{1-q^n}[d_{n-1}(\tau) - q^{n-1}(1-\tau)d_{n-1}(q\tau)] \\
&= \frac{1}{1-q^n}\sum_{m=1}^{n-1}\frac{q^{(m-1)m}}{(q,q)_m}\begin{bmatrix} n-2 \\ m-1 \end{bmatrix}_q \tau^m - \frac{q^{n-1}}{1-q^n}(1-\tau)\sum_{m=1}^{n-1}\frac{q^{(m-1)m}}{(q,q)_m}\begin{bmatrix} n-2 \\ m-1 \end{bmatrix}_q (q\tau)^m \\
&= \frac{1}{1-q^n}\sum_{m=1}^{n-1}\frac{q^{(m-1)m}}{(q,q)_m}\begin{bmatrix} n-2 \\ m-1 \end{bmatrix}_q (1-q^{n+m-1})\tau^m \\
&\quad + \frac{1}{1-q^n}\sum_{m=1}^{n-1}\frac{q^{(m-1)m}}{(q,q)_m}\begin{bmatrix} n-2 \\ m-1 \end{bmatrix}_q q^{n+m-1}\tau^{m+1} \\
&= \frac{1}{1-q^n}\left\{\sum_{m=1}^{n-1}\frac{q^{(m-1)m}}{(q,q)_m}\begin{bmatrix} n-2 \\ m-1 \end{bmatrix}_q (1-q^{n+m-1})\tau^m \right. \\
&\qquad\qquad \left. + \sum_{m=2}^{n}\frac{q^{(m-2)m+n}}{(q,q)_{m-1}}\begin{bmatrix} n-2 \\ m-2 \end{bmatrix}_q \tau^m\right\} \\
&= \frac{1}{1-q^n}\sum_{m=1}^{n}\frac{q^{(m-1)m}}{(q,q)_m}\left\{\begin{bmatrix} n-2 \\ m-1 \end{bmatrix}_q (1-q^{n+m-1}) + q^{n-m}\begin{bmatrix} n-2 \\ m-2 \end{bmatrix}_q (1-q^m)\right\}\tau^m \\
&= \sum_{m=1}^{n}\frac{q^{(m-1)m}}{(q,q)_m}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q \tau^m,
\end{aligned}
$$

as can be confirmed by straightforward calculation. We thus obtained the left-hand side of (4.5). In other words, the functions $d_n$ in (4.4) and (4.5) obey the same recurrence relation, Since they match for $n = 1$, an inductive proof follows. □

Since $(\tau, q)_\infty/(q\tau, q)_\infty = 1-\tau$, the outcome of our analysis is the rational expansion

$$
H_1(\zeta, \tau, q) = (1-\tau)\frac{\sum_{n=0}^{\infty} d_n(q\tau)\zeta^n}{\sum_{n=0}^{\infty} d_n(\tau)\zeta^n}, \tag{4.6}
$$

where alternative expressions for $d_n$ have been given in (4.4) and (4.5).

Bearing in mind the representation (3.7), combining the values of $H_1$ at $z/\alpha$ and at $\alpha/z$, it is perhaps more illuminating to consider (4.6) not as a rational expansion in $\zeta$ about the origin but as a Fourier expansion on circles of radii $|\alpha| = q^{1/2}$ and $|\alpha|^{-1} = q^{-1/2}$.

## 4.3 An expansion of the generating function

The above expressions of the function $F$ provides an expansion of the generating function in $z$ and $z^{-1}$. It is enough to taking into account the expression (3.3) and to recall that we have:

$$F(\alpha^{-1}z;|c|^2q^N,q)=\frac{(|c|^2,q)_N}{(|c|^2,q)_\infty}\left[1+\sum_{n=1}^{\infty}d_n(|c|^2q^N)\left(\frac{z}{\alpha}\right)^n\right],$$
$$F(\alpha z^{-1};|c|^2q^N,q)=\frac{(|c|^2,q)_N}{(|c|^2,q)_\infty}\left[1+\sum_{n=1}^{\infty}d_n(|c|^2q^N)\left(\frac{\alpha}{z}\right)^n\right],$$

where

$$d_n(x)=\sum_{m=1}^{n}\frac{q^{(m-1)m}}{(q,q)_m}\left[\begin{array}{c}n-1\\ m-1\end{array}\right]_q x^m,\qquad n\in\mathbb{N}.$$

Also, the function $F$ provides a first approximation to the sequence of OPUC as a expansion in $z$ and $z^{-1}$ using (3.4)

$$\phi_N(z)=\alpha^N\beta_1(z)F(\alpha z^{-1};|c|^2q^N,q)+z^N\beta_2(z)F(\alpha^{-1}z;|c|^2q^N,q),\qquad n\in\mathbb{Z}_+,$$

where $\beta_1$ and $\beta_2$ are determined from the initial conditions $\phi_0\equiv 1$ and $\phi_1(z)=c\alpha+z$. Therefore,

$$\beta_1(z)=\frac{zF(\alpha^{-1}z;|c|^2q,q)-(c\alpha+z)F(\alpha^{-1}z;|c|^2,q)}{zF(\alpha z^{-1};|c|^2,q)F(\alpha^{-1}z;|c|^2q,q)-\alpha F(\alpha z^{-1};|c|^2q,q)F(\alpha^{-1}z;|c|^2,q)},$$
$$\beta_2(z)=\frac{(c\alpha+z)F(\alpha z^{-1};|c|^2,q)-\alpha F(\alpha z^{-1};|c|^2q,q)}{zF(\alpha z^{-1};|c|^2,q)F(\alpha^{-1}z;|c|^2q,q)-\alpha F(\alpha z^{-1};|c|^2q,q)F(\alpha^{-1}z;|c|^2,q)}.$$

Incidentally, for all $N\in\mathbb{N}$ we have

$$\phi_N(0)=c\alpha^N\frac{F(\infty;|c|^2q^N,q)}{F(\infty;|c|^2q,q)}=c\alpha^N,$$

because $F(\infty;|c|^2q^N,q)\equiv 1$ according to our computation.

# 5 A representation of the OPUC as $q$-Bessel functions

In this section we obtain an explicit representation of the OPUC sequence $\{\phi_n\}$ as a linear combination of the $q$-Bessel functions $J_\nu^{(2)}$ (cf. (Gasper & Rahman 2004, p. 4) for the definition of $q$-Bessel functions). This is very much in line with the numerous explicit representations of orthogonal polynomials on the real line in terms of hypergeometric and $q$-hypergeometric functions (Chihara 1978, Ismail 2005). Bearing in mind the definition of $F$ and the representation (3.4) of the OPUC $\{\phi_n\}$, a simple calculation leads to

$$F(\zeta,\tau,q)-F(\zeta,q\tau,q)=\sum_{m=1}^{\infty}\frac{(1-q^m)\tau^m}{(q,q)_m(\zeta,q)_m}=\sum_{m=1}^{\infty}\frac{\tau^m}{(q,q)_{m-1}(\zeta,q)_m}$$
$$=\sum_{m=0}^{\infty}\frac{\tau^{m+1}}{(q,q)_m(\zeta,q)_{m+1}}=\frac{\tau}{1-\zeta}F(q\zeta,\tau,q).$$

We thus deduce the functional equation

$$F(\zeta,\tau,q)=F(\zeta,q\tau,q)+\frac{\tau}{1-\zeta}F(q\zeta,\tau,q), \tag{5.1}$$

given in tandem with the initial condition $F(\zeta,0,q)\equiv 1$.

Note, incidentally, that the function

$$\tilde{F}(\zeta,\tau,q)=\frac{1}{(\zeta,q)_\infty(\tau,q)_\infty}$$

is a solution of (5.1), as can be verified easily by direct substitution. Needless to say, $\tilde{F}\neq F$ (cf. (4.2)), but then we have never claimed that (5.1) has a unique solution.

We now start similarly to Subsection 4.2, yet progress differently,

$$\begin{aligned}F(\zeta,\tau,q)-F(q\zeta,\tau,q)&=\sum_{m=1}^{\infty}\frac{\tau^m}{(q,q)_m(\zeta,q)_{m+1}}[(1-\zeta q^m)-(1-\zeta)]\\&=\zeta\sum_{m=1}^{\infty}\frac{\tau^m}{(q,q)_{m-1}(\zeta,q)_{m+1}}=\frac{\zeta\tau}{(1-\zeta)(1-q\zeta)}F(q^2\zeta,\tau,q).\end{aligned}$$

Let

$$\chi_r=F(q^r\zeta,\tau,q),\qquad r\in\mathbb{Z}_+.$$

(Needless to say, $\chi_r=\chi_r(\zeta,\tau,q)$, but it is convenient to use a more economical notation.) We have just proved that

$$\chi_0=\chi_1+\frac{\zeta\tau}{(\zeta,q)_2}\chi_2$$

and, replacing $\zeta$ with $q^r\zeta$ for $r\in\mathbb{Z}_+$, we deduce the recurrence

$$\chi_r=\chi_{r+1}+\frac{\zeta q^r\tau}{(\zeta q^r,q)_2}\chi_{r+2},\qquad r\in\mathbb{Z}_+. \tag{5.2}$$

**Proposition 5** *For every $s\in\mathbb{Z}_+$ it is true that*

$$\chi_0=\sum_{\ell=0}^{s}\begin{bmatrix}s\\ \ell\end{bmatrix}_q\frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta,q)_\ell(q^s\zeta,q)_\ell}\chi_{s+\ell}. \tag{5.3}$$

*Proof* By induction on $s$. The statement is trivial for $s=0$ and reduces to (5.2) for $s=1$. In general, we assume (5.3) for $s$ and use (5.2),

$$\begin{aligned}\chi_0&=\sum_{\ell=0}^{s}\begin{bmatrix}s\\ \ell\end{bmatrix}_q\frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta,q)_\ell(q^s\zeta,q)_\ell}\left[\chi_{s+1+\ell}+\frac{q^{s+\ell}\zeta\tau}{(q^{s+\ell}\zeta,q)_2}\chi_{s+\ell+2}\right]\\&=\sum_{\ell=0}^{s}\begin{bmatrix}s\\ \ell\end{bmatrix}_q\frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta,q)_\ell(q^s\zeta,q)_\ell}\chi_{s+1+\ell}\\&\quad+\sum_{\ell=1}^{s+1}\frac{q^{s+\ell-1}}{(q^{s+\ell-1}\zeta,q)_2}\begin{bmatrix}s\\ \ell-1\end{bmatrix}_q\frac{q^{(\ell-2)(\ell-1)}(\zeta\tau)^\ell}{(\zeta,q)_{\ell-1}(q^s\zeta,q)_{\ell-1}}\chi_{s+1+\ell}.\end{aligned}$$

Let us examine the $\ell$th term (we restrict our attention to $1 \le \ell \le s$, cases $\ell = 0$ and $\ell = s+1$ being trivial):

$$\begin{aligned}
&\begin{bmatrix} s \\ \ell \end{bmatrix}_q \frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta,q)_\ell(q^s\zeta,q)_\ell} + \begin{bmatrix} s \\ \ell-1 \end{bmatrix}_q \frac{q^{s+\ell-1+(\ell-2)(\ell-1)}(\zeta\tau)^\ell}{(\zeta,q)_{\ell-1}(q^s\zeta,q)_{\ell+1}} \\
&= \frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta,q)_\ell(q^s\zeta,q)_{\ell+1}} \left\{ \begin{bmatrix} s \\ \ell \end{bmatrix}_q (1-q^{s+\ell}\zeta) + \begin{bmatrix} s \\ \ell-1 \end{bmatrix}_q q^{s-\ell+1}(1-q^{\ell-1}\zeta) \right\}.
\end{aligned}$$

But

$$\begin{aligned}
&\begin{bmatrix} s \\ \ell \end{bmatrix}_q (1-q^{s+\ell}\zeta) + \begin{bmatrix} s \\ \ell-1 \end{bmatrix}_q q^{s-\ell+1}(1-q^{\ell-1}\zeta) \\
&= \frac{(q,q)_s}{(q,q)_\ell(q,q)_{s+1-\ell}} \left[(1-q^{s+1-\ell})(1-q^{s+\ell}\zeta) + (1-q^\ell)(q^{s-\ell+1}-q^s\zeta)\right] \\
&= \frac{(q,q)_s}{(q,q)_\ell(q,q)_{s+1-\ell}}(1-q^{s+1})(1-q^s\zeta) = \begin{bmatrix} s+1 \\ \ell \end{bmatrix}_q (1-q^s\zeta),
\end{aligned}$$

therefore the $\ell$th term is

$$\frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta,q)_\ell(q^s\zeta,q)_{\ell+1}} \begin{bmatrix} s+1 \\ \ell \end{bmatrix}_q (1-q^s\zeta) = \begin{bmatrix} s+1 \\ \ell \end{bmatrix}_q \frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta,q)_\ell(q^{s+1}\zeta,q)_\ell}.$$

This is precisely (5.3) for $s+1$ and the inductive proof is complete. $\square$

We now let $s \to \infty$ in (5.3),

$$\begin{aligned}
\lim_{s\to\infty} \chi_{s+\ell} &= \lim_{s\to\infty} F(q^{s+\ell}\zeta,\tau,q) = F(0,\tau,q) = \mathrm{e}_q(\tau) = \frac{1}{(\tau,q)_\infty}, \\
\lim_{s\to\infty} (q^s\zeta,q)_\ell &= (0,q)_\ell = 1, \\
\lim_{s\to\infty} \begin{bmatrix} s \\ \ell \end{bmatrix}_q &= \lim_{s\to\infty} \frac{(q,q)_s}{(q,q)_\ell(q,q)_{s-\ell}} = \frac{(q,q)_\infty}{(q,q)_\ell(q,q)_\infty} = \frac{1}{(q,q)_\ell}.
\end{aligned}$$

Therefore

$$F(\zeta,\tau,q) = \mathrm{e}_q(\tau)\sum_{\ell=0}^\infty q^{(\ell-1)\ell}\frac{(\zeta\tau)^\ell}{(q,q)_\ell(\zeta,q)_\ell} = \frac{1}{(\tau,q)_\infty}{}_0\phi_1\left[\begin{matrix} -; \\ \zeta; \end{matrix}\; q, \zeta\tau\right].$$

There are several generalizations of Bessel functions into the real of $q$-functions. In particular, the second $q$-Bessel function is

$$J_\nu^{(2)}(x,q) = \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}\left(\frac{x}{2}\right)^\nu {}_0\phi_1\left[\begin{matrix} -; \\ q^{\nu+1}; \end{matrix}\; q, -\tfrac{1}{4}x^2q^{\nu+1}\right]$$

(Gasper & Rahman 2004, p. 25). Letting

$$\mu = \frac{\log\zeta}{\log q}$$

(in other words, $q^{\mu} = \zeta$) we thus have

$$J^{(2)}_{\mu-1}(2\sqrt{\tau}, q) = \frac{(\zeta, q)_\infty}{(q, q)_\infty} \tau^{(\mu-1)/2} {}_0\phi_1 \left[ \begin{array}{c} \text{—}; \\ \zeta; \end{array} \; q, -\zeta\tau \right]$$

and we conclude that

$$F(\zeta, \tau, q) = \frac{(q, q)_\infty}{(\tau, q)_\infty (\zeta, q)_\infty} (-\tau)^{-(\mu-1)/2} J^{(2)}_{\mu-1}(2\mathrm{i}\sqrt{\tau}, q). \tag{5.4}$$

We now use (3.4) to obtain an explicit representation of the $\phi_m$s in terms of $q$-Bessel functions. To this end we note that we need to reckon for both $F(\alpha z^{-1}, q^m\tau, q)$ and $F(\alpha^{-1}z, q^m\tau, q)$. However, if $q^{\mu(z)} = \alpha^{-1}z$ then $q^{-\mu(z)} = \alpha z^{-1}$. Therefore,

$$\begin{aligned} \phi_m(z) = \frac{(q, q)_\infty}{(q^m\tau, q)_\infty} \Bigg\{ & \frac{\alpha^m \beta_1(z)}{(\alpha z^{-1}, q)_\infty} (-q^m\tau)^{[\mu(z)+1]/2} J^{(2)}_{-\mu(z)-1}(2\mathrm{i}(q^m\tau)^{1/2}, q) \\ & + \frac{z^m \beta_2(z)}{(\alpha^{-1}z, q)_\infty} (-q^m\tau)^{[-\mu(z)+1]/2} J^{(2)}_{\mu(z)-1}(2\mathrm{i}(q^m\tau)^{1/2}, q) \Bigg\}, \qquad m \in \mathbb{Z}_+. \end{aligned}$$

# 6 Limiting behaviour

In this section we consider the three cases when the Schur parameters $a_n = c\alpha^n$ are allowed to approach their limiting values, which correspond to known OPUC: $\alpha = 0$ (Lebesgue), $c = 1$ (Rogers–Szegő) and $\alpha = 1$ (Geronimus). The first is trivial, while the third case confronts us with the greatest difficulty.

In the Lebesgue case the pantograph equation (2.1) becomes the (trivial) ODE $\Phi'' = z\Phi'$ which, in tandem with the initial conditions $\Phi(0) = 1$, $\Phi'(0) = z$, results in the explicit solution $\Phi(t, z) = \mathrm{e}^{tz}$. Hence $\phi_m(z) = z^m$ – no great surprise here! To deduce this from directly from the representation (3.5), we note first that in the current case $F(\zeta, \tau, q) = (1 - \tau)^{-1}$ is independent of $\zeta$ and this implies that also $H_m(\zeta, \tau, q) = (1 - \tau)^{-1}$. Therefore, by (3.6), $\eta_2(z) \equiv 1 - \tau$ and, $\alpha$ being zero, we recover $\phi_m(z) = z^m$ from (3.5).

## 6.1 Rogers–Szegő polynomials

The OPUC whose sequence of Schur parameters is given by $a_n = \alpha^n = (-1)^n q^{n/2}$, where $q \in (0, 1)$, are the familiar *Rogers–Szegő* polynomials (Simon 2005), whose explicit form is well known,

$$\phi_m(z) = \sum_{j=0}^{m} (-1)^{m-j} \left[ \begin{array}{c} m \\ j \end{array} \right]_q q^{\frac{1}{2}(m-j)} z^j, \qquad m \in \mathbb{Z}_+. \tag{6.1}$$

Setting $\alpha = -q^{1/2}$ presents absolutely no problems in our analysis – recall that $q = |\alpha|^2$, consistently with the current setting. Thus, we can readily deduce from (4.2) that

$$F(\zeta, q^m, q) = \frac{1}{(\zeta, q)_\infty (q^m, q)_\infty} \sum_{\ell=0}^{\infty} (-1)^{\ell} \left[ \begin{array}{c} m + \ell - 1 \\ \ell \end{array} \right]_q q^{\frac{1}{2}(\ell-1)\ell} \zeta^{\ell}, \qquad m \in \mathbb{N}.$$

In particular,

$$
\begin{aligned}
F(\zeta, q, q) &= \frac{1}{(\zeta, q)_\infty (q, q)_\infty} r(\zeta), \\
F(\zeta, q^2, q) &= \frac{1}{(\zeta, q)_\infty (q^2, q)_\infty} \sum_{\ell=0}^{\infty} (-1)^\ell \frac{1-q^{\ell+1}}{1-q} q^{\frac{1}{2}(\ell-1)\ell} \zeta^\ell \\
&= \frac{1}{(\zeta, q)_\infty (q, q)_\infty} [r(\zeta) - q r(q\zeta)],
\end{aligned}
$$

where

$$
r(\zeta) = G(\zeta, q, q) = \sum_{\ell=0}^{\infty} (-1)^\ell q^{\frac{1}{2}(\ell-1)\ell} \zeta^\ell
$$

is an entire function of order zero: all of the analysis in Subsection 4.2 applies here. With greater generality, it follows from (Gasper & Rahman 2004, p. 235) that

$$
F(\zeta, q^m, q) = \frac{1}{(\zeta, q)_\infty (q, q)_\infty} \sum_{j=0}^{m-1} \begin{bmatrix} m-1 \\ j \end{bmatrix}_q q^{\frac{1}{2}j(j+1)} r(q^j \zeta), \qquad m \in \mathbb{N}.
$$

Therefore

$$
H_m(\zeta, q, q) = \frac{F(\zeta, q^{m+1}, q)}{F(\zeta, q, q)} = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q q^{\frac{1}{2}j(j+1)} \frac{r(q^j \zeta)}{r(\zeta)}. \tag{6.2}
$$

It can be verified at once by elementary algebra that

$$
r(q\zeta) = \frac{1 - r(\zeta)}{\zeta}, \qquad \zeta \in \mathbb{C},
$$

therefore, by induction,

$$
r(q^m \zeta) = (-1)^m q^{-\frac{1}{2}(m-1)m} \zeta^{-m} \left[ r(\zeta) - \sum_{\ell=0}^{m-1} (-1)^\ell q^{\frac{1}{2}(\ell-1)\ell} \zeta^\ell \right].
$$

Therefore, in principle (6.2) can be reformulated employing just $r(\zeta)$, but this adds little to our understanding.

Intriguingly, the function $r$ resembles a Jacobi theta function. Specifically, $r(\zeta) + r(\zeta^{-1}) = -1 + \sum_{\ell=-\infty}^{\infty} (-1)^\ell q^{\frac{1}{2}(\ell-1)\ell} \zeta^\ell$. But, letting $p = q^{1/2}$, we have

$$
\sum_{\ell=-\infty}^{\infty} (-1)^\ell q^{\frac{1}{2}(\ell-1)\ell} \zeta^\ell = p^{-1/4} \sum_{\ell=-\infty}^{\infty} (-1)^\ell p^{(\ell-\frac{1}{2})^2} \zeta^\ell = p^{-1/4} \sum_{\ell=-\infty}^{\infty} p^{(\ell+\frac{1}{2})^2} (-\zeta^{-1})^\ell.
$$

Let $z = \log(-\zeta)/(2\mathrm{i})$. Then

$$
\sum_{\ell=-\infty}^{\infty} (-1)^\ell q^{\frac{1}{2}(\ell-1)\ell} \zeta^\ell = p^{-1/4} \mathrm{e}^{-\mathrm{i}z} \theta_2(z, p),
$$

where $\theta_2$ is the second *Jacobi theta function* (Rainville 1960, p. 316). Unfortunately, this intriguing connection with theta functions does not provide, insofar as we can see, much insight into Rogers–Szegő polynomials.

Abandoning the theta connection, we substitute (6.2) into (3.5) to recover an alternative representation of Rogers–Szegő polynomials, substituting (6.2) into

$$\phi_m(z) = (-1)^m q^{m/2}\eta_1(z)H_m(-q^{1/2}z^{-1}, q^{m+1}, q) + z^m\eta_2(z)H_m(-q^{-1/2}z, q^{m+1}, q),$$

where $\eta_1$ and $\eta_2$ can be also expressed using the form for $H_1$ from (6.2).

## 6.2 Geronimus polynomials

The limiting case $\alpha = 1$, therefore $q = 1$, corresponding to *Geronimus polynomials,* is substantially more complicated, because the $q$-factorials $(q, q)_m$, littering our denominators, become zero, hence naive progression to the limit does not work.

We recall from Section 2 that in this case the generating function $\Phi$ obeys an ODE with the explicit solution (2.3). using the notation therein, we let

$$\varrho_+^*(z) = z\bar{\varrho}_+(z^{-1})$$

and observe that, conjugation flipping the sign of a square root, it is true that $\varrho_+^*(z) = \varrho_-(z)$. Consequently, for Geronimus polynomials,

$$\phi_m(z) = \beta_+(z)\varrho_+^m(z) + \beta_-(z)\varrho_+^{*\,m}(z), \qquad m \in \mathbb{Z}_+, \tag{6.3}$$

where we recall that

$$\varrho_+(z) = \tfrac{1}{2}[1 + z + \sqrt{(1-z)^2 + 4|c|^2 z}].$$

Our intent is to demonstrate that, as $\alpha \to 1$, the expression (3.7) tends to (6.3). This is not a straightforward statement since, as $\alpha \to 1$, so does $q$ and the function $F$ becomes unbounded. Fortunately the functions $F(\zeta, q\tau, q)$ and $F(\zeta, \tau, q)$, at the numerator and denominator of $H_1$ respectively, blow up at a commensurate rate and their quotient $H_1(\zeta, \tau, q)$ remains bounded.

**Lemma 6** *Bearing in mind that $q = |\alpha|^2$, it is true that*

$$\lim_{\alpha\to 1} H_1(\alpha^{-1}z, \tau, q) = \frac{\varrho_-(z)}{z} = \frac{\varrho_+^*(z)}{z}. \tag{6.4}$$

*Proof* Set

$$R(\zeta, \tau, q) = \frac{F(q\zeta, \tau, q)}{F(\zeta, \tau, q)}$$

and denote

$$H_1^\circ(z, c) = \lim_{\alpha\to 1} H_1(\alpha^{-1}z, \tau, q), \qquad R^\circ(z, c) = \lim_{\alpha\to 1} R(\alpha^{-1}z, \tau, q).$$

(Recall that $\tau = q|c|^2$.) We have

$$\begin{aligned} F(\zeta,\tau,q) - F(\zeta,q\tau,q) &= \sum_{m=1}^{\infty} \frac{\tau^m}{(q,q)_{m-1}(\zeta,q)_m} = \frac{\tau}{1-\zeta}\sum_{m=0}^{\infty}\frac{\tau^m}{(q,q)_m(q\zeta,q)_m} \\ &= \frac{\tau}{1-\zeta}F(q\zeta,\tau,q), \end{aligned}$$

while we have already proved in Section 5 that

$$F(\zeta,\tau,q) - F(q\zeta,\tau,q) = \frac{\zeta\tau}{(1-\zeta)(1-q\zeta)}F(q^2\zeta,\tau,q).$$

Dividing the first identity by $F(\zeta,\tau,q)$, we have

$$1 - H_1(\zeta,\tau,q) = \frac{\tau}{1-\zeta}\frac{F(q\zeta,\tau,q)}{F(\zeta,\tau,q)} \qquad \Rightarrow \qquad H_1(\zeta,\tau,q) = 1 - \frac{\tau}{1-\zeta}R(\zeta,\tau,q),$$

while similar division in the second identity yields

$$\begin{aligned} 1 - R(\zeta,\tau,q) &= \frac{\zeta\tau}{(1-\zeta)(1-q\zeta)}\frac{F(q\zeta,\tau,q)}{F(\zeta,\tau,q)} \times \frac{F(q^2\zeta,\tau,q)}{F(q\zeta,\tau,q)} \\ &= \frac{\zeta\tau}{(1-\zeta)(1-q\zeta)}R(\zeta,\tau,q)R(q\zeta,\tau,q). \end{aligned}$$

Letting $\alpha \to 1$, hence $q \to 1$, $\zeta \to z$ and $\tau \to |c|^2$, we obtain the quadratic equation

$$|c|^2 z R^{\circ 2}(z,c) + (1-z)^2 R^\circ(z,c) - (1-z)^2 = 0,$$

therefore

$$R^\circ(z,c) = -\frac{1-z}{2|c|^2 z}[(1-z) \pm \sqrt{(1-z)^2 + 4|c|^2 z}]$$

and analyticity at the origin means that we need to take a minus sign inside the square brackets. Therefore

$$\begin{aligned} H_1^\circ(z,c) &= \lim_{\alpha\to 1}\left[1 - \frac{\tau}{1-\zeta}R(\zeta,\tau,q)\right] = 1 - \frac{|c|^2}{1-z}R^\circ(z,c) \\ &= 1 + \frac{(1-z) - \sqrt{(1-z)^2 + 4|c|^2 z}}{2z} = \frac{1+z-\sqrt{(1-z)^2+4|c|^2 z}}{2z} = \frac{\varrho_-(z)}{z} \end{aligned}$$

and the proof follows. □

Formulæ (2.3) and (3.7) are both linear combinations of two components: for (2.3) these are powers of $\varrho_+$ and $\varrho_-$. Let us restrict the attention to the curve of orthogonality, $|z| = 1$. The functions $\varrho_\pm$ have two branch points, $1 - 2|c|^2 \pm \mathrm{i}|c|\sqrt{1-|c|^2}$, both of unit modulus. Since conjugations flips the sign of a square root, it follows from (6.4) that

$$H_1(\alpha e^{\mathrm{i}\theta},\tau,q) = \overline{H_1(\alpha^{-1}e^{\mathrm{i}\theta},\tau,q)} \xrightarrow{\alpha\to 1} \overline{\frac{e^{-\mathrm{i}\theta}}{2}\left[1 + e^{\mathrm{i}\theta} - \sqrt{(1-e^{\mathrm{i}\theta})^2 + 4|c|^2 e^{\mathrm{i}\theta}}\right]} = \varrho_+(e^{\mathrm{i}\theta})$$

for every $\theta \in [-\pi, \pi]$. We deduce that

$$\lim_{\alpha\to1} \phi_m(z) = [\lim_{\alpha\to1} \eta_1(z)]\lambda_+^m(z) + [\lim_{\alpha\to1} \eta_2(z)]\lambda_-^m(z), \qquad m \in \mathbb{Z}_+.$$

Although it is possible to prove directly (and messily) that $\lim_{\alpha\to1} \eta_1 = \beta_+$ and $\lim_{\alpha\to1} \eta_2 = \beta_-$, this is not necessary, because $\eta_{1,2}$ and $\beta_\pm$ are determined by the equations

$$\beta_+(z) + \beta_-(z) = 1, \qquad \beta_+(z)\varrho_+(z) + \beta_-(z)\varrho_-(z) = z + c$$

and

$$\eta_1(z) + \eta_2(z) = 1, \qquad \alpha H_1\left(\frac{\alpha}{z}, \tau, q\right)\eta_1(z) + zH_1\left(\frac{z}{\alpha}, \tau, q\right)\eta_2(z) = z + c\alpha.$$

Thus, once $\alpha \to 1$, the second set of equations tends to the first, we obtain the right limits to $\eta_1$ and $\eta_2$ and our polynomials indeed converge to Geronimus polynomials.

## 6.3 A set of Geronimus-like polynomials

Another limiting case of the Schur parameters

$$a_n = \begin{cases} 1, & n = 0, \\ c\alpha^n, & n \in \mathbb{N}, \end{cases}$$

is when $|\alpha| = 1$. In other words, $|a_n|^2 = |c|^2$, $n \in \mathbb{N}$, the recurrence relation (1.1) becomes

$$(\alpha + z)\phi_n(z) = \phi_{n+1}(z) + \alpha z(1 - |c|^2)\phi_{n-1}(z), \qquad n \in \mathbb{Z}_+$$

and the generating function $\Phi$ obeys the ODE

$$\Phi''(t) - (\alpha + z)\Phi'(t) + \alpha z(1 - |c|^2)\Phi(t) = 0,$$

whose solution with the initial conditions $\Phi(0) = \phi_0(z) = 1$, $\Phi'(0) = \phi_1(z) = z + c\alpha$ is

$$\Phi(t) = \tfrac{1}{2}\left[1 + \frac{(2c-1)\alpha + z}{\sqrt{(\alpha - z)^2 + 4\alpha|c|^2 z}}\right] \mathrm{e}^{\frac{1}{2}[\alpha+z+\sqrt{(\alpha-z)^2+4\alpha|c|^2 z}]t}$$
$$+ \tfrac{1}{2}\left[1 - \frac{(2c-1)\alpha + z}{\sqrt{(\alpha - z)^2 + 4\alpha|c|^2 z}}\right] \mathrm{e}^{-\frac{1}{2}[\alpha+z+\sqrt{(\alpha-z)^2+4\alpha|c|^2 z}]t}.$$

Therefore,

$$\phi_n(z) = \Phi^{(n)}(0) = \tfrac{1}{2}\left[1 + \frac{(2c-1)\alpha + z}{\sqrt{(\alpha - z)^2 + 4\alpha|c|^2 z}}\right]\left[\frac{\alpha + z + \sqrt{(\alpha - z)^2 + 4\alpha|c|^2 z}}{2}\right]^n$$
$$+ \tfrac{1}{2}\left[1 - \frac{(2c-1)\alpha + z}{\sqrt{(\alpha - z)^2 + 4\alpha|c|^2 z}}\right]\left[\frac{\alpha + z - \sqrt{(\alpha - z)^2 + 4\alpha|c|^2 z}}{2}\right]^n, \quad n \in \mathbb{Z}_+.$$

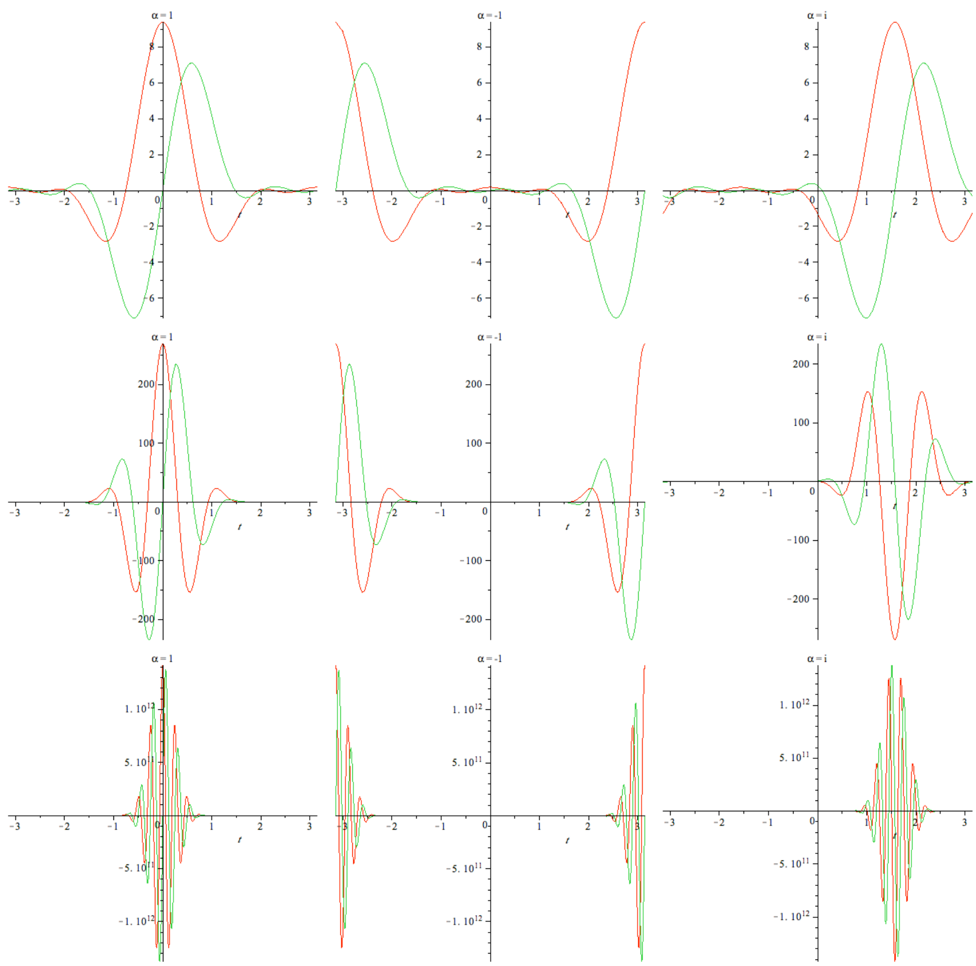



Figure 6.1: The real and imaginary parts of $\phi_n(e^{i\theta})$ for $n = 4$ (top row), $n = 10$ (second row) and $n = 50$ (bottom row), with $c = \frac{3}{4}$ and different values of $|\alpha| = 1$.

Letting $\alpha = 1$, we recover the Geronimus polynomials. Note that letting $|\alpha| \to 1$ in (3.7) presents no difficulties as long as the limit is not a root of unity. However, this can be accomplished for all such $\alpha$ using the methodology of the previous subsection.

In Fig. 6.1 we display the real and imaginary parts of $\phi_n$ for $c = \frac{3}{4}$ and different values of $n$ and $|\alpha| = 1$, plotted on the unit circle. Note that the first column corresponds to $\alpha = 1$, i.e. to classical Geronimus polynomials. It is quite evident that for large $n$ the polynomials (relatively to their maxima) decay very rapidly in part of the unit circle.

Using the explicit form of the $\phi_n$s that we have just derived, we can compute another generating function,

$$
\begin{aligned}
\Psi(t) &= \sum_{n=0}^{\infty} \phi_n(z) t^n \\
&= \left[1 + \frac{(2c-1)\alpha + z}{\sqrt{(\alpha-z)^2 + 4\alpha|c|^2 z}}\right] \frac{1}{2 - [\alpha + z + \sqrt{(\alpha-z)^2 + 4\alpha|c|^2 z}]t} \\
&\quad + \left[1 - \frac{(2c-1)\alpha + z}{\sqrt{(\alpha-z)^2 + 4\alpha|c|^2 z}}\right] \frac{1}{2 - [\alpha + z - \sqrt{(\alpha-z)^2 + 4\alpha|c|^2 z}]t} \\
&= \frac{1 + (c-1)\alpha t}{1 - (\alpha+z)t + \alpha(1-|c|^2) z t^2}.
\end{aligned}
$$

Another interesting observation is the following. Let us compute explicitly $\phi_n^*$. Since $|\alpha| = 1$, we have $\bar{\alpha} = \alpha^{-1}$ and

$$
\begin{aligned}
\phi_n^*(z) = \frac{z^n}{2} &\left[1 + \frac{\dfrac{2\bar{c}-1}{\alpha} + \dfrac{1}{z}}{\sqrt{\left(\dfrac{1}{\alpha} - \dfrac{1}{z}\right)^2 + \dfrac{4|c|^2}{\alpha z}}}\right] \frac{1}{2^n} \left[\frac{1}{\alpha} + \frac{1}{z} + \sqrt{\left(\frac{1}{\alpha} - \frac{1}{z}\right)^2 + \frac{4|c|^2}{\alpha z}}\right]^n \\
&+ \frac{z^n}{2} \left[1 - \frac{\dfrac{2\bar{c}-1}{\alpha} + \dfrac{1}{z}}{\sqrt{\left(\dfrac{1}{\alpha} - \dfrac{1}{z}\right)^2 + \dfrac{4|c|^2}{\alpha z}}}\right] \frac{1}{2^n} \left[\frac{1}{\alpha} + \frac{1}{z} - \sqrt{\left(\frac{1}{\alpha} - \frac{1}{z}\right)^2 + \frac{4|c|^2}{\alpha z}}\right]^n \\
= \tfrac{1}{2} &\left[1 + \frac{(2\bar{c}-1)z + \alpha}{\sqrt{(\alpha-z)^2 + 4\alpha|c|^2 z}}\right] \frac{1}{(2\alpha)^n} [\alpha + z + \sqrt{(\alpha-z)^2 + 4\alpha|c|^2 z}]^n \\
&+ \tfrac{1}{2} \left[1 - \frac{(2\bar{c}-1)z + \alpha}{\sqrt{(\alpha-z)^2 + 4\alpha|c|^2 z}}\right] \frac{1}{(2\alpha)^n} [\alpha + z - \sqrt{(\alpha-z)^2 + 4\alpha|c|^2 z}]^n.
\end{aligned}
$$

Therefore, once we let $c = 1$, we have $\alpha^n \phi_n^*(z) = \phi_n(z)$, in other words $\phi_n^*$ is a rotation of $\phi_n$. Moreover, for every $c \in \mathbb{C} \setminus \{0\}$ it is true that $\phi_n^*(0) \equiv 1$, $n \in \mathbb{Z}_+$.

Finally, it follows from (2.3) that

$$
\begin{aligned}
\phi_n(\alpha z) &= \tfrac{1}{2} \left[1 + \frac{2c - 1 + z}{\sqrt{(1-z)^2 + 4|c|^2 z}}\right] \left\{\frac{\alpha[1 + z + \sqrt{(1-z)^2 + 4|c|^2 z}]}{2}\right\}^n \\
&\quad + \tfrac{1}{2} \left[1 - \frac{2c - 1 + z}{\sqrt{(1-z)^2 + 4|c|^2 z}}\right] \left\{\frac{\alpha[1 + z - \sqrt{(1-z)^2 + 4|c|^2 z}]}{2}\right\}^n \\
&= \alpha^n \check{\phi}_n(z),
\end{aligned}
$$

where $\check{\phi}_n$ is the standard Geronimus polynomial. This relation means that the orthogonality measure of the polynomials $\phi_n$ is a rotation by parameter $\alpha$ of the orthogonality measure of Geronimus polynomials. It is very well known that this orthogonality measure is supported in an arc of the unit circle and, under certain condition for the parameter $c$, the support additionally includes a single pure point (Simon 2005, p. 84).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## References


Chihara, T. S. (1978), *An Introduction to Orthogonal Polynomials*, Gordon & Breach, New York.

Gasper, G. & Rahman, M. (2004), *Basic Hypergeometric Series*, 2nd edn, Cambridge University Press, Cambridge.

Hille, E. (1962), *Analytic Function Theory*, Vol. II, Blaisdell, Waltham, MA.

Iserles, A. (1993), 'On the generalized pantograph functional-differential equation', *Europ. J. Appl. Maths* **4**, 1–38.

Ismail, M. E. H. (2005), *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, Cambridge.

Rainville, E. D. (1960), *Special Functions*, Macmillan, New York.

Simon, B. (2005), *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, Amer. Math. Soc., Providence, RI.