# On a curious $q$-hypergeometric identity

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**Abstract**

In this paper we examine the limiting behaviour of solutions to an infinite set of recursions involving $q$-factorial terms as $q \to 1$. The underlying problem is sensitive to small perturbations and the very existence of a limit, to say nothing of its precise form, is surprising. We determine it by showing that the task in hand is equivalent to the convergence of one set of orthogonal polynomials on the unit circle to another such set, Geronimus polynomials, as $q \to 1$.

## 1 Statement of the problem

The subject matter of this paper is a curious fact pertaining to the solution of an infinite triangular set of linear algebraic equations with $q$-factorial coefficients. Specifically, we concern ourselves with the equations

$$\begin{aligned} a_0 &= 1 \\ \sum_{l=0}^{m} \frac{a_{m-l}}{(q,q)_l (z,q)_l} &= \frac{q^m}{(q,q)_m (z,q)_m}, \qquad m = 1, 2, \ldots, \end{aligned} \tag{1}$$

where $z, q \in \mathbb{C}$, $|q| < 1$, and the $q\text{-}factorial symbol$ $(b,q)_m$ is defined as

$$(b,q)_m = \prod_{k=0}^{m-1} (1 - q^k b), \qquad b \in \mathbb{C}, \quad m \in \mathbb{Z}_+ \cup \{\infty\}$$

(Gasper & Rahman 2004).

Since the coefficients of $a_m$ in (1) is one, the system always has a solution, which can be obtained recursively. Thus,

$$\begin{aligned}
a_1 &= -\frac{1}{1-z}, \\
a_2 &= \frac{z}{(1-z)^2(1-qz)}, \\
a_3 &= -\frac{(1+q)z^2}{(1-z)^3(1-qz)(1-q^2z)}, \\
a_4 &= \frac{[(1+2q+q^2+q^3)-q(1+q+2q^2+q^3)z]z^3}{(1-z)^4(1-qz)^2(1-q^2z)(1-q^3z)}, \\
a_5 &= -\frac{(1+q)[(1+2q+q^2+2q^3+q^5)-q(1+2q^2+q^3+2q^4+q^5)z]z^4}{(1-z)^5(1-qz)^2(1-q^2z)(1-q^3z)(1-q^4z)}
\end{aligned}$$

and so on. One face of it, the expressions are getting increasingly more complex, without any general rule. However, it is our contention in this paper that

$$\lim_{q\to 1} a_m = (-1)^m \frac{(2m-2)!}{(m-1)!m!} \frac{z^{m-1}}{(1-z)^{2m-1}}, \qquad m \in \mathbb{N}. \tag{2}$$

This identity is surprising, not least because just about everything in (1), except for the $l = 0$ term, blows up as $q \to 1$. Thus, the terms need to blow up in a perfect balance!

The volatility of (1) means that what appear to be very minor and harmless amendments completely change the solution, typically leading to blow-up as $q \to 1$.The most striking is also the most obvious along the route of seeking to prove (2): It is very well known that for $q \to 1$ we have $(q, q)_s \approx s!(1-q)^s$ and $(z, q)_s \approx (z)_s$, where $(z)_s = z(z+1)\cdots(z+s-1)$, $s \in \mathbb{Z}_+$, is the *Pochhammer symbol* (Rainville 1960). Consequently, (1) is 'approximated' by

$$\sum_{l=0}^{m} \frac{\tilde{a}_{m-l}}{l!(z)_l}(1-q)^{2(m-l)} = \frac{q^m}{m!(z)_m} \qquad m \in \mathbb{N}, \tag{3}$$

with $\tilde{a}_0 = 1$. However, the solution of (3) blows up as $q \to 1$ - we do not need to iterate much since already $\tilde{a}_1 = -(1-q)^{-1}z^{-1}$.

Even less drastic changes to (1) result either in a blow-up or in a ver radical change to its character. Thus, the solution of both

$$\sum_{l=0}^{\infty} \frac{\tilde{a}_{m-l}}{(q,q)_l(z,q)_l)} = \frac{1}{(q,q)_m(z,q)_m}, \qquad m \in \mathbb{N}$$

and of

$$\sum_{l=0}^{\infty} \frac{q^l \tilde{a}_{m-l}}{(q,q)_l(z,q)_l)} = \frac{q^m}{(q,q)_m(z,q)_m)}, \qquad m \in \mathbb{N}$$

is, trivially, $\tilde{a}_m \equiv 0$, $l \in \mathbb{N}$, while the solution of

$$\sum_{l=0}^{\infty} \frac{\tilde{a}_{m-l}}{(q,q)_l(z,q)_l)} = \frac{q^{\frac{1}{2}(m-1)m}}{(q,q)_m(z,q)_m}, \qquad m \in \mathbb{N},$$

bolws up as $q \to 1$, since

$$\tilde{a}_2 = -\frac{1}{(1-q^2)(1-z)(1-qz)}.$$

The very fact that the solution of (1) stays bounded as $q \to 1$ and that it approaches the fairly complicated expression limit (2) is part of the magic of $q$-hypergeometric functions. The delicate filigree of this set of equations and their orderly progression to an unusual limit is worthy of Ramanujan. So should be the proof of (2): in and ideal world it would be beautiful, directly, short and crisp. Unfortunately, such a proof is beyond the wit of the authors. Instead, we present a roundabout proof of (2), which is anchored on our work in the theory of *orthogonal polynomials on the unit circle* (OPUC) (Cantero & Iserles 2011).

## 2 From OPUC to the $q$-hypergeometric identities

A set of monic polynomials $\{\phi_n\}_{n\in\mathbb{Z}_+}$, orthogonal on the unit circle with respect to some measure, can be formally characterised by the set of its *Schur parameters* $a_n = \phi_n(0)$, $n \in \mathbb{Z}_+$ (Simon 2005). Specifically, the OPUC $\{\phi_n\}_{n\in\mathbb{Z}_+}$ obeys the recurrence relation

$$a_n\phi_{n+1}(z) = (a_{n+1} + a_n z)\phi_n(z) - (1 - |a_n|^2)a_{n+1} z\phi_{n-1}(z), \qquad n \in \mathbb{N},$$

with the initial conditions $\phi_0(z) \equiv 1$, $\phi_1(z) \equiv z + a_1$. In (Cantero & Iserles 2011) we addressed the OPUC with the Schur parameters

$$a_n = \begin{cases} 1, & n = 0, \\ c\alpha^n, & n \in \mathbb{N}, \end{cases} \tag{4}$$

where $c, \alpha \in \mathbb{C}$, $0 < |c|, |\alpha| < 1$. Such OPUC fills the space spanned by the arguably the three most important sets of OPUC:*Lebesgue polynomials* $\phi_n(z) = z^n$ $(c = 0)$, *Geronimus polynomials* $(\alpha = 1)$ and *Rogers-Szegő polynomials* $(c = 1)$. The generating function of the OPUC with the parameters (4),

$$\Phi_z(t) = \sum_{n=0}^{\infty} \frac{\phi_n(z)}{n!} t^n,$$

obeys the *pantograph-type* functional differential equation

$$\Phi_z''(t) = (\alpha + z)\Phi_z'(t) - \alpha\tau z\Phi_z(qt), \qquad t \ge 0, \tag{5}$$

with the initial conditions $\Phi_z(0) = 1$, $\Phi_z'(0) = z + c\alpha$, where $q = |\alpha|^2 \in (0,1)$ and $\tau = q|c|^2 \in (0,1)$. Solutions of pantograph-type equations can be expanded into Dirichlet series (Iserles 1993) and this has led in (Cantero & Iserles 2011) to the explicit expansion

$$\Phi_z(t) = \beta_1(z) \sum_{m=0}^{\infty} \frac{\tau^m e^{\alpha q^m t}}{(q,q)_m(\alpha/z,q)_m} + \beta_2(z) \sum_{m=0}^{\infty} \frac{\tau^m e^{zq^m t}}{(q,q)_m(z/\alpha,q)_m} \tag{6}$$

where $\beta_1$ and $\beta_2$ are determined by the initial conditions.

Let

$$F(\zeta,\tau,q)=\sum_{m=0}^{\infty}\frac{\tau^m}{(q,q)_m(\zeta,q)_m},\quad H(\zeta,\tau,q)=\frac{F(\zeta,q\tau,q)}{F(\zeta,\tau,q)}$$

-both functions clearly converge since $|\tau|\le q<1$. Repeatedly differentiating (6), it has been proved in (Cantero & Iserles 2011) that

$$\phi_m(z)=\alpha^m\eta_1(z)\prod_{l=1}^{m}H(\alpha z^{-1},q^l\tau,q)+z^m\eta_2(z)\prod_{l=1}^{m}H(\alpha^{-1}z,q^l\tau,q),\qquad m\in\mathbb{Z}_+,\ (7)$$

where $\eta_1(z)=\beta_1(z)F(\alpha z^{-1},\tau,q)$, $\eta_2(z)=\beta_2(z)F(\alpha^{-1}z,\tau,q)$ can also been expressed explicitly in terms of the function $H$.

Let us consider the case $\alpha\to 1$, hence also $q\to 1$ and $\tau\to|c|^2$. This corresponds to the *Geronimus polynomials* $\{\psi_m\}_{m\in\mathbb{Z}_+}$, with the explicit representation

$$\psi_m(z)=\left[\frac{1}{2}-\frac{(1-z)-2c}{\sqrt{(1-z)^2+4|c|^2z}}\right]\left[\frac{1+z+\sqrt{(1-z)^2+4|c|^2z}}{2}\right]^m \qquad (8)$$

$$+\left[\frac{1}{2}+\frac{(1-z)-2c}{\sqrt{(1-z)^2+4|c|^2z}}\right]\left[\frac{1+z-\sqrt{(1-z)^2+4|c|^2z}}{2}\right]^m,\qquad m\in\mathbb{Z}_+$$

(Simon 2005, p.87). It is true that, as $\alpha\to 1$, (7) tends to (8)? To this end, it is sufficient to prove that

$$H^0(z,c)=\lim_{\alpha\to 1}H(\alpha^{-1}z,\tau,q)=\frac{1+z-\sqrt{(1-z)^2+4|c|^2z}}{2z} \qquad (9)$$

(Cantero & Iserles 2011). To this end, let us consider the power series in $\tau$ of the function $\tau$. Since

$$H^0(z,c)=\sum_{m=0}^{\infty}a_m\tau^m\qquad\Rightarrow\qquad F(\zeta,q\tau,q)=F(\zeta,\tau,q)\sum_{m=0}^{\infty}a_m\tau^m,$$

a substitution of the power-series definition of $F$ and a straightforward multiplication of infinite series and a comparison of equal powers of $\tau$ results in the infinite set (1) of recurrence relations for the $a_m$s. Moreover, expanding the square rot in (9) in powers of $|c|^2$ yields

$$\frac{1+z-\sqrt{(1-z)^2+4|c|^2z}}{2z}=1-\frac{1}{2}\sum_{m=1}^{\infty}(-1)^m\frac{\left(-\frac{1}{2}\right)_m}{m!}\frac{4^mz^{m-1}|c|^{2m}}{(1-z)^{2m-1}}$$

$$=1+\sum_{m=0}^{\infty}(-1)^m\frac{(2m-2)!}{(m-1)!m!}\frac{z^{m-1}}{(1-z)^{2m-1}}|c|^{2m}.$$

Since $\lim_{\alpha\to 1}\tau=|c|^2$, term by term comparison results in (2). In other words, our contention that (2) is true is equivalent to the statement that $\lim_{q\to 1}\phi_m(z)=\psi_m(z)$, $m\in\mathbb{N}$.

# 3 From the OPUC to Geronimus polynomials

Our aim is to demonstrate that (9) is true, since, by the analysis of the last section, this proves (2). Revisiting the work of (Cantero & Iserles 2011), let

$$R(\zeta,\tau,q)=\frac{F(q\zeta,\tau,q)}{F(\zeta,\tau,q)},\qquad R^0(z,c)=\lim_{\alpha\o 1}R(\alpha^{-1}z,\tau,q).$$

However,

$$\begin{aligned}F(\zeta,\tau,q)-F(\zeta,q\tau,q)&=\sum_{m=1}^{\infty}\frac{(1-q^m)\tau^m}{(q,q)_m(\zeta,q)_m}=\frac{\tau}{1-\zeta}\sum_{m=0}^{\infty}\frac{\tau^m}{(q,q)_m(q\zeta,q)_m}\\&=\frac{\tau}{1-\zeta}F(q\zeta,\tau,q)\end{aligned}$$

and, dividing by $F(\zeta,\tau,q)$, we obtain, after elementary algebra,

$$H(\zeta,\tau,q)=1-\frac{\tau}{1-\zeta}R(\zeta,\tau,q). \tag{10}$$

Moreover,

$$\begin{aligned}F(\zeta,\tau,q)-F(q\zeta,\tau,q)&=\sum_{m=1}^{\infty}\frac{\tau^m}{(q,q)_m(\zeta,q)_{m+1}}\left[(1-q^m\zeta)-(1-\zeta)\right]\\&=\zeta\sum_{m=1}^{\infty}\frac{\tau^m}{(q,q)_{m-1}(\zeta,q)_{m+1}}=\frac{\zeta\tau}{(1-\zeta)(1-q\zeta)}F(q^2\zeta,\tau,q).\end{aligned}$$

Dividing by $F(\zeta,\tau,q)$, we thus have

$$\begin{aligned}1-R(\zeta,\tau,q)&=\frac{\zeta\tau}{(1-\zeta)(1-q\zeta)}\frac{F(q\zeta,\tau,q)}{F(\zeta,\tau,q)}\times\frac{F(q^2\zeta,\tau,q)}{F(q\zeta,\tau,q)}\\&=\frac{\zeta\tau}{(1-\zeta)(1-q\zeta)}R(\zeta,\tau,q)R(q\zeta,\tau,q).\end{aligned}$$

It is perfectly safe to let $\alpha\to 1$ (hence also $q\to 1$ and $\tau\to|c|^2$) in the last expression, the outcome being the quadratic equation

$$z|c|^2R^{o^2}(z,c)+(1-z)^2R^o(z,c)-(1-z)^2=0.$$

Since $R^o(z,c)=1$, its solution is

$$R^o(z,c)=\frac{1-z}{2z|c|^2}[-(1-z)+\sqrt{(1-z)^2+4z|c|^2}]$$

and substitution in (10) results in (9). Therefore, the proof of the limit (2) follows in a roundabout manner and our work is done.

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