Asymptotic solvers for ordinary differential equations with multiple frequencies

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Abstract

We construct asymptotic expansions for ordinary differential equations with highly oscillatory forcing terms, focussing on the case of multiple, non-commensurate frequencies. We derive an asymptotic expansion in inverse powers of the oscillatory parameters and use its truncation as an exceedingly effective means to discretize the differential equation in question. Numerical examples illustrate the effectiveness of the method.

1 Introduction

The subject matter of this paper is discretization methods for highly oscillatory ordinary differential equations (ODEs) of the form

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}) + G(\boldsymbol{y}) \sum_{m=1}^{\infty} \boldsymbol{a}_m(t) \mathrm{e}^{\mathrm{i}\omega_m t}, \qquad t \geq 0, \tag{1.1}$$

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accompanied by the initial condition $\boldsymbol{y}(0) = \boldsymbol{y}_0$, where $\boldsymbol{y} : \mathbb{R}_+ \to \mathbb{C}^d$, $\boldsymbol{f} : \mathbb{C}^d \to \mathbb{C}^d$ and $G : \mathbb{C}^d \to \mathbb{C}^{d\times d}$ are smooth. The *frequencies* ω_m are all nonzero real numbers and we assume that $\sup_m |\omega_m|$ is large, possibly infinite.

Special cases of (1.1) include $\omega_{2m-1} = m\omega$, $\omega_{2m} = -m\omega$, $m \in \mathbb{N}$, where $\omega > 0$, which corresponds to

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}) + G(\boldsymbol{y}) \sum_{k\neq 0} \boldsymbol{b}_k(t) \mathrm{e}^{\mathrm{i}k\omega t}, \qquad t \geq 0, \tag{1.2}$$

which has been already analysed at some length in (Condon, Deaño & Iserles 2009*a*). With greater generality, given any $s \in \mathbb{N}$ and distinct variables $\omega^{(1)}, \ldots, \omega^{(s)}$, we set

$$\omega_{2sm-2s+2j-1} = m\omega^{(j)}, \quad \omega_{2sm-2s+2j} = -m\omega^{(j)}, \qquad j = 1, \ldots, s, \;\; m \in \mathbb{N},$$

whereby (1.1) becomes

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}) + G(\boldsymbol{y}) \sum_{j=1}^{s} \sum_{k\neq 0} \boldsymbol{b}_{s,k} \mathrm{e}^{\mathrm{i}k\omega^{(j)} t}, \qquad t \geq 0.$$

Note that the highly oscillatory term in (1.2) is periodic in $t\omega$: the main difference with our model (1.1) is that we allow the more general setting of *almost periodic* terms (Besicovitch 1932). It is justified by important applications, not least in the modelling of nonlinear circuits (Giannini & Leuzzi 2004, Ramírez, Suárez, Lizarraga & Collantes 2010).

In this paper we advance gradually from the simple to the more intricate. Thus, in Section 2 we let $\boldsymbol{f}(\boldsymbol{y}) = A\boldsymbol{y}$, $\omega_1 = \omega$, $\omega_2 = -\omega$ and $\boldsymbol{a}_1 \equiv 1/(2\mathrm{i})\mathbf{1}$, $\boldsymbol{a}_2 \equiv -1/(2\mathrm{i})\mathbf{1}$, $\boldsymbol{a}_m \equiv \mathbf{0}$ for $m \geq 3$ and $G(\boldsymbol{y})\mathbf{1} = \boldsymbol{g}(\boldsymbol{y})$ – in other words, consider the equation

$$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y}) \sin \omega t, \qquad t \geq 0. \tag{1.3}$$

This is a special case of a framework already introduced in (Condon et al. 2009*a*) and we include it in the present paper to introduce our notation and methodology in a simplified manner.

In Section 3 we consider the linear equation

$$\boldsymbol{y}' = A\boldsymbol{y} + \sum_{q=1}^{s} \sum_{m=-\infty}^{\infty} \boldsymbol{a}_m^q(t) \mathrm{e}^{\mathrm{i}m\omega^{(q)} t}, \qquad t \geq 0 \tag{1.4}$$

– in other words, $\boldsymbol{f}(\boldsymbol{y}) = A\boldsymbol{y}$, $G(\boldsymbol{y}) \equiv I$ and there is just a finite number of non-commensurate frequencies $\omega^{(1)}, \ldots, \omega^{(s)}$. Finally, in Section 4 we address the equation

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}) + \sum_{m=1}^{\infty} \boldsymbol{a}_m \mathrm{e}^{\mathrm{i}\omega_m t}, \qquad t \geq 0. \tag{1.5}$$

The only simplification in (1.5), in comparison with (1.1), is that we set $G(\boldsymbol{y})$ equal to the identity matrix. The expansion in Section 4 is already very elaborate, in particular once we seek high-order terms. Extending the framework to (1.1) presents no conceptual difficulties but would have rendered the expansion considerably more complicated.

This paper follows upon a methodology originally introduced in (Condon, Deaño & Iserles 2009*b*, Condon, Deaño & Iserles 2010*b*, Condon, Deaño & Iserles 2010*a*). Thus, instead of solving (1.1) with a standard numerical method, it is expanded into asymptotic series in inverse powers of the underlying frequencies,

$$\boldsymbol{y}(t) = \boldsymbol{p}_0(t) + \sum_{r=1}^{\infty} \sum_{\boldsymbol{\ell} \in \mathbb{L}_r} \frac{1}{\boldsymbol{\omega}^{\boldsymbol{\ell}}} \sum_{\boldsymbol{m} \in \mathbb{Z}_+^{\infty}} \boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}}(t) \mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top \boldsymbol{\omega} t}, \tag{1.6}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots)$, $\boldsymbol{\ell} = (\ell_1, \ell_2, \ldots)$ and we use the multi-index notation $\boldsymbol{\omega}^{\boldsymbol{\ell}} = \omega_1^{\ell_1} \omega_2^{\ell_2} \cdots$, and

$$\mathbb{L}_r = \left\{ \boldsymbol{\ell} \in \mathbb{Z}_+^{\infty} \, : \, \sum_{j=1}^{\infty} \ell_j = r \right\}.$$

The coefficients $\boldsymbol{p}_0$ and $\boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}}$ can be derived by either solving *non-oscillatory* ODEs or by recursion, thereby avoiding all problems usually associated with high oscillation. Indeed, such methods actually *improve* once frequencies grow.

The expansion (1.6) can be easily converted into a numerical method, restricting the first sum to $1 \leq r \leq R$ for some fairly small natural number R. We thus commit an error of $\mathcal{O}\left(\omega_{\min}^{-R-1}\right)$, where $\omega_{\min} = \min_j |\omega_j|$. Note that, in general, this requires the solution of $R+1$ non-oscillatory (indeed, independent of $\boldsymbol{\omega}$) ODE systems: this can be easily accomplished by standard numerical software.

The methodology in (Condon et al. 2009*b*, Condon et al. 2010*b*, Condon et al. 2010*a*) has been restricted to a single frequency: in our terminology, to all ω_ms being integer multiples of a single parameter ω. In the current paper we explore the highly non-trivial complications once multiple, non-commensurate frequencies are allowed in (1.1).

2 The basic sine oscillator

Inasmuch as this section revisits material that has been already introduced in (Condon et al. 2009*a*), it serves as a valuable warm-up exercise and allows us to introduce a raft of new concepts and notation – indeed, to sketch a new paradigm for the numerical discretization of highly oscillatory phenomena.

In this section we consider the ODE (1.3), namely

$$\boldsymbol{y}'(t) = \boldsymbol{A}\boldsymbol{y}(t) + \boldsymbol{g}(\boldsymbol{y})\sin\omega t, \quad t \ge 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{2.1}$$

where $\omega \gg 1$, $\omega \notin \sigma(A)$.

2.1 The fully linear case

Let us commence from a special case, $\boldsymbol{g}(\boldsymbol{y}) \equiv \boldsymbol{b}$. In that case the equation (2.1) can be integrated explicitly using standard variation-of-constants formula. Thus,

$$\begin{aligned}
\boldsymbol{y}(t) &= \mathrm{e}^{tA}\boldsymbol{y}_0 + \int_0^t \mathrm{e}^{(t-\tau)A}\sin\omega\tau \,\mathrm{d}\tau \boldsymbol{b} \\
&= \mathrm{e}^{tA}\boldsymbol{y}_0 + \frac{1}{2\mathrm{i}}\mathrm{e}^{tA}\int_0^t [\mathrm{e}^{\tau(\mathrm{i}\omega - A)} - \mathrm{e}^{\tau(-\mathrm{i}\omega - A)}]\,\mathrm{d}\tau\boldsymbol{b} \\
&= \mathrm{e}^{tA}\boldsymbol{y}_0 + \frac{\mathrm{e}^{tA}}{2\mathrm{i}}\{(\mathrm{i}\omega I - A)^{-1}[\mathrm{e}^{t(\mathrm{i}\omega - A)} - I] + (\mathrm{i}\omega I + A)^{-1}[\mathrm{e}^{-t(\mathrm{i}\omega + A)} - I]\}\boldsymbol{b} \\
&= \mathrm{e}^{tA}\boldsymbol{y}_0 - (\omega^2 I + A^2)^{-1}A\boldsymbol{b}\sin\omega t - \omega(\omega^2 I + A^2)^{-1}\boldsymbol{b}\cos\omega t \\
&\quad + \omega(\omega^2 I + A^2)^{-1}\mathrm{e}^{tA}\boldsymbol{b}.
\end{aligned}$$

For reasons that will become clear in the sequel, we next expand $\boldsymbol{y}$ in inverse powers of ω. Assuming that $\omega > \|A\|$, we can expand $(I + \omega^{-2}A^2)^{-1}$ in Taylor series, therefore

$$\begin{aligned}
\boldsymbol{y}(t) &= \mathrm{e}^{tA}\boldsymbol{y}_0 - \frac{\cos\omega t}{\omega}(I + \omega^{-2}A^2)^{-1}\boldsymbol{b} + \frac{\mathrm{e}^{tA}}{\omega}(I + \omega^{-2}A^2)^{-1}\boldsymbol{b} \\
&\quad - \frac{\sin\omega t}{\omega^2}(I + \omega^{-2}A^2)^{-1}A\boldsymbol{b} \\
&= \mathrm{e}^{tA}\boldsymbol{y}_0 + \sum_{r=0}^{\infty}\frac{(-1)^r}{\omega^{2r+1}}(\mathrm{e}^{tA} - \cos\omega t I)A^{2r}\boldsymbol{b} - \sin\omega t\sum_{r=0}^{\infty}\frac{(-1)^r}{\omega^{2r+2}}A^{2r+1}\boldsymbol{b}.
\end{aligned} \tag{2.2}$$

It does not take much to bring (2.2) into the form (1.6). Thus, $\boldsymbol{p}_0(t) = \mathrm{e}^{tA}\boldsymbol{y}_0$ and, letting $\omega_{2m-1} = m\omega$, $\omega_{2m} = -m\omega$, we have

$$\begin{aligned}
\boldsymbol{p}_{\boldsymbol{0},(r+1)\boldsymbol{e}_1+r\boldsymbol{e}_2} &= -\tfrac{1}{2}\mathrm{e}^{tA}A^{2r}\boldsymbol{b}, \qquad \boldsymbol{p}_{\boldsymbol{0},r\boldsymbol{e}_1+(r+1)\boldsymbol{e}_2} = \tfrac{1}{2}\mathrm{e}^{tA}A^{2r}\boldsymbol{b},\\
\boldsymbol{p}_{\boldsymbol{e}_1,(r+1)\boldsymbol{e}_1+r\boldsymbol{e}_2} &= -\tfrac{1}{2}A^{2r}\boldsymbol{b}, \qquad \boldsymbol{p}_{\boldsymbol{e}_2,r\boldsymbol{e}_1+(r+1)\boldsymbol{e}_2} = \tfrac{1}{2}A^{2r}\boldsymbol{b},\\
\boldsymbol{p}_{\boldsymbol{e}_1,(r+1)\boldsymbol{e}_1+(r+1)\boldsymbol{e}_2} &= -\frac{\mathrm{i}}{2}A^{2r+1}\boldsymbol{b}, \quad \boldsymbol{p}_{\boldsymbol{e}_2,(r+1)\boldsymbol{e}_1+(r+1)\boldsymbol{e}_2} = \frac{\mathrm{i}}{2}A^{2r+1}\boldsymbol{b}, \qquad r \in \mathbb{Z}_+,
\end{aligned}$$

where $\boldsymbol{e}_k$ is the kth unit vector, otherwise $\boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}} \equiv \boldsymbol{0}$.

Note that we have a measure of freedom in 'translating' (2.2) to (1.6), because $\omega_2 = -\omega_1$ and $\omega_k = 0$ for $k \geq 3$. Thus, for example, all we really need is

$$\sum_{j=0}^{2r+1}(-1)^j\boldsymbol{p}_{\boldsymbol{0},j\boldsymbol{e}_1+(2r+1-j)\boldsymbol{e}_2} = (-1)^{r+1}\mathrm{e}^{tA}A^{2r}\boldsymbol{b}.$$

In the best mathematical tradition, we use this freedom to endow our expansion with maximal symmetry and simplicity.

2.2 A general expansion

The expansion (2.2) fits a more general pattern which is true for all equations of the form (2.1), namely

$$\boldsymbol{y}(t) = \boldsymbol{p}_0(t) + \sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m=-\infty}^{\infty}\boldsymbol{p}_{r,m}(t)\mathrm{e}^{\mathrm{i}m\omega t}. \tag{2.3}$$

(It is easy to cast this in the form (1.6), like we have just done to the expansion (2.2).) To prove this, we assume the *ansatz* (2.3) and equate

$$\boldsymbol{y}' = \boldsymbol{p}_0' + \mathrm{i}\sum_{m\neq 0}m\boldsymbol{p}_{1,m}\mathrm{e}^{\mathrm{i}m\omega t} + \sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m=-\infty}^{\infty}(\boldsymbol{p}_{r,m}' + \mathrm{i}m\boldsymbol{p}_{r+1,m})\mathrm{e}^{\mathrm{i}m\omega t}$$

with

$$\begin{aligned}
&A\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y})\sin\omega t\\
&= A\left(\boldsymbol{p}_0 + \sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m=-\infty}^{\infty}\boldsymbol{p}_{r,m}\mathrm{e}^{\mathrm{i}m\omega t}\right) + \sin\omega t\,\boldsymbol{g}\left(\boldsymbol{p}_0 + \sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m=-\infty}^{\infty}\boldsymbol{p}_{r,m}\mathrm{e}^{\mathrm{i}m\omega t}\right),
\end{aligned}$$

while expanding $\boldsymbol{g}$ about $\boldsymbol{p}_0$. Therefore,

$$\begin{aligned}
&\boldsymbol{p}_0' + \mathrm{i}\sum_{m\neq 0} m\boldsymbol{p}_{1,m}\mathrm{e}^{\mathrm{i}m\omega t} + \sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m=-\infty}^{\infty}(\boldsymbol{p}_{r,m}' + \mathrm{i}m\boldsymbol{p}_{r+1,m})\mathrm{e}^{\mathrm{i}m\omega t}\\
&= A\boldsymbol{p}_0 + \sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m=-\infty}^{\infty}A\boldsymbol{p}_{r,m}\mathrm{e}^{\mathrm{i}m\omega t} + \sin\omega t\,\boldsymbol{g}(\boldsymbol{p}_0)\\
&\quad + \sin\omega t\sum_{k=1}^{\infty}\frac{1}{k!}\boldsymbol{g}_k\Bigg(\boldsymbol{p}_0, \sum_{r_1=1}^{\infty}\frac{1}{\omega^{r_1}}\sum_{m_1=-\infty}^{\infty}\boldsymbol{p}_{r_1,m_1}\mathrm{e}^{\mathrm{i}m_1\omega t}, \sum_{r_2=1}^{\infty}\frac{1}{\omega^{r_2}}\sum_{m_2=-\infty}^{\infty}\boldsymbol{p}_{r_2,m_2}\mathrm{e}^{\mathrm{i}m_2\omega t},\\
&\qquad \ldots, \sum_{r_k=1}^{\infty}\frac{1}{\omega^{r_k}}\sum_{m_k=-\infty}^{\infty}\boldsymbol{p}_{r_k,m_k}\mathrm{e}^{\mathrm{i}m_k\omega t}\Bigg),
\end{aligned}$$

where $\boldsymbol{g}_k$ is the kth derivative at $\boldsymbol{p}_0$,

$$g_{k,\ell}(\boldsymbol{p}_0, \boldsymbol{z}, \ldots, \boldsymbol{z}) = \sum_{i_1=1}^{d}\cdots\sum_{i_k=1}^{d}\frac{\partial^k g_\ell(\boldsymbol{p}_0)}{\partial y_{i_1}\cdots\partial y_{i_k}}z_{i_1}\cdots z_{i_k}, \qquad \ell = 1, \ldots, d.$$

Note that $\boldsymbol{g}_k$ is linear in all its variables except for $\boldsymbol{p}_0$. Therefore

$$\begin{aligned}
&\boldsymbol{p}_0' + \mathrm{i}\sum_{m\neq 0} m\boldsymbol{p}_{1,m}\mathrm{e}^{\mathrm{i}m\omega t} + \sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m=-\infty}^{\infty}(\boldsymbol{p}_{r,m}' + \mathrm{i}m\boldsymbol{p}_{r+1,m})\mathrm{e}^{\mathrm{i}m\omega t}\\
&= A\boldsymbol{p}_0 + \sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m=-\infty}^{\infty}A\boldsymbol{p}_{r,m}\mathrm{e}^{\mathrm{i}m\omega t} + \sin\omega t\,\boldsymbol{g}(\boldsymbol{p}_0)\\
&\quad + \sin\omega t\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{r_1=1}^{\infty}\sum_{r_2=-\infty}^{\infty}\cdots\sum_{r_k=1}^{\infty}\frac{1}{\omega^{r_1+r_2+\cdots+r_k}}\sum_{m_1=-\infty}^{\infty}\\
&\qquad \sum_{m_2=-\infty}^{\infty}\cdots\sum_{m_k=-\infty}^{\infty}\boldsymbol{g}_k(\boldsymbol{p}_0, \boldsymbol{p}_{r_1,m_1}, \boldsymbol{p}_{r_2,m_2}\cdots, \boldsymbol{p}_{r_k,m_k})\mathrm{e}^{\mathrm{i}(m_1+m_2+\cdots+m_k)\omega t}\\
&= A\boldsymbol{p}_0 + \sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m=-\infty}^{\infty}A\boldsymbol{p}_{r,m}\mathrm{e}^{\mathrm{i}m\omega t} + \sin\omega t\,\boldsymbol{g}(\boldsymbol{p}_0)\\
&\quad + \sin\omega t\sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{k=1}^{r}\frac{1}{k!}\sum_{\boldsymbol{n}\in\mathbb{I}_{k,r}}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{k,m}}\boldsymbol{g}_k(\boldsymbol{p}_0, \boldsymbol{p}_{n_1,\ell_1}, \ldots, \boldsymbol{p}_{n_k,\ell_k})\mathrm{e}^{\mathrm{i}m\omega t},
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{I}_{k,r} &= \{\boldsymbol{n}\in\mathbb{N}^k \,:\, n_1+n_2+\cdots+n_k = r\}, \qquad 1\le k\le r,\\
\mathbb{J}_{k,m} &= \{\boldsymbol{\ell}\in\mathbb{Z}^k \,:\, \ell_1+\ell_2+\cdots+\ell_k = m\}, \qquad k\in\mathbb{N}, \quad m\in\mathbb{Z}.
\end{aligned}$$

Since $\sin\omega t = (\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{e}^{-\mathrm{i}\omega t})/(2\mathrm{i})$, we deduce that

$$
\begin{aligned}
&\boldsymbol{p}_0' + \mathrm{i}\sum_{m\neq 0} m\boldsymbol{p}_{1,m}\mathrm{e}^{\mathrm{i}m\omega t} + \sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m=-\infty}^{\infty}(\boldsymbol{p}_{r,m}' + \mathrm{i}m\boldsymbol{p}_{r+1,m})\mathrm{e}^{\mathrm{i}m\omega t} \\
&= A\boldsymbol{p}_0 + \sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m=-\infty}^{\infty}A\boldsymbol{p}_{r,m}\mathrm{e}^{\mathrm{i}m\omega t} + \frac{1}{2\mathrm{i}}\boldsymbol{g}(\boldsymbol{p}_0)(\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{e}^{-\mathrm{i}\omega t}) \qquad (2.4)\\
&\quad + \frac{1}{2\mathrm{i}}\sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{k=1}^{r}\frac{1}{k!}\sum_{\boldsymbol{n}\in\mathbb{I}_{k,r}}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{k,m-1}}\boldsymbol{g}_k(\boldsymbol{p}_0,\boldsymbol{p}_{n_1,\ell_1},\ldots,\boldsymbol{p}_{n_k,\ell_k})\mathrm{e}^{\mathrm{i}m\omega t}\\
&\quad - \frac{1}{2\mathrm{i}}\sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{k=1}^{r}\frac{1}{k!}\sum_{\boldsymbol{n}\in\mathbb{I}_{k,r}}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{k,m+1}}\boldsymbol{g}_k(\boldsymbol{p}_0,\boldsymbol{p}_{n_1,\ell_1},\ldots,\boldsymbol{p}_{n_k,\ell_k})\mathrm{e}^{\mathrm{i}m\omega t}.
\end{aligned}
$$

The next step consists of separating scales in (2.4). Note that there are two scales: firstly, we need to separate different powers of ω and then different frequencies $\mathrm{e}^{\mathrm{i}\omega m t}$. We commence from $\mathcal{O}(1)$ terms in ω, i.e.

$$
\boldsymbol{p}_0' + \mathrm{i}\sum_{m\neq 0} m\boldsymbol{p}_{1,m}\mathrm{e}^{\mathrm{i}m\omega t} = A\boldsymbol{p}_0 + \frac{1}{2\mathrm{i}}\boldsymbol{g}(\boldsymbol{p}_0)(\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{e}^{-\mathrm{i}\omega t}).
$$

Separating frequencies and endowing $\boldsymbol{p}_0$ with the original initial conditions for $\boldsymbol{y}$ results in a *non-oscillatory* ordinary differential equation,

$$
\boldsymbol{p}_0' = A\boldsymbol{p}_0, \quad t \geq 0, \qquad \boldsymbol{p}_0(0) = \boldsymbol{y}_0,
$$

whose solution is

$$
\boldsymbol{p}_0(t) = \mathrm{e}^{tA}\boldsymbol{y}_0, \qquad (2.5)
$$

as well as the recurrences

$$
\boldsymbol{p}_{1,\pm 1}(t) = -\tfrac{1}{2}\boldsymbol{g}(\boldsymbol{p}_0(t)), \qquad \boldsymbol{p}_{1,m} \equiv \boldsymbol{0}, \quad |m| \geq 2. \qquad (2.6)
$$

Subsequently, we separate terms of size $\mathcal{O}(\omega^{-r})$ for $r = 1, 2, \ldots$. This yields

$$
\begin{aligned}
&\sum_{m=-\infty}^{\infty}(\boldsymbol{p}_{r,m}' + \mathrm{i}m\boldsymbol{p}_{r+1,m})\mathrm{e}^{\mathrm{i}m\omega t} = \sum_{m=-\infty}^{\infty}A\boldsymbol{p}_{r,m}\mathrm{e}^{\mathrm{i}m\omega t}\\
&\quad + \frac{1}{2\mathrm{i}}\sum_{k=1}^{r}\frac{1}{k!}\sum_{\boldsymbol{n}\in\mathbb{I}_{k,r}}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{k,m-1}}\boldsymbol{g}_k(\boldsymbol{p}_0,\boldsymbol{p}_{n_1,\ell_1},\ldots,\boldsymbol{p}_{n_k,\ell_k})\mathrm{e}^{\mathrm{i}m\omega t}\\
&\quad - \frac{1}{2\mathrm{i}}\sum_{k=1}^{r}\frac{1}{k!}\sum_{\boldsymbol{n}\in\mathbb{I}_{k,r}}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{k,m+1}}\boldsymbol{g}_k(\boldsymbol{p}_0,\boldsymbol{p}_{n_1,\ell_1},\ldots,\boldsymbol{p}_{n_k,\ell_k})\mathrm{e}^{\mathrm{i}m\omega t}.
\end{aligned}
$$

We first gather non-oscillatory (i.e., $m=0$) terms: this results in the non-oscillatory ODE

$$\begin{aligned}\boldsymbol{p}_{r,0}' = A\boldsymbol{p}_{r,0} &+ \frac{1}{2\mathrm{i}}\sum_{k=1}^{r}\frac{1}{k!}\sum_{\boldsymbol{n}\in\mathbb{I}_{k,r}}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{k,-1}}\boldsymbol{g}_k(\boldsymbol{p}_0,\boldsymbol{p}_{n_1,\ell_1},\ldots,\boldsymbol{p}_{n_k,\ell_k})\\ &- \frac{1}{2\mathrm{i}}\sum_{k=1}^{r}\frac{1}{k!}\sum_{\boldsymbol{n}\in\mathbb{I}_{k,r}}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{k,+1}}\boldsymbol{g}_k(\boldsymbol{p}_0,\boldsymbol{p}_{n_1,\ell_1},\ldots,\boldsymbol{p}_{n_k,\ell_k}).\end{aligned}\tag{2.7}$$

Note that $\boldsymbol{p}_0$ and p_{n_i,ℓ_i} for $\boldsymbol{n}\in\mathbb{I}_{k,r}$ and $\boldsymbol{\ell}\in\mathbb{J}_{k,\pm1}$ are already known by this stage, thus $\boldsymbol{p}_{r,0}$ is the only unknown. To equip (2.7) with an initial condition we impose the requirement that

$$\sum_{m=-\infty}^{\infty}\boldsymbol{p}_{r,m}(0)=\boldsymbol{0},\qquad r\in\mathbb{N}\tag{2.8}$$

– this, in tandem with $\boldsymbol{p}_0(0)=\boldsymbol{y}_0$, implies that the original initial conditions in (2.1) are satisfied. Consequently,

$$\boldsymbol{p}_{r,0}(0)=-\sum_{m\neq0}\boldsymbol{p}_{r,m}(0),$$

where the right-hand side is already known.

Finally, for $m\neq0$ we obtain the recursion

$$\begin{aligned}\boldsymbol{p}_{r+1,m} = \frac{1}{\mathrm{i}m}A\boldsymbol{p}_{r,m} &- \frac{1}{2m}\sum_{k=1}^{r}\frac{1}{k!}\sum_{\boldsymbol{n}\in\mathbb{I}_{k,r}}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{k,m-1}}\boldsymbol{g}_k(\boldsymbol{p}_0,\boldsymbol{p}_{n_1,\ell_1},\ldots,\boldsymbol{p}_{n_k,\ell_k})\\ &+ \frac{1}{2m}\sum_{k=1}^{r}\frac{1}{k!}\sum_{\boldsymbol{n}\in\mathbb{I}_{k,r}}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{k,m+1}}\boldsymbol{g}_k(\boldsymbol{p}_0,\boldsymbol{p}_{n_1,\ell_1},\ldots,\boldsymbol{p}_{n_k,\ell_k}) - \frac{1}{\mathrm{i}m}\boldsymbol{p}_{r,m}'.\end{aligned}\tag{2.9}$$

To sum up, for each 'magnitude' $\mathcal{O}(\omega^{-r})$ we obtain a non-oscillatory ODE for $\boldsymbol{p}_{r,0}$ and recurrences for $\boldsymbol{p}_{r+1,m}$, $m\neq0$. We can continue this procedure for as long as necessary.

Recalling that $\boldsymbol{p}_{1,m}\equiv\boldsymbol{0}$ for $|m|\geq2$, we note from (2.9) that, for $r=1$,

$$\boldsymbol{p}_{2,m}=\frac{1}{\mathrm{i}m}A\boldsymbol{p}_{1,m}-\frac{1}{\mathrm{i}m}\boldsymbol{p}_{1,m}'-\frac{1}{2m}\boldsymbol{g}_1(\boldsymbol{p}_0,\boldsymbol{p}_{1,m-1})+\frac{1}{2m}\boldsymbol{g}_1(\boldsymbol{p}_0,\boldsymbol{p}_{1,m+1}),\quad m\neq0,$$

implies $\boldsymbol{p}_{2,m} \equiv \mathbf{0}$, $|m| \geq 3$. Likewise, for $r = 2$,

$$\boldsymbol{p}_{3,m} = \frac{1}{\mathrm{i}m} A\boldsymbol{p}_{2,m} - \frac{1}{2m}\left[\boldsymbol{g}_1(\boldsymbol{p}_0, \boldsymbol{p}_{2,m-1}) + \tfrac{1}{2}\sum_{\ell=-\infty}^{\infty} \boldsymbol{g}_2(\boldsymbol{p}_0, \boldsymbol{p}_{1,\ell}, \boldsymbol{p}_{1,m-1-\ell})\right]$$
$$+ \frac{1}{2m}\left[\boldsymbol{g}_1(\boldsymbol{p}_0, \boldsymbol{p}_{2,m+1}) + \tfrac{1}{2}\sum_{\ell=-\infty}^{\infty} \boldsymbol{g}_2(\boldsymbol{p}_0, \boldsymbol{p}_{1,\ell}, \boldsymbol{p}_{1,m+1-\ell})\right] - \frac{1}{\mathrm{i}m}\boldsymbol{p}'_{2,m}$$

for $m \neq 0$ means that $\boldsymbol{p}_{3,m} \equiv \mathbf{0}$, $|m| \geq 4$. In general, it is not difficult to prove that $\boldsymbol{p}_{r,m} \equiv \mathbf{0}$ for $|m| \geq r + 1$. In other words, while energy can jump across frequencies, the bandwidth at each magnitude $\mathcal{O}(\omega^{-r})$ remains restricted to frequencies $|m| \leq r$.

2.3 Numerical experiments

We commence from a simple scalar ODE

$$y'(t) = 2\mathrm{i}y(t) + y^2(t)\sin\omega t, \quad t \geq 0, \qquad y(0) = 1. \tag{2.10}$$

In other words, $A = 2\mathrm{i}$ and $g(y) = y^2$.

In Fig. 2.1 we display the real parts of the error committed by the standard `rkf45` method of Maple for $\omega = 500$ and $\omega = 1000$. It is readily seen that the method, typical of standard, Taylor expansion-based ODE solvers, becomes worse once the oscillatory parameter ω increases. Of course, the precision of adaptive software can be increased by setting more demanding tolerances: in Fi. 2.2 we display similar information once we set `AbsErr` $= 10^{10}$ and `RelErr` $= 10^{-16}$. Note that the Maple software delivers a solution of roughly right accuracy, but the cost is substantial: 55342 steps for $\omega = 500$ and 95045 steps for $\omega = 1000$. Unsurprisingly, the average time step scales like ω^{-1}, which is precisely the situation we wish to avoid with our approch.

To illustrate the performance of the asymptotic expansion approximation for the solution $y(t)$, we compute the asymptotic terms compared with the exact solution. The global error for any $R \in \mathbb{Z}_+$ is defined as the difference

$$e_R(t) = y(t) - p_{0,0}(t) - \sum_{r=1}^{R} \frac{1}{\omega^r} \sum_{|m|\leq R} p_{r,m}(t) e^{\mathrm{i}m\omega t}.$$

In Fig. 2.3 we display the real part of the error for $R = 0, 1, 2$ with $\omega = 500$ and $\omega = 1000$. It is readily observed that even these modest values of R result in an outstanding performance and rapidly decreasing global error. Moreover,

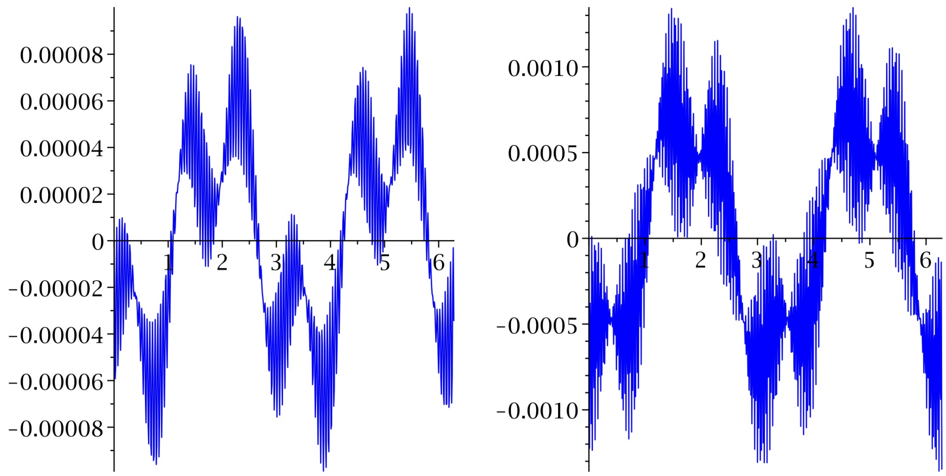

Figure 2.1: Real parts of the error with $\omega = 500$ (the left) and $\omega = 1000$ (the right), as computed with `rkf45`.

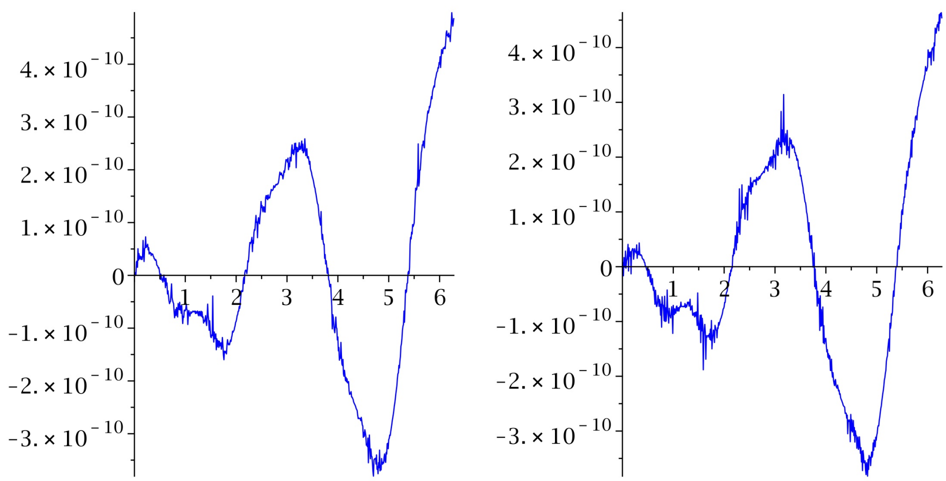

Figure 2.2: The same as Fig. 2.1, except that `AbsErr` $= 10^{-10}$ and `RelErr` $= 10^{-16}$, computed with 32 significant digits.

in a manner characteristic of asymptotic-numerical expansions, and completely at odds with classical numerical analysis, the quality of approximation rapidly improves for increasing ω.

Of course, our method requires numerical computations, specifically the solution of *non-oscillatory* ODE systems for $\boldsymbol{p}_{r,0}$, $r = 0, 1, \ldots, R$. This, however, is fairly cheap. Specifically, for $R = 2$ (the rightmost column of Fig. 2.3), set-

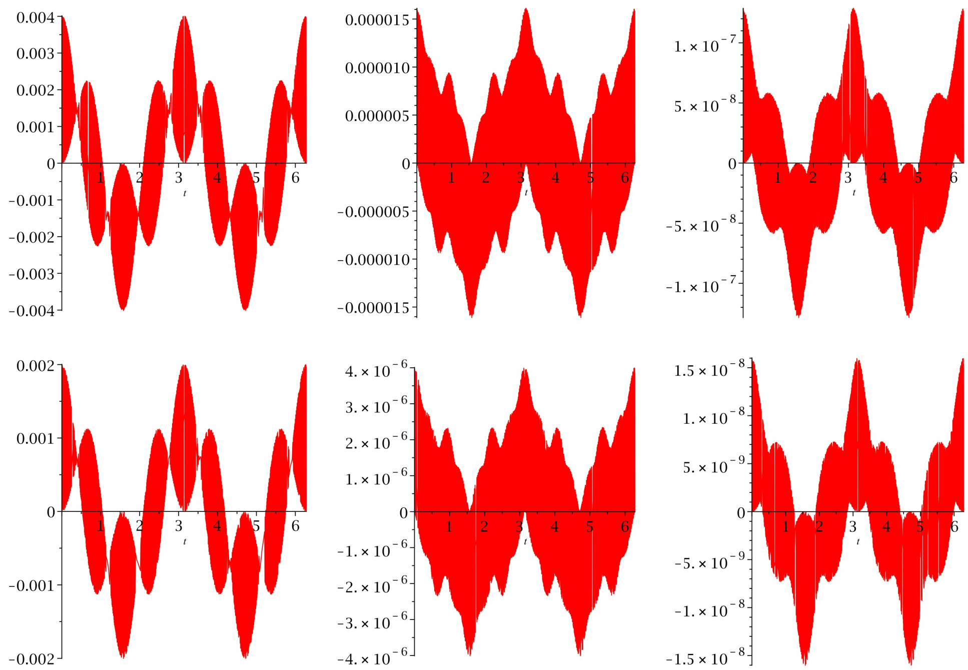

Figure 2.3: The top row: the real part of e_0(on the left), e_1(in the centre), e_2(on the right) for $\omega = 500$; Underneath, the real parts of e_0, e_1 and e_2 for $\omega = 1000$.

ting `AbsErr` $= 10^{-8}$, `rkf45` needed $1090 + 1848 = 2938$ steps (independently of ω), while similar accuracy with classical approach would have required 36051 and 86960 steps for $\omega = 500$ and $\omega = 1000$, respectively.

3 The linear multiple-frequency case

In this section we consider the linear highly oscillatory system (1.4), that is

$$\boldsymbol{y}'(t) = \boldsymbol{A}y(t) + \sum_{q=1}^{s} \sum_{m=-\infty}^{\infty} \boldsymbol{a}_m^q(t)\mathrm{e}^{\mathrm{i}m\omega^{(q)}t}, \qquad t \geq 0, \tag{3.1}$$

where $s \in \mathbb{N}$, while $\boldsymbol{a}_m^q : \mathbb{R}_+ \to \mathbb{C}^d$ for all m and q are sufficiently smooth and $\omega^{(q)} \gg 1$. In particular, we stipulate that $\|A\| < \min_q \omega^{(q)}$, where $\|\cdot\|$ is the Euclidean matrix norm. We have already commented in Section 1 that this can be recast in the general form (1.1).

The term $\sum_{m=-\infty}^{\infty} \boldsymbol{a}_m^q(t)\mathrm{e}^{\mathrm{i}m\omega^{(q)}t}$, periodic in $\omega^{(q)}t$ but not in t, is known as a *modulated Fourier expansion* (Cohen, Hairer & Lubich 2005, Hairer, Lubich & Wanner 2006).

3.1 The asymptotic expansion

It will be now demonstrated that the natural generalisation of the *ansatz* (2.3) is valid in the current case,

$$\boldsymbol{y}(t) = \boldsymbol{p}_0(t) + \sum_{r=1}^{\infty} \sum_{q=1}^{s} \frac{1}{{\omega^{(q)}}^r} \sum_{m=-\infty}^{\infty} \boldsymbol{p}_{r,m}^q(t) \mathrm{e}^{\mathrm{i}m\omega^{(q)}t}, \tag{3.2}$$

where $\boldsymbol{p}_0, \boldsymbol{p}_{r,m}^q : \mathbb{R}_+ \to \mathbb{C}^d$ are smooth, independent of $\boldsymbol{\omega}$ (hence non-oscillatory) and can be obtained by either solving non-oscillatory ODEs or by recursion. Note that (3.2) can be easily rewritten in the form (1.6), but we prefer to employ notation that makes good use of the salient features of the problem (3.1).

We commence by imposing the initial conditions

$$\boldsymbol{p}_0(0) = \boldsymbol{y}_0, \qquad \boldsymbol{p}_{r,0}^q(0) = -\sum_{m\neq 0} \boldsymbol{p}_{r,m}^q(0), \quad r \in \mathbb{N},\ q = 1, \ldots, s. \tag{3.3}$$

Differentiating (3.2), we have

$$\boldsymbol{y}' = \boldsymbol{p}_0' + \mathrm{i} \sum_{q=1}^{s} \sum_{m\neq 0} m \boldsymbol{p}_{1,m}^q \mathrm{e}^{\mathrm{i}m\omega^{(q)}t} + \sum_{r=1}^{\infty} \sum_{q=1}^{s} \frac{1}{{\omega^{(q)}}^r} \sum_{m=-\infty}^{\infty} ({\boldsymbol{p}_{r,m}^q}' + \mathrm{i}m \boldsymbol{p}_{r+1,m}^q) \mathrm{e}^{\mathrm{i}m\omega^{(q)}t}$$

and substitution in (3.1) results in

$$\begin{aligned} &\boldsymbol{p}_0' + \mathrm{i} \sum_{q=1}^{s} \sum_{m\neq 0} m \boldsymbol{p}_{1,m}^q \mathrm{e}^{\mathrm{i}m\omega^{(q)}t} + \sum_{r=1}^{\infty} \sum_{q=1}^{s} \frac{1}{{\omega^{(q)}}^r} \sum_{m=-\infty}^{\infty} ({\boldsymbol{p}_{r,m}^q}' + \mathrm{i}m \boldsymbol{p}_{r+1,m}^q) \mathrm{e}^{\mathrm{i}m\omega^{(q)}t} \\ &= A\boldsymbol{p}_0 + \sum_{r=1}^{\infty} \sum_{q=1}^{s} \frac{1}{{\omega^{(q)}}^r} \sum_{m=-\infty}^{\infty} A\boldsymbol{p}_{r,m}^q \mathrm{e}^{\mathrm{i}m\omega^{(q)}t} + \sum_{q=1}^{s} \sum_{m=-\infty}^{\infty} \boldsymbol{a}_m^q \mathrm{e}^{\mathrm{i}m\omega^{(q)}t}. \end{aligned}$$

As in Section 2, we separate scales: first orders of magnitude $\mathcal{O}\left({\omega^{(q)}}^{-r}\right)$ and subsequently frequencies. We commence from $\mathcal{O}(1)$ terms,

$$\boldsymbol{p}_0' + \mathrm{i} \sum_{q=1}^{s} \sum_{m\neq 0} m \boldsymbol{p}_{1,m}^q \mathrm{e}^{\mathrm{i}m\omega^{(q)}t} = A\boldsymbol{p}_0 + \sum_{q=1}^{s} \sum_{m=-\infty}^{\infty} \boldsymbol{a}_m^q \mathrm{e}^{\mathrm{i}m\omega^{(q)}t}.$$

Separating frequencies, we thus obtain a non-oscillatory ODE,

$$\boldsymbol{p}_0' = A\boldsymbol{p}_0 + \sum_{q=1}^{s} \boldsymbol{a}_0^q, \quad t \ge 0, \qquad \boldsymbol{p}_0(0) = \boldsymbol{y}_0, \tag{3.4}$$

as well as the recurrences

$$\boldsymbol{p}_{1,m}^q = \frac{1}{\mathrm{i}m}\boldsymbol{a}_m^q, \qquad m \ne 0, \quad q = 1, 2, \ldots, s. \tag{3.5}$$

Note that (3.4) and (3.5) are the counterparts of (2.5) and (2.6), respectively.

Turning our attention to $\mathcal{O}\left(\omega^{(q)-r}\right)$ terms for $r \in \mathbb{N}$ and $q \in \{1, 2, \ldots, s\}$, we have

$$\sum_{m \ne 0} (\boldsymbol{p}_{r,m}^q{}' + \mathrm{i}m\boldsymbol{p}_{r+1,m}^q)\mathrm{e}^{\mathrm{i}m\omega^{(q)}t} + \boldsymbol{p}_{r,0}^q{}' = \sum_{m=-\infty}^{\infty} A\boldsymbol{p}_{r,m}^q \mathrm{e}^{\mathrm{i}m\omega^{(q)}t}$$

and separation of frequencies, in tandem with the initial condition (3.3), result in

$$\boldsymbol{p}_{r,0}^q = -\mathrm{e}^{tA} \sum_{m \ne 0} \boldsymbol{p}_{r,m}^q(0), \tag{3.6}$$

$$\boldsymbol{p}_{r+1,m}^q = \frac{1}{\mathrm{i}m}(A\boldsymbol{p}_{r,m}^q - \boldsymbol{p}_{r,m}^q{}'), \qquad m \ne 0. \tag{3.7}$$

The pattern is clear: for each $r \in \mathbb{N}$ we already know $\boldsymbol{p}_{k,0}^q$ for $0 \le k \le r-1$ and $\boldsymbol{p}_{k,m}^q$, $m \ne 0$, for $1 \le k \le r$. This is sufficient to carry out both (3.6) and (3.7). The outcome confirms the *ansatz* (3.2), as well as providing a constructive means to compute it.

3.2 The constant-coefficient case

Once the $\boldsymbol{a}_m^q$s are constant vectors, rather than functions of t, the expansion simplifies considerably. In that case the $\boldsymbol{p}_{r,m}^q$s for $m \ne 0$ (but not the $\boldsymbol{p}_{r,0}^q$s) are all constant: it is easy to verify directly from (3.7) by induction on r that

$$\boldsymbol{p}_{r,m}^q \equiv \frac{1}{(\mathrm{i}m)^r} A^{r-1}\boldsymbol{a}_m^q, \qquad m \ne 0, \quad r \in \mathbb{N}, \quad q \in \{1, 2, \ldots, s\}$$

and it follows from (3.7) that

$$\boldsymbol{p}_{r,0}^q = -\mathrm{e}^{tA} A^{r-1} \sum_{m \ne 0} \frac{1}{(\mathrm{i}m)^r}\boldsymbol{a}_m^q, \qquad r \in \mathbb{N}, \quad q \in \{1, \ldots, s\}.$$

3.3 Numerical experiments

We first consider the two-frequencies scalar ODE

$$y'(t) = \mathrm{i}y(t) + \mathrm{e}^{\mathrm{i}\omega_1 t} + \mathrm{e}^{\mathrm{i}\omega_2 t}, \quad t \geq 0, \qquad y(0) = 1, \tag{3.8}$$

whose solution,

$$\begin{aligned} y(t) &= \mathrm{e}^{\mathrm{i}t} + \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\omega_1 t} - \mathrm{e}^{\mathrm{i}t}}{1-\omega_1} + \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\omega_2 t} - \mathrm{e}^{\mathrm{i}t}}{1-\omega_2} \\ &= \mathrm{e}^{\mathrm{i}t} - \mathrm{i}\sum_{r=1}^{\infty}\frac{1}{\omega_1^r}(\mathrm{e}^{\mathrm{i}\omega_1 t} - \mathrm{e}^{\mathrm{i}t}) - \mathrm{i}\sum_{r=1}^{\infty}\frac{1}{\omega_2^r}(\mathrm{e}^{\mathrm{i}\omega_2 t} - \mathrm{e}^{\mathrm{i}t}), \end{aligned}$$

is displayed in Fig. 3.1 for $\omega_1 = 100$ and $\omega_2 = 500$.

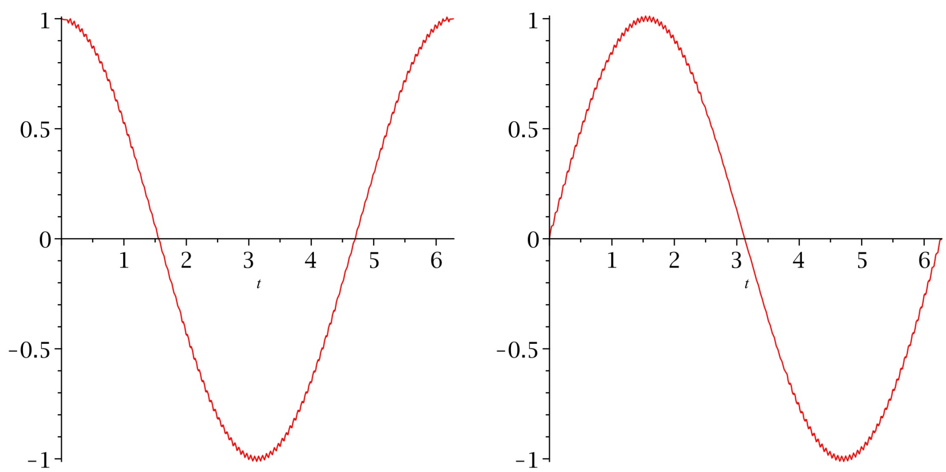

Figure 3.1: The real (the left) and imaginary parts of the solution of (3.8) for $\boldsymbol{\omega} = (100, 500)$.

A cursory examination of Fig. 3.1 might create the impression that the solution is 'nice' and non-oscillatory. Nothing, of course, can be further from the truth: as can be seen both from the exact solution and from the general form (3.2), the non-oscillatory sinusoidal wave forming the basis of the solution is overlaid with a rapidly oscillating, small-amplitude signal. Although this might escape a human eye, examining Fig. 3.1, this rapid oscillation impedes any classical numerical time-stepping solver, which requires minute steps, scaling like $1/\max \omega_k$ (Condon et al. 2009*b*).

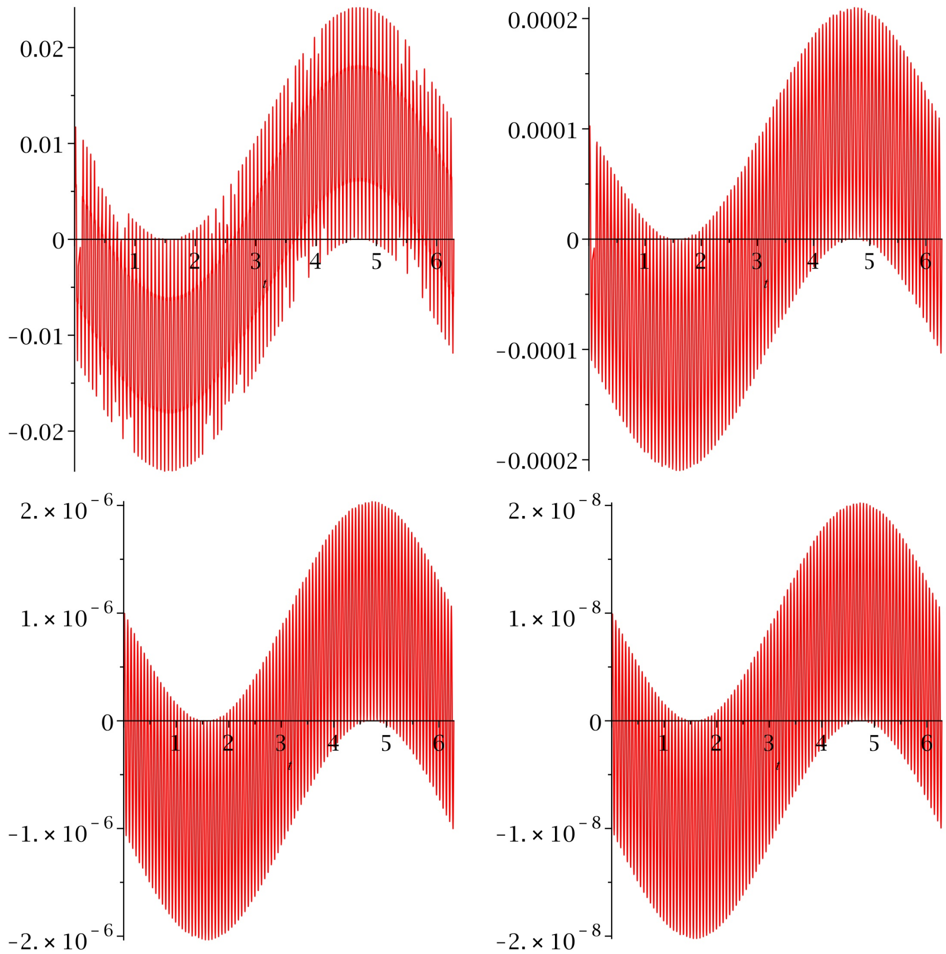

Figure 3.2: The top row: the real parts of e_0 (on the left) and e_1 for $\omega_1 = 100$, underneath, the real parts of e_2 and e_3, all for $\boldsymbol{\omega} = (100, 500)$.

Let

$$\boldsymbol{e}_R = \boldsymbol{y}(t) - \boldsymbol{p}_0(t) - \sum_{r=1}^{R} \sum_{q=1}^{s} \frac{1}{\omega^{(q)^r}} \sum_{m=-\infty}^{\infty} \boldsymbol{p}_{r,m}^{q}(t) \mathrm{e}^{\mathrm{i}m\omega^{(q)}t}.$$

In Fig. 3.2 we have plotted the real part of the (scalar) e_R for equation (3.8), $R = 0, 1, 2, 3$ and $\boldsymbol{\omega} = (100, 500)$. The rate of decay in the amplitude of the error is clear – indeed, to remarkable accuracy, the error's amplitude is divided by $\min\{\omega_1, \omega_2\} = 100$ in each consecutive R.

In Fig. 3.3 we have compared the computation of (3.1) using the MAPLE routine `rkf45` (with standard tolerances, `abserr` $= 10^{-7}$ and `relerr` $= 10^{-6}$)

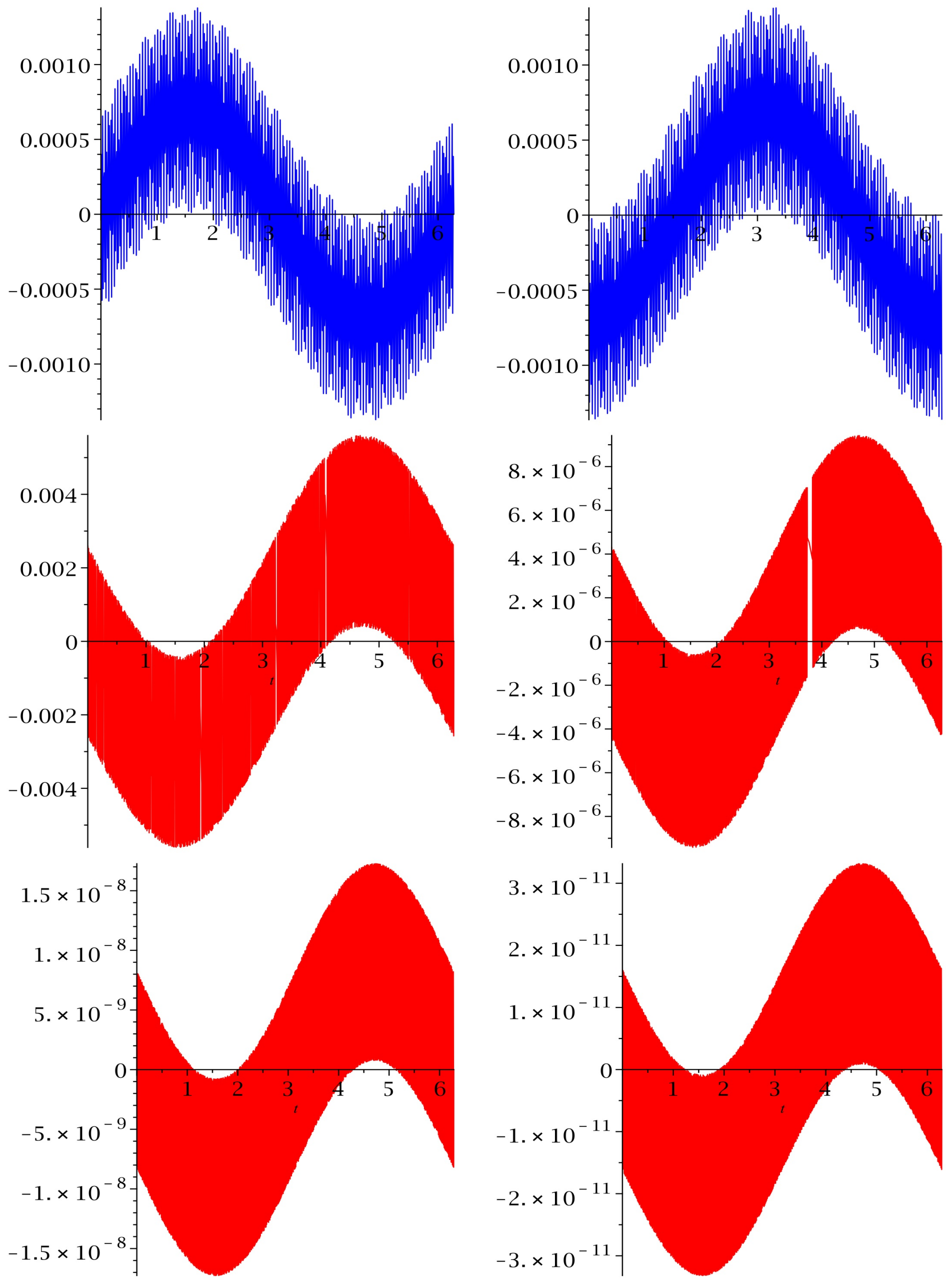

Figure 3.3: The top row: the real (the left) and imaginary (the right) parts of the error committed by `rkf45`. In the middle row and bottom row: real parts of e_R for $R = 0, 1, 2, 3$ for $\boldsymbol{\omega} = (500, 1000)$.

with the use of a truncated asymptotic-numerical expansion. The difference is striking. The classical Runge–Kutta–Fehlberg method (whose run-time is substantially longer) drastically underperforms and its error control breaks down in the presence of high oscillation: the global error is four significant digits worse than the absolute tolerance. Not so the asymptotic-numerical algorithm which, even without error control, delivers tiny error. For each consecutive R the error decays roughly by a factor of 500, in line with the theoretical prediction.

As an example of an equation whose oscillatory forcing term is a genuine modulated Fourier series, we consider a minor variation on (3.8),

$$y'(t) = \mathrm{i}y(t) + t\mathrm{e}^{\mathrm{i}\omega_1 t} + t^2\mathrm{e}^{\mathrm{i}\omega_2 t}, \quad t \geq 0, \qquad y(0) = 1. \tag{3.9}$$

The explicit solution of (3.9) is

$$y(t) = \mathrm{e}^{\mathrm{i}t} + \frac{\mathrm{i}t\mathrm{e}^{\mathrm{i}\omega_1 t}}{1-\omega_1} + \frac{\mathrm{i}t^2\mathrm{e}^{\mathrm{i}\omega_2 t}}{1-\omega_2} + \frac{\mathrm{e}^{\mathrm{i}\omega_1 t} - \mathrm{e}^{\mathrm{i}t}}{(1-\omega_1)^2} + \frac{2t\mathrm{e}^{\mathrm{i}\omega_2 t}}{(1-\omega_2)^2} - \frac{2\mathrm{i}(\mathrm{e}^{\mathrm{i}\omega_2 t} - \mathrm{e}^{\mathrm{i}t})}{(1-\omega_2)^3}$$

and, bearing in mind that $A = \mathrm{i}$, $s = 2$, $\boldsymbol{a}_1^1(t) = t$, $\boldsymbol{a}_1^2(t) = t^2$ and $\boldsymbol{a}_m^q \equiv 0$ for $m \neq 1$, it can be easily brought into the form (3.2). Specifically, we have

$$\begin{aligned}
&p_{0,0}(t) = \mathrm{e}^{\mathrm{i}t},\\
&p_{1,0}^1, p_{1,0}^2 \equiv 0, \qquad p_{1,1}^1(t) = -\mathrm{i}t, \qquad p_{1,1}^2(t) = -\mathrm{i}t^2,\\
&p_{2,0}^1(t) = -\mathrm{e}^{\mathrm{i}t}, \qquad p_{2,0}^2 \equiv 0, \qquad p_{2,1}^1(t) = 1 - \mathrm{i}t, \qquad p_{2,1}^2(t) = 2t - \mathrm{i}t^2,\\
&p_{r,0}^1(t) = -(r-1)\mathrm{e}^{\mathrm{i}t}, \qquad p_{r,0}^2(t) = -\mathrm{i}(r-2)(r-1)\mathrm{e}^{\mathrm{i}t}, \qquad p_{r,1}^1(t) = r - 1 - \mathrm{i}t,\\
&\qquad p_{r,1}^2(t) = \mathrm{i}(r-2)(r-1) + 2(r-1)t - \mathrm{i}t^2, \qquad r \geq 3.
\end{aligned}$$

Fig. 3.4 displays the error committed by `rkf45` and by the asymptotic-numerical approximation for $\boldsymbol{\omega} = (100, 500)$, while Fig. 3.5 reports the error of the asymptotic-numerical method for $\boldsymbol{\omega} = (500, 1000)$. The conclusions are fully consistent with our analysis. Thus, the classical time-stepping method clearly exhibits failure of the error control mechanism in the presence of high oscillation. The truncated asymptotic expansion, on the other hand, delivers (at an exceedingly modest cost) very high precision and demonstrates increased accuracy as $\min \omega^{(q)}$ grows.

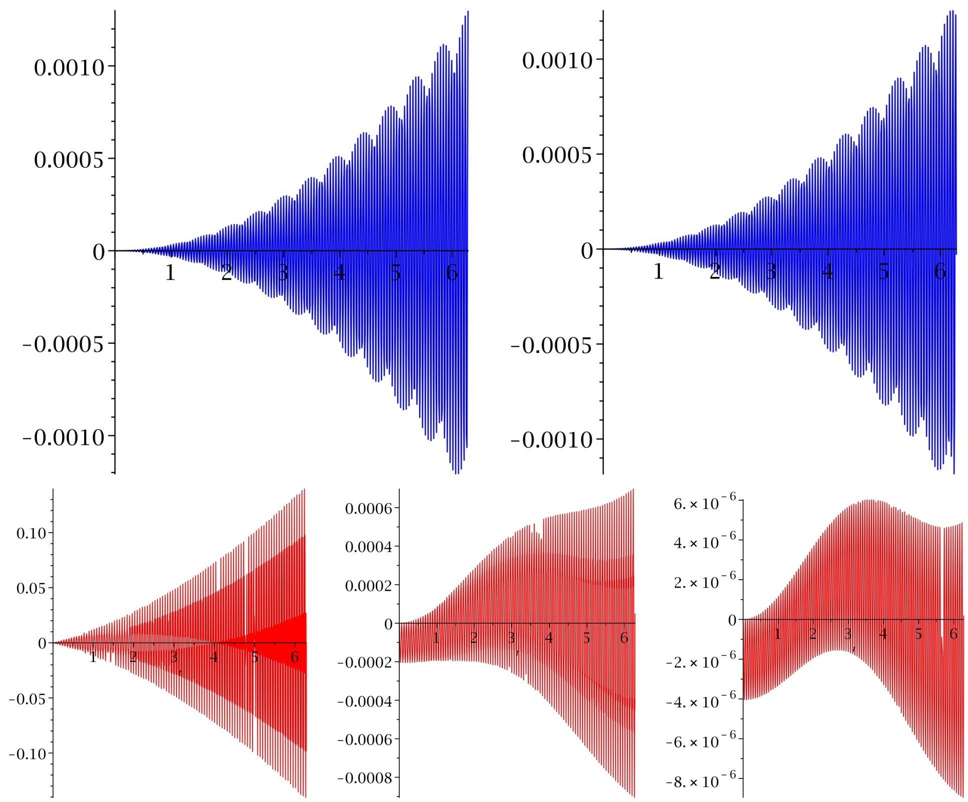

Figure 3.4: The top row: the real (the left) and imaginary parts of the error of `rkf45`. Underneath, from the left, real parts of e_0, e_1 and e_2(the right) for equation (3.9) and $\boldsymbol{\omega} = (100, 500)$.

3.4 Modulated forcing term

An alternative to the model (3.1) is the linear ODE

$$\boldsymbol{y}' = A\boldsymbol{y} + \sum_{m=1}^{\infty} \boldsymbol{a}_m(t)\mathrm{e}^{\mathrm{i}\omega_m t}, \qquad t \geq 0, \tag{3.10}$$

where the $\boldsymbol{a}_m$s vary with t, while $\|A\| < \min|\omega_m|$. The framework of this section can be readily extended to this setting. Thus, by variation of constants,

$$\boldsymbol{y}(t) = \mathrm{e}^{tA}\boldsymbol{y}_0 + \sum_{m=1}^{\infty} \mathrm{e}^{tA} \int_0^t \mathrm{e}^{x(-A+\mathrm{i}\omega_m I)} \boldsymbol{a}_m(x)\,\mathrm{d}x.$$

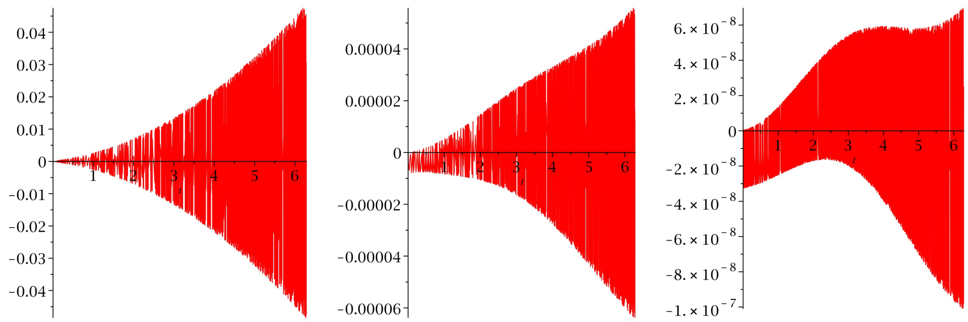

Figure 3.5: From the left, the real parts of e_0, e_1 and e_2 for equation (3.9) and $\boldsymbol{\omega} = (500, 1000)$.

Using the theory of (Iserles, Nørsett & Olver 2006), it is easy to prove that, formally (i.e., disregarding issues of convergence)

$$\int_0^t \mathrm{e}^{xB}\boldsymbol{c}(x)\,\mathrm{d}x = -\sum_{k=0}^{\infty}(-B)^{-k-1}[\mathrm{e}^{tB}\boldsymbol{c}^{(k)}(t) - \boldsymbol{c}^{(k)}(0)],$$

where B is a nonsingular matrix. In our case, setting $B = -A + \mathrm{i}\omega_m I$ (which is non-singular since $\|A\| < |\omega_m|$ and convergent in an asymptotic sense for sufficiently large $|\omega_m|$) we have

$$\boldsymbol{y}(t) = \mathrm{e}^{tA}\boldsymbol{y}_0 - \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(-\mathrm{i}\omega_m)^{k+1}}\left(I - \frac{1}{\mathrm{i}\omega_m}A\right)^{-k-1}[\mathrm{e}^{\mathrm{i}\omega_m t}\boldsymbol{a}_m^{(k)}(t) - \mathrm{e}^{tA}\boldsymbol{a}_m(0)].$$

Since $(1-z)^{-k-1} = \sum_{r=0}^{\infty}\binom{k+r}{k}z^r$, we have, after easy algebra,

$$\begin{aligned}\boldsymbol{y}(t) &= \mathrm{e}^{tA}\boldsymbol{y}_0 + \sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(\mathrm{i}\omega_m)^r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}A^{r-k-1}[\mathrm{e}^{\mathrm{i}\omega_m t}\boldsymbol{a}_m^{(k)}(t) - \mathrm{e}^{tA}\boldsymbol{a}_m^{(k)}(0)]\\ &= \mathrm{e}^{tA}\boldsymbol{y}_0 + \sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-\mathrm{i})^r}{\boldsymbol{\omega}^{r\boldsymbol{e}_m}}\left[\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}A^{r-k-1}\boldsymbol{a}_m^{(k)}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{e}_m^\top\boldsymbol{\omega}t}\right.\\ &\qquad\left. - \mathrm{e}^{tA}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}A^{r-k-1}\boldsymbol{a}_m^{(k)}(0)\right],\end{aligned}$$

a form consistent with (1.6).

The equations (3.1) and (3.10) being linear, there is no transfer of energy across frequencies. As we have already seen in Section 2 and will observe again in the next section, this is definitely not the case in the nonlinear setting.

4 The nonlinear equation

4.1 The multi-frequency model

Let distinct ω_j, $j \in \mathbb{N}$, be given, where $\sup_j |\omega_j| \gg 1$ (and might be infinite). In this section we consider a general model (1.5) with a multiple-frequency forcing term,

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}) + \sum_{m=1}^{\infty} \boldsymbol{a}_m(t)\mathrm{e}^{\mathrm{i}\omega_m t}, \quad t \geq 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{4.1}$$

where all functions concerned are smooth.

We assume that the solution of (4.1) is of the form (1.6), namely

$$\boldsymbol{y}(t) = \boldsymbol{p}_0(t) + \sum_{r=1}^{\infty} \sum_{\boldsymbol{\ell} \in \mathbb{L}_r} \frac{1}{\boldsymbol{\omega}^{\boldsymbol{\ell}}} \sum_{\boldsymbol{m} \in \mathbb{Z}_+^\infty} \boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top \boldsymbol{\omega} t}, \tag{4.2}$$

where

$$\mathbb{L}_r = \left\{ \boldsymbol{\ell} \in \mathbb{Z}_+^\infty \,:\, \sum_{i=1}^{\infty} \ell_i = r \right\} = \left\{ \sum_{j=1}^{r} \boldsymbol{e}_{\alpha_j} \,:\, \alpha_1, \ldots, \alpha_r \neq 0 \right\}, \quad r \in \mathbb{N},$$

$\boldsymbol{e}_\alpha$ being the αth unit vector. The coefficients $\boldsymbol{p}_0$ and $\boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}}$ will be determined algorithmically in the sequel, except that we stipulate at the outset that

$$\exists i \in \{1, 2, \ldots, r\} \text{ such that } \ell_i = 0,\ m_i \neq 0 \qquad \Rightarrow \qquad \boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}} = \boldsymbol{0}. \tag{4.3}$$

Expressions of the form $\mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top \boldsymbol{\omega} t}$ will be henceforth called *oscillators*.

4.2 Computing the expansion coefficients

We commence by evaluating the derivative of $\boldsymbol{y}$. Differentiating (4.2),

$$\boldsymbol{y}' = \boldsymbol{p}_0' + \sum_{r=1}^{\infty} \sum_{\boldsymbol{\ell} \in \mathbb{L}_r} \frac{1}{\boldsymbol{\omega}^{\boldsymbol{\ell}}} \sum_{\boldsymbol{m} \in \mathbb{Z}_+^\infty} (\boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}}' + \mathrm{i}\boldsymbol{m}^\top \boldsymbol{\omega} \boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}})\mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top \boldsymbol{\omega} t}.$$

While

$$\boldsymbol{p}_0' + \sum_{r=1}^{\infty} \sum_{\boldsymbol{\ell} \in \mathbb{L}_r} \frac{1}{\boldsymbol{\omega}^{\boldsymbol{\ell}}} \sum_{\boldsymbol{m} \in \mathbb{Z}_+^\infty} \boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}}' \mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top \boldsymbol{\omega} t}$$

presents no difficulties, we need to devote more attention to

$$
\begin{aligned}
&\sum_{r=1}^{\infty}\sum_{\boldsymbol{\ell}\in\mathbb{L}_r}\frac{1}{\boldsymbol{\omega}^{\boldsymbol{\ell}}}\sum_{\boldsymbol{m}\in\mathbb{Z}_+^\infty}\mathrm{i}\boldsymbol{m}^\top\boldsymbol{\omega}\boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}}\mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top\boldsymbol{\omega}t}\\
={}&\sum_{\alpha\neq0}\frac{1}{\omega_\alpha}\sum_{\boldsymbol{m}\in\mathbb{Z}_+^\infty}\sum_{j=-\infty}^{\infty}\mathrm{i}m_j\omega_j\boldsymbol{p}_{\boldsymbol{m},\boldsymbol{e}_\alpha}\mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top\boldsymbol{\omega}t}\\
&+\sum_{\alpha,\beta\neq0}\frac{1}{\omega_\alpha\omega_\beta}\sum_{\boldsymbol{m}\in\mathbb{Z}_+^\infty}\sum_{j=-\infty}^{\infty}\mathrm{i}m_j\omega_j\boldsymbol{p}_{\boldsymbol{m},\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta}\mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top\boldsymbol{\omega}t}\\
&+\sum_{\alpha,\beta,\gamma\neq0}\frac{1}{\omega_\alpha\omega_\beta\omega_\gamma}\sum_{\boldsymbol{m}\in\mathbb{Z}_+^\infty}\sum_{j=-\infty}^{\infty}\mathrm{i}m_j\omega_j\boldsymbol{p}_{\boldsymbol{m},\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta+\boldsymbol{e}_\gamma}\mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top\boldsymbol{\omega}t}\\
&+\cdots.
\end{aligned}
$$

It follows from (4.3) that, unless $\boldsymbol{p}_{\boldsymbol{m},\boldsymbol{e}_{\alpha_1}+\cdots\boldsymbol{e}_{\alpha_r}}\equiv\boldsymbol{0}$, necessarily $\boldsymbol{m}=m_{\alpha_1}\boldsymbol{e}_{\alpha_1}+\cdots+m_{\alpha_r}\boldsymbol{e}_{\alpha_r}$ for some $m_{\alpha_1},\ldots,m_{\alpha_r}\in\mathbb{Z}_+$, therefore, using symmetry,

$$
\begin{aligned}
&\sum_{r=1}^{\infty}\sum_{\boldsymbol{\ell}\in\mathbb{L}_r}\frac{1}{\boldsymbol{\omega}^{\boldsymbol{\ell}}}\sum_{\boldsymbol{m}\in\mathbb{Z}_+^\infty}\mathrm{i}\boldsymbol{m}^\top\boldsymbol{\omega}\boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}}\mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top\boldsymbol{\omega}t}\\
={}&\sum_{\alpha\neq0}\sum_{m_1\in\mathbb{Z}_+}\mathrm{i}m_1\boldsymbol{p}_{m_1\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha}\mathrm{e}^{\mathrm{i}m_1\omega_\alpha t}\\
&+2\sum_{\alpha,\beta\neq0}\frac{1}{\omega_\alpha}\sum_{m_1,m_2\in\mathbb{Z}_+}\mathrm{i}m_2\boldsymbol{p}_{m_1\boldsymbol{e}_\alpha+m_2\boldsymbol{e}_\beta,\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta}\mathrm{e}^{\mathrm{i}(m_1\omega_\alpha+m_2\omega_\beta)t}\\
&+3\sum_{\alpha,\beta,\gamma\neq0}\frac{1}{\omega_\alpha\omega_\beta}\sum_{m_1,m_2,m_3\in\mathbb{Z}_+}\mathrm{i}m_3\boldsymbol{p}_{m_1\boldsymbol{e}_\alpha+m_2\boldsymbol{e}_\beta+m_3\boldsymbol{e}_\gamma,\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta+\boldsymbol{e}_\gamma}\\
&\qquad\times\mathrm{e}^{\mathrm{i}(m_1\omega_\alpha+m_2\omega_\beta+m_3\omega_\gamma)t}+\cdots.
\end{aligned}
$$

Next we expand $\boldsymbol{f}(\boldsymbol{y})$. Using the linearity of the derivative operators $\boldsymbol{f}_k$ in all but the first variable, we have

$$\begin{aligned}
\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{p}_0) &+ \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{r_1=1}^{\infty} \cdots \sum_{r_k=1}^{\infty} \sum_{\boldsymbol{\ell}_1 \in \mathbb{L}_{r_1}} \cdots \sum_{\boldsymbol{\ell}_k \in \mathbb{L}_{r_k}} \frac{1}{\boldsymbol{\omega}^{\boldsymbol{\ell}_1+\cdots+\boldsymbol{\ell}_k}} \sum_{\boldsymbol{m}_1 \in \mathbb{Z}_+^\infty} \cdots \sum_{\boldsymbol{m}_k \in \mathbb{Z}_+^\infty} \\
&\qquad \boldsymbol{f}_k(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{m}_1,\boldsymbol{\ell}_1}, \ldots, \boldsymbol{p}_{\boldsymbol{m}_k,\boldsymbol{\ell}_k}) \mathrm{e}^{\mathrm{i}(\boldsymbol{m}_1+\cdots+\boldsymbol{m}_k)^\top \boldsymbol{\omega} t} \\
= \boldsymbol{f}(\boldsymbol{p}_0) &+ \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{r=1}^{\infty} \sum_{\boldsymbol{n} \in \mathbb{I}_{k,r}} \sum_{\boldsymbol{\ell} \in \mathbb{L}_r} \frac{1}{\boldsymbol{\omega}^{\boldsymbol{\ell}}} \sum_{(\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k) \in \mathbb{K}_{k,\boldsymbol{n},\boldsymbol{\ell}}} \sum_{\boldsymbol{m} \in \mathbb{Z}_+^\infty} \sum_{(\boldsymbol{\rho}_1,\ldots,\boldsymbol{\rho}_k) \in \mathbb{M}_{k,\boldsymbol{m}}} \\
&\qquad \boldsymbol{f}_k(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{\rho}_1,\boldsymbol{j}_1}, \ldots, \boldsymbol{p}_{\boldsymbol{\rho}_k,\boldsymbol{j}_k}) \mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top \boldsymbol{\omega} t} \\
= \boldsymbol{f}(\boldsymbol{p}_0) &+ \sum_{r=1}^{\infty} \sum_{\boldsymbol{\ell} \in \mathbb{L}_r} \frac{1}{\boldsymbol{\omega}^{\boldsymbol{\ell}}} \sum_{\boldsymbol{m} \in \mathbb{Z}_+^\infty} \sum_{k=1}^{r} \frac{1}{k!} \sum_{\boldsymbol{n} \in \mathbb{I}_{k,r}} \sum_{(\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k) \in \mathbb{K}_{k,\boldsymbol{n},\boldsymbol{\ell}}} \sum_{(\boldsymbol{\rho}_1,\ldots,\boldsymbol{\rho}_k) \in \mathbb{M}_{k,\boldsymbol{m}}} \\
&\qquad \boldsymbol{f}_k(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{\rho}_1,\boldsymbol{j}_1}, \ldots, \boldsymbol{p}_{\boldsymbol{\rho}_k,\boldsymbol{j}_k}) \mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top \boldsymbol{\omega} t},
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{K}_{k,\boldsymbol{n},\boldsymbol{\ell}} &= \left\{ (\boldsymbol{j}_1, \ldots, \boldsymbol{j}_k) \,:\, \boldsymbol{j}_1 \in \mathbb{L}_{n_1}, \ldots, \boldsymbol{j}_k \in \mathbb{L}_{n_k},\ \sum_{i=1}^{k} \boldsymbol{j}_i = \boldsymbol{\ell} \right\}, \\
\mathbb{M}_{k,\boldsymbol{m}} &= \left\{ (\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_k) \,:\, \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_k \in \mathbb{Z}_+^\infty,\ \sum_{i=1}^{k} \boldsymbol{\rho}_k = \boldsymbol{m} \right\}.
\end{aligned}$$

We now substitute the expansions into into (4.1),

$$\begin{aligned}
&\boldsymbol{p}_0' + \sum_{\boldsymbol{m} \in \mathbb{Z}_+^\infty \setminus \{\boldsymbol{0}\}} \sum_{j=-\infty}^{\infty} \mathrm{i} m_j \boldsymbol{p}_{\boldsymbol{m},\boldsymbol{e}_j} \mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top \boldsymbol{\omega} t} \\
&\quad + \sum_{r=1}^{\infty} \sum_{\boldsymbol{\ell} \in \mathbb{L}_r} \frac{1}{\boldsymbol{\omega}^{\boldsymbol{\ell}}} \sum_{\boldsymbol{m} \in \mathbb{Z}_+^\infty} \left(\boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}}' + \sum_{j=-\infty}^{\infty} \mathrm{i} m_j \boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}+\boldsymbol{e}_j} \right) \mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top \boldsymbol{\omega} t} \\
&= \boldsymbol{f}(\boldsymbol{p}_0) + \sum_{r=1}^{\infty} \sum_{\boldsymbol{\ell} \in \mathbb{L}_r} \frac{1}{\boldsymbol{\omega}^{\boldsymbol{\ell}}} \sum_{\boldsymbol{m} \in \mathbb{Z}_+^\infty} \sum_{k=1}^{r} \frac{1}{k!} \sum_{\boldsymbol{n} \in \mathbb{I}_{k,r}} \sum_{(\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k) \in \mathbb{K}_{k,\boldsymbol{n},\boldsymbol{\ell}}} \sum_{(\boldsymbol{\rho}_1,\ldots,\boldsymbol{\rho}_k) \in \mathbb{M}_{k,\boldsymbol{m}}} \\
&\qquad \boldsymbol{f}_k(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{\rho}_1,\boldsymbol{j}_1}, \ldots, \boldsymbol{p}_{\boldsymbol{\rho}_k,\boldsymbol{j}_k}) \mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top \boldsymbol{\omega} t} + \sum_{m=1}^{\infty} \boldsymbol{a}_m \mathrm{e}^{\mathrm{i}\omega_m t}.
\end{aligned}$$

The next step is clear: separating scales and frequencies. This is fairly straightforward for $\boldsymbol{m} = \boldsymbol{0}$, when for every $r \in \mathbb{Z}_+$ and $\boldsymbol{\ell} \in \mathbb{L}_r$ we have the non-oscillatory ODE

$$\boldsymbol{p}_{\boldsymbol{0},\boldsymbol{\ell}}' = \sum_{k=1}^{r} \frac{1}{k!} \sum_{\boldsymbol{n}\in\mathbb{I}_{k,r}} \sum_{(\boldsymbol{j}_{,}\ldots,\boldsymbol{j}_k)\in\mathbb{K}_{k,\boldsymbol{n},\boldsymbol{\ell}}} \sum_{(\boldsymbol{\rho}_1,\ldots,\boldsymbol{\rho}_k)\in\mathbb{M}_{k,\boldsymbol{0}}} \boldsymbol{f}_k(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{\rho}_1,\boldsymbol{j}_1}, \ldots, \boldsymbol{p}_{\boldsymbol{\rho}_k,\boldsymbol{j}_k}) \quad (4.4)$$

with the initial condition

$$\boldsymbol{p}_{\boldsymbol{0},\boldsymbol{\ell}}(0) = -\sum_{\boldsymbol{m}\neq\boldsymbol{0}} \boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}}(0).$$

For $r = 0$ this is simply

$$\boldsymbol{p}_0' = \boldsymbol{f}(\boldsymbol{p}_0), \quad t \geq 0, \qquad \boldsymbol{p}_0(0) = \boldsymbol{y}_0,$$

Note that, having reached $r \in \mathbb{N}$, we already know $\boldsymbol{p}_{\boldsymbol{m},\boldsymbol{\ell}}$ for $\boldsymbol{m} \neq \boldsymbol{0}$ and $\boldsymbol{\ell} \in \mathbb{L}_\alpha$ $\alpha \leq r$. This means that the right-hand side of the initial condition is known. Moreover, $\mathbb{I}_{1,r} = \{r\}$, $\mathbb{K}_{1,r,\boldsymbol{\ell}} = \{\boldsymbol{\ell}\}$ and $\mathbb{M}_{1,\boldsymbol{0}} = \{\boldsymbol{0}\}$, therefore (4.4) can be rewritten in the form

$$\begin{aligned}\boldsymbol{p}_{\boldsymbol{0},\boldsymbol{\ell}}' &= \boldsymbol{f}_1(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{\ell}}) \\ &\quad + \sum_{k=2}^{r} \frac{1}{k!} \sum_{\boldsymbol{n}\in\mathbb{I}_{k,r}} \sum_{(\boldsymbol{j}_{,}\ldots,\boldsymbol{j}_k)\in\mathbb{K}_{k,\boldsymbol{n},\boldsymbol{\ell}}} \sum_{(\boldsymbol{\rho}_1,\ldots,\boldsymbol{\rho}_k)\in\mathbb{M}_{k,\boldsymbol{0}}} \boldsymbol{f}_k(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{\rho}_1,\boldsymbol{j}_1}, \ldots, \boldsymbol{p}_{\boldsymbol{\rho}_k,\boldsymbol{j}_k}),\end{aligned}$$

where the multiple sum on the right consists of quantities that are already known.

We next address the case $\boldsymbol{m} \in \mathbb{Z}_+^\infty \setminus \{\boldsymbol{0}\}$ in some detail.

4.2.1 $r = 0$

At the first instance we examine the $\mathcal{O}(1)$ terms, whence

$$\boldsymbol{p}_0' + \sum_{\alpha\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \mathrm{i} m \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha} \mathrm{e}^{\mathrm{i}m\omega_\alpha t} = \boldsymbol{f}(\boldsymbol{p}_0) + \sum_{m=-\infty}^{\infty} \boldsymbol{a}_m \mathrm{e}^{\mathrm{i}\omega_m t}$$

and, bearing (4.3) in mind, we obtain the recurrences

$$\boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha} = -\mathrm{i}\boldsymbol{a}_\alpha \qquad \boldsymbol{p}_{\boldsymbol{m},\boldsymbol{e}_j} \equiv \boldsymbol{0}, \quad j \neq \alpha, \quad \boldsymbol{m} \neq \boldsymbol{e}_\alpha, \boldsymbol{0}, \qquad \alpha \in \mathbb{Z}.$$

4.2.2 $r = 1$

Next, equating both scales and frequencies, for every $r \in \mathbb{N}$ and $\alpha \in \mathbb{Z} \setminus \{0\}$ we have

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} \boldsymbol{p}'_{m\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha} \mathrm{e}^{\mathrm{i}m\omega_\alpha t} \\
&\quad + 2 \sum_{\beta\in\mathbb{Z}\setminus\{0\}} \sum_{m_1,m_2\in\mathbb{Z}} \mathrm{i}m_2 \boldsymbol{p}_{m_1\boldsymbol{e}_\alpha+m_2\boldsymbol{e}_\beta,\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta} \mathrm{e}^{\mathrm{i}(m_1\omega_\alpha+m_2\omega_\beta)t} \\
= &\sum_{\boldsymbol{m}\in\mathbb{Z}^+} \boldsymbol{f}_1(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{m},\boldsymbol{e}_\alpha}) \mathrm{e}^{\mathrm{i}\boldsymbol{m}^\top\boldsymbol{\omega}t} = \boldsymbol{f}_1(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha}) \mathrm{e}^{\mathrm{i}\omega_\alpha t}.
\end{aligned}$$

The only choice that results in a nonzero term in the sum is $\beta = \alpha$, $m_1 = 0$, $m_2 = 1$, whence

$$\boldsymbol{p}_{\boldsymbol{e}_\alpha,2\boldsymbol{e}_\alpha} = \mathrm{i}[\boldsymbol{p}'_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha} - \boldsymbol{f}_1(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha})], \qquad \alpha \in \mathbb{Z}.$$

We set $\boldsymbol{p}_{\boldsymbol{m},\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta} \equiv \boldsymbol{0}$ for all the remaining choices of $\alpha, \beta \neq 0$.

4.2.3 $r = 2$

We now have for every $\alpha, \beta \neq 0$

$$\begin{aligned}
&3 \sum_{m_1,m_2,m_3\in\mathbb{Z}} \sum_{\gamma\neq 0} \mathrm{i}m_3 \boldsymbol{p}_{m_1\boldsymbol{e}_\alpha+m_2\boldsymbol{e}_\beta+m_3\boldsymbol{e}_\gamma,\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta+\boldsymbol{e}_\gamma} \mathrm{e}^{\mathrm{i}(m_1\omega_\alpha+m_2\omega_\beta+m_3\omega_\gamma)t} \\
= &- \sum_{m_1,m_2\in\mathbb{Z}} \boldsymbol{p}'_{m_1\boldsymbol{e}_\alpha+m_2\boldsymbol{e}_\beta,\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta} \mathrm{e}^{\mathrm{i}(m_1\omega_\alpha+m_2\omega_\beta)t} \\
&+ \sum_{m_1,m_2\in\mathbb{Z}} \boldsymbol{f}_1(\boldsymbol{p}_0, \boldsymbol{p}_{m_1\boldsymbol{e}_\alpha+m_2\boldsymbol{e}_\beta,\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta}) \mathrm{e}^{\mathrm{i}(m_1\omega_\alpha+m_2\omega_\beta)t} \\
&+ \tfrac{1}{2} \sum_{m_1,m_2\in\mathbb{Z}} \sum_{\boldsymbol{j}_1+\boldsymbol{j}_2=\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta} \sum_{\boldsymbol{\rho}_1+\boldsymbol{\rho}_2=m_1\boldsymbol{e}_\alpha+m_2\boldsymbol{e}_\beta} \boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{\rho}_1,\boldsymbol{j}_1}, \boldsymbol{p}_{\boldsymbol{\rho}_2,\boldsymbol{j}_2}) \mathrm{e}^{\mathrm{i}(m_1\omega_\alpha+m_2\omega_\beta)t},
\end{aligned} \tag{4.5}$$

where in the last sum either $\boldsymbol{\rho}_i = \boldsymbol{j}_i$ or $\boldsymbol{\rho}_i = \boldsymbol{0}$, otherwise $\boldsymbol{p}_{\boldsymbol{\rho}_i,\boldsymbol{\rho}_i} = \boldsymbol{0}$, rendering the $\boldsymbol{f}_2$ function zero.

We commence from the case $\alpha = \beta$, hence $\boldsymbol{j}_1 = \boldsymbol{j}_2 = \boldsymbol{e}_\alpha$, and examine first the right-hand side. If $\boldsymbol{\rho}_1 = \boldsymbol{e}_\alpha$ then if either $(m_1, m_2) = (1, 0)$ or $(m_1, m_2) = (0, 1)$ the oscillator is $\mathrm{e}^{\mathrm{i}\omega_\alpha t}$ and we obtain $-\boldsymbol{p}'_{\boldsymbol{e}_\alpha,2\boldsymbol{e}_\alpha} + \boldsymbol{f}_1(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha,2\boldsymbol{e}_\alpha})$, while once $(m_1, m_2) \in \{(2, 0), (1, 1), (0, 2)\}$, the oscillator is $\mathrm{e}^{2\mathrm{i}\omega_\alpha t}$ and the relevant term is $\frac{1}{2}\boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha})$.

In addition, if $\boldsymbol{\rho}_1 = \boldsymbol{0}$ then $\boldsymbol{\rho}_2 = \boldsymbol{e}_\alpha$, hence $(m_1, m_2) = (1, 0)$, with oscillator $\mathrm{e}^{\mathrm{i}\omega_\alpha t}$ and the term $\frac{1}{2}\boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha})$. Because of symmetry of the second and third terms of $\boldsymbol{f}_2$, an identical term is obtained once we take $\boldsymbol{\rho}_2 = \boldsymbol{0}$.

We need to identify the above two frequencies on the left-hand side. Insofar as $\mathrm{e}^{\mathrm{i}\omega_\alpha t}$ is concerned, nonzero terms occur only when $\gamma = \alpha$ and $(m_1, m_2, m_3) = (0, 0, 1)$, this results in $3\mathrm{i}\boldsymbol{p}_{\boldsymbol{e}_\alpha, 3\boldsymbol{e}_\alpha}$. There are three combinations that yield $\mathrm{e}^{2\mathrm{i}\omega_\alpha t}$, namely $(m_1, m_2, m_3) \in \{(0, 0, 2), (1, 0, 1), (0, 1, 1)\}$, which together give $12\mathrm{i}\boldsymbol{p}_{2\boldsymbol{e}_\alpha, 3\boldsymbol{e}_\alpha}$. We thus deduce that

$$\begin{aligned}\boldsymbol{p}_{\boldsymbol{e}_\alpha, 3\boldsymbol{e}_\alpha} &= \frac{\mathrm{i}}{3}[\boldsymbol{p}'_{\boldsymbol{e}_\alpha, 2\boldsymbol{e}_\alpha} - \boldsymbol{f}_1(\boldsymbol{p}_m, \boldsymbol{p}_{\boldsymbol{e}_\alpha, 2\boldsymbol{e}_\alpha}) - \boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0}, \boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha, \boldsymbol{e}_\alpha})],\\ \boldsymbol{p}_{2\boldsymbol{e}_\alpha, 3\boldsymbol{e}_\alpha} &= -\frac{\mathrm{i}}{24}\boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha, \boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha, \boldsymbol{e}_\alpha})\end{aligned}$$

for every $\alpha \in \mathbb{N}$.

We next address the case $\alpha \neq \beta$, starting as before on the right-hand side of (4.5). The first two sums vanish, because we have already seen that $\boldsymbol{p}_{\boldsymbol{m}, \boldsymbol{e}_\alpha + \boldsymbol{e}_\beta} \equiv \boldsymbol{0}$ for $\alpha \neq \beta$, while in the third sum $\boldsymbol{\rho}_1 = \boldsymbol{j}_1 \in \{\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta\}$ imply that necessarily $m_1 = m_2 = 1$ and the oscillator is $\mathrm{e}^{\mathrm{i}(\omega_\alpha + \omega_\beta)t}$. Specifically, $\boldsymbol{j}_1 = \boldsymbol{e}_\alpha$ results in the term $\frac{1}{2}\boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha, \boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\beta, \boldsymbol{e}_\beta})$, while $\boldsymbol{j}_1 = \boldsymbol{e}_\beta$ yields $\frac{1}{2}\boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\beta, \boldsymbol{e}_\beta}, \boldsymbol{p}_{\boldsymbol{e}_\alpha, \boldsymbol{e}_\alpha})$. Note, however, that $\boldsymbol{f}_2$ is symmetric in its second and third arguments, hence the entire contribution is $\boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha, \boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\beta, \boldsymbol{e}_\beta})$.

Moreover, letting $\boldsymbol{\rho}_1 = \boldsymbol{0}$ implies that $\boldsymbol{\rho}_2 = m_1\boldsymbol{e}_\alpha + m_2\boldsymbol{e}_\beta$ – if $\boldsymbol{j}_1 = \boldsymbol{e}_\alpha$ then we need $(m_1, m_2) = (0, 1)$ and obtain the oscillator $\mathrm{e}^{\mathrm{i}\omega_\beta t}$ and the term $\frac{1}{2}\boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0}, \boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\beta, \boldsymbol{e}_\beta})$. Likewise, when $\boldsymbol{j}_1 = \boldsymbol{e}_\beta$ then $(m_1, m_2) = (1, 0)$, the oscillator $\mathrm{e}^{\mathrm{i}\omega_\alpha t}$ and the term $\frac{1}{2}\boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0}, \boldsymbol{e}_\beta}, \boldsymbol{p}_{\boldsymbol{e}_\alpha, \boldsymbol{e}_\alpha})$. By symmetry, we have exactly the same taking $\boldsymbol{\rho}_2 = \boldsymbol{0}$.

Turning our attention to the left-hand side of (4.5), we first identify $\mathrm{e}^{\mathrm{i}\omega_\alpha t}$ terms. For this necessarily $\gamma = \alpha$ and $(m_1, m_2, m_3) = (0, 0, 1)$, therefore

$$\boldsymbol{p}_{\boldsymbol{e}_\alpha, 2\boldsymbol{e}_\alpha + \boldsymbol{e}_\beta} = -\frac{\mathrm{i}}{3}\boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0}, \boldsymbol{e}_\beta}, \boldsymbol{p}_{\boldsymbol{e}_\alpha, \boldsymbol{e}_\alpha}), \qquad \alpha \neq \beta.$$

To obtain $\mathrm{e}^{\mathrm{i}(\omega_\alpha + \omega_\beta)t}$ we need necessarily take either $\gamma = \alpha$ or $\gamma = \beta$, and in either case $m_3 = 1$. Specifically, either $\gamma = \alpha$, $(m_1, m_2, m_3) = (0, 1, 1)$, or $\gamma = \beta$, $(m_1, m_2, m_3) = (1, 0, 1)$: these choices result in $3\mathrm{i}\boldsymbol{p}_{\boldsymbol{e}_\alpha + \boldsymbol{e}_\beta, 2\boldsymbol{e}_\alpha + \boldsymbol{e}_\beta}$ and in $3\mathrm{i}\boldsymbol{p}_{\boldsymbol{e}_\alpha + \boldsymbol{e}_\beta, \boldsymbol{e}_\alpha + 2\boldsymbol{e}_\beta}$ respectively. Therefore

$$3\mathrm{i}(\boldsymbol{p}_{\boldsymbol{e}_\alpha + \boldsymbol{e}_\beta, 2\boldsymbol{e}_\alpha + \boldsymbol{e}_\beta} + \boldsymbol{p}_{\boldsymbol{e}_\alpha + \boldsymbol{e}_\beta, \boldsymbol{e}_\alpha + 2\boldsymbol{e}_\beta}) = \boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha, \boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\beta, \boldsymbol{e}_\beta})$$

and, by symmetry,

$$\boldsymbol{p}_{\boldsymbol{e}_\alpha + \boldsymbol{e}_\beta, 2\boldsymbol{e}_\alpha + \boldsymbol{e}_\beta} = -\frac{\mathrm{i}}{6}\boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha, \boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\beta, \boldsymbol{e}_\beta}), \qquad \alpha \neq \beta.$$

All the remaining values of $\boldsymbol{p}_{\boldsymbol{m}, \boldsymbol{\ell}}$, $\boldsymbol{\ell} \in \mathbb{L}_3$, $\boldsymbol{m} \neq \boldsymbol{0}$, are set to zero.

4.2.4 General $r \in \mathbb{N}$

Scaling up of our recurrences to general $r \in \mathbb{N}$ is just a matter of increasing complexity, rather than of principle. In general, we have

$$\begin{aligned}(r+1) & \sum_{\alpha_1,\ldots,\alpha_{r+1}\in\mathbb{N}} \frac{1}{\omega_{\alpha_1}\cdots\omega_{\alpha_r}} \sum_{m_1,\ldots,m_{r+1}\in\mathbb{Z}_+} \mathrm{i} m_{r+1} \boldsymbol{p}_{m_1\boldsymbol{e}_{\alpha_1}+\cdots+m_{r+1}\boldsymbol{e}_{\alpha_{r+1}},\boldsymbol{e}_{\alpha_1}+\cdots+\boldsymbol{e}_{\alpha_{r+1}}} \\ & \qquad \mathrm{e}^{\mathrm{i}(m_1\omega_{\alpha_1}+\cdots+m_{r+1}\omega_{\alpha_{r+1}})t} \\ = & -\sum_{\alpha_1,\ldots,\alpha_r\in\mathbb{N}} \frac{1}{\omega_{\alpha_1}\cdots\omega_{\alpha_r}} \sum_{m_1,\ldots,m_r\in\mathbb{Z}_+} \boldsymbol{p}'_{m_1\boldsymbol{e}_{\alpha_1}+\cdots+m_r\boldsymbol{e}_{\alpha_r},\boldsymbol{e}_{\alpha_1}+\cdots+\boldsymbol{e}_{\alpha_r}} \mathrm{e}^{\mathrm{i}(m_1\omega_{\alpha_1}+\cdots+m_r\omega_{\alpha_r})t} \\ & + \sum_{\alpha_1,\ldots\alpha_1\in\mathbb{N}} \frac{1}{\omega_{\alpha_1}\cdots\omega_{\alpha_r}} \sum_{m_1,\ldots,m_r\in\mathbb{Z}_+} \sum_{k=1}^{r} \frac{1}{k!} \sum_{\boldsymbol{n}\in\mathbb{I}_{k,r}} \sum_{\boldsymbol{j}_1+\cdots+\boldsymbol{j}_k=\boldsymbol{e}_{\alpha_1}+\cdots+\boldsymbol{e}_{\alpha_r}} \\ & \sum_{\boldsymbol{\rho}_1+\cdots+\boldsymbol{\rho}_k=m_1\boldsymbol{e}_{\alpha_1}+\cdots+m_r\boldsymbol{e}_{\alpha_r}} \boldsymbol{f}_k(\boldsymbol{p}_0,\boldsymbol{p}_{\boldsymbol{\rho}_1,\boldsymbol{j}_1},\ldots,\boldsymbol{p}_{\boldsymbol{\rho}_k,\boldsymbol{j}_k}) \mathrm{e}^{\mathrm{i}(m_1\omega_{\alpha_1}+\cdots+m_r\omega_{\alpha_r})t}.\end{aligned}$$

therefore, for every $\alpha_1,\ldots,\alpha_r \in \mathbb{N}$,

$$\begin{aligned}(r+1) & \sum_{\alpha_{r+1}\in\mathbb{N}} \sum_{m_1,\ldots,m_{r+1}\in\mathbb{Z}_+} \mathrm{i} m_{r+1} \boldsymbol{p}_{m_1\boldsymbol{e}_{\alpha_1}+\cdots+m_{r+1}\boldsymbol{e}_{\alpha_{r+1}},\boldsymbol{e}_{\alpha_1}+\cdots+\boldsymbol{e}_{\alpha_{r+1}}} \\ & \qquad \mathrm{e}^{\mathrm{i}(m_1\omega_{\alpha_1}+\cdots+m_{r+1}\omega_{\alpha_{r+1}})t} \\ = & -\sum_{m_1,\ldots,m_r\in\mathbb{Z}_+} \boldsymbol{p}'_{m_1\boldsymbol{e}_{\alpha_1}+\cdots+m_r\boldsymbol{e}_{\alpha_r},\boldsymbol{e}_{\alpha_1}+\cdots+\boldsymbol{e}_{\alpha_r}} \mathrm{e}^{\mathrm{i}(m_1\omega_{\alpha_1}+\cdots+m_r\omega_{\alpha_r})t} \\ & + \sum_{m_1,\ldots,m_r\in\mathbb{Z}_+} \sum_{k=1}^{r} \frac{1}{k!} \sum_{\boldsymbol{j}_1+\cdots+\boldsymbol{j}_k=\boldsymbol{e}_{\alpha_1}+\cdots+\boldsymbol{e}_{\alpha_r}} \\ & \sum_{\boldsymbol{\rho}_1+\cdots+\boldsymbol{\rho}_k=m_1\boldsymbol{e}_{\alpha_1}+\cdots+m_r\boldsymbol{e}_{\alpha_r}} \boldsymbol{f}_k(\boldsymbol{p}_0,\boldsymbol{p}_{\boldsymbol{\rho}_1,\boldsymbol{j}_1},\ldots,\boldsymbol{p}_{\boldsymbol{\rho}_k,\boldsymbol{j}_k}) \mathrm{e}^{\mathrm{i}(m_1\omega_{\alpha_1}+\cdots+m_r\omega_{\alpha_r})t}.\end{aligned} \tag{4.6}$$

As before, we commence on the right-hand side of (4.6), identifying all frequencies that result in nonzero terms, next matching them on the left-hand side. This requires separate treatment for different partitions of the natural number r. Thus, for $r = 3$ we need to consider separately the cases $\alpha_1 = \alpha_2 = \alpha_3$, $\alpha_1 = \alpha_2 \neq \alpha_3$ and $\{\alpha_1, \alpha_2, \alpha_3\}$ distinct. In the first instance, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha \geq 1$, say, we obtain for every $m_1, m_2, m_3 \in \mathbb{Z}_+$,

$m_1+m_2+m_3 \geq 1$, the oscillator $\mathrm{e}^{\mathrm{i}(m_1+m_2+m_3)\omega_\alpha t}$, multiplying the term

$$\begin{aligned}
&-\boldsymbol{p}'_{(m_1+m_2+m_3)\boldsymbol{e}_\alpha,3\boldsymbol{e}_\alpha} + \boldsymbol{f}_1(\boldsymbol{p}_0, \boldsymbol{p}_{(m_1+m_2+m_3)\boldsymbol{e}_\alpha,3\boldsymbol{e}_\alpha}) \\
&+ \tfrac{1}{2} \sum_{\boldsymbol{j}_1+\boldsymbol{j}_2=3\boldsymbol{e}_\alpha} \sum_{\boldsymbol{\rho}_1+\boldsymbol{\rho}_2=(m_1+m_2+m_3)\boldsymbol{e}_\alpha} \boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{\rho}_1,\boldsymbol{j}_1}, \boldsymbol{p}_{\boldsymbol{\rho}_2,\boldsymbol{j}_2}) \\
&+ \tfrac{1}{6} \sum_{\boldsymbol{j}_1+\boldsymbol{j}_2+\boldsymbol{j}_3=3\boldsymbol{e}_\alpha} \sum_{\boldsymbol{\rho}_1+\boldsymbol{\rho}_2+\boldsymbol{\rho}_3=(m_1+m_2+m_3)\boldsymbol{e}_\alpha} \boldsymbol{f}_3(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{\rho}_1,\boldsymbol{j}_1}, \boldsymbol{p}_{\boldsymbol{\rho}_2,\boldsymbol{j}_2}, \boldsymbol{p}_{\boldsymbol{\rho}_3,\boldsymbol{j}_3}).
\end{aligned}$$

The only nonzero coefficients in the above expression are $\boldsymbol{p}_{\boldsymbol{0},k\boldsymbol{e}_\alpha}$ and $\boldsymbol{p}_{\boldsymbol{e}_\alpha,k\boldsymbol{e}_\alpha}$ for $k=1,2,3$ and $\boldsymbol{p}_{2\boldsymbol{e}_\alpha,3\boldsymbol{e}_\alpha}$.

There is a number of possibilities. Firstly, $(m_1,m_2,m_3)=(1,0,0)$ yields $\mathrm{e}^{\mathrm{i}\omega_\alpha t}$ – given that $(m_1,m_2,m_3)=(0,1,0)$ and $(m_1,m_2,m_3)=(0,0,1)$ yield the same outcome, we need to scale the outcome of our computation by a factor of three. After long algebra, we have

$$\begin{aligned}
&-\boldsymbol{p}'_{\boldsymbol{e}_\alpha,3\boldsymbol{e}_\alpha} + \boldsymbol{f}_1(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha,3\boldsymbol{e}_\alpha}) + \tfrac{1}{2}[\boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha,2\boldsymbol{e}_\alpha}) + \boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0},2\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha}) \\
&\qquad + \boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{0},2\boldsymbol{e}_\alpha}) + \boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha,2\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha})] \\
&\quad + \tfrac{1}{6}[\boldsymbol{f}_3(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha}) + \boldsymbol{f}_3(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha}) \\
&\qquad + \boldsymbol{f}_3(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha})] \\
&= -\boldsymbol{p}'_{\boldsymbol{e}_\alpha,3\boldsymbol{e}_\alpha} + \boldsymbol{f}_1(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha,3\boldsymbol{e}_\alpha}) + [\boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha,2\boldsymbol{e}_\alpha}) + \boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0},2\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha})] \\
&\quad + \tfrac{1}{2}\boldsymbol{f}_3(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha}).
\end{aligned}$$

Over to the left-hand side of (4.6). The computation becomes easier: we must have $m_4 \in \mathbb{N}$ and the only choice consistent with the oscillator $\mathrm{e}^{\mathrm{i}\omega_\alpha t}$ is $(m_1,m_2,m_3,m_4)=(0,0,0,1)$ – the outcome is the term $4\mathrm{i}\boldsymbol{p}_{\boldsymbol{e}_\alpha,4\boldsymbol{e}_\alpha}$. Bearing in mind that the right-hand side needs be scaled by 3, we thus obtain

$$\begin{aligned}
\frac{4\mathrm{i}}{3}\boldsymbol{p}_{\boldsymbol{e}_\alpha,4\boldsymbol{e}_\alpha} &= -\boldsymbol{p}'_{\boldsymbol{e}_\alpha,3\boldsymbol{e}_\alpha} + \boldsymbol{f}_1(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{e}_\alpha,3\boldsymbol{e}_\alpha}) + [\boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha,2\boldsymbol{e}_\alpha}) \\
&\quad + \boldsymbol{f}_2(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0},2\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha})] + \tfrac{1}{2}\boldsymbol{f}_3(\boldsymbol{p}_0, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{0},\boldsymbol{e}_\alpha}, \boldsymbol{p}_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha}).
\end{aligned}$$

Second possibility on the right is the oscillator $\mathrm{e}^{2\mathrm{i}\omega_\alpha t}$, for which we have two options: either $(m_1,m_2,m_3)=(2,0,0)$ or $(m_1,m_2,m_3)=(1,1,0)$ – each has a symmetry group of dimension 3 (e.g., the second option stands for the choices $(1,1,0),(1,0,1),(0,1,1)$), hence the outcome need be scaled by 3. Finally, the oscillator $\mathrm{e}^{3\mathrm{i}\omega_\alpha t}$ corresponds to $(m_1,m_2,m_3) \in \{(3,0,0),(2,1,0),(1,1,1)\}$, of

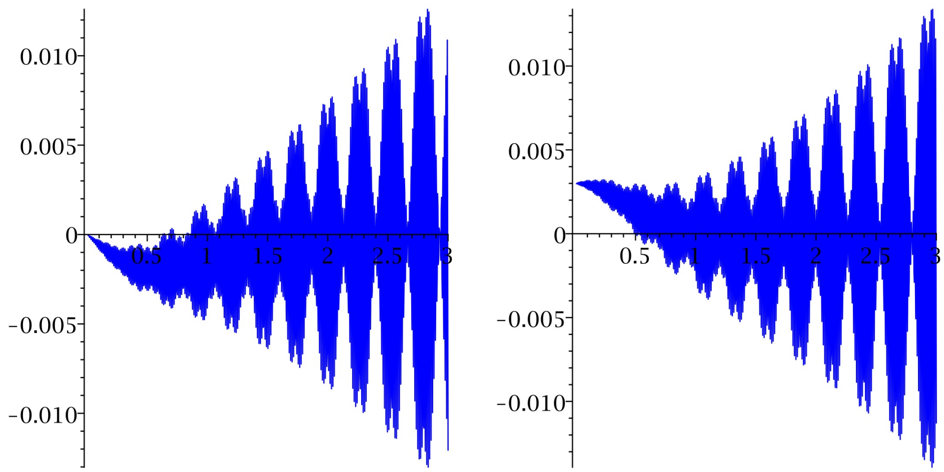

Figure 4.1: The real and imaginary parts of the error, as computed by MAPLE's `rkf45` routine, applied to the ODE (4.7).

symmetry-group dimensions 3, 6 and 1 respectively. The computation of the corresponding terms on the right and the left presents no conceptual difficulties, and neither do the cases of $\alpha_1 = \alpha_2 \neq \alpha_3$ (with symmetry group of dimension 3) and of $\alpha_1, \alpha_2, \alpha_3$ being all distinct, although all this calls for a great deal of very careful algebra.

4.3 A numerical experiment

To illustrate our expansion, we present a simple concrete example, the two-frequency nonlinear ODE

$$y' = \mathrm{i}y^2 + t\mathrm{e}^{\mathrm{i}\omega_1 t} + t^2\mathrm{e}^{\mathrm{i}\omega_2 t}, \quad t \geq 0, \qquad y(0) = 1. \tag{4.7}$$

In all our experiments we have taken $\omega_1 = 500$ and $\omega_2 = 500\pi$.

In Fig. 4.1 we have displayed the real and imaginary parts of the error committed by `rkf45` when applied to (4.7). Evidently, high oscillation plays havoc with the error control mechanism and the solution falls way short of reasonable expectations. This is true even when the routine is applied with a relative error bound of 10^{-16}, at the very limit of IEEE computer arithmetic.

As an alternative, we compute the first few terms of the asymptotic expansion (4.2). As before, we denote by e_R the error committed when (4.2) is truncated at $r = R$.

It follows from (4.4) that

$$p_0(t) = \frac{1}{1 - \mathrm{i}t}.$$

In the top row of Fig. 4.2 we have displayed the error committed just by taking the non-oscillatory solution p_0 as an approximation to y – it can be seen that it is already better than `rkf45`, no matter how small the tolerances imposed on the latter.

For $r = 1$ we have (singling for attention only nonzero coefficients)

$$p_{\boldsymbol{e}_1,\boldsymbol{e}_1}(t) = -\mathrm{i}t, \qquad p_{\boldsymbol{e}_2,\boldsymbol{e}_2} = -\mathrm{i}t^2$$

– note that, since $p'_{\boldsymbol{0},\boldsymbol{e}_1} = 2\mathrm{i}p_0 p_{\boldsymbol{0},\boldsymbol{e}_1}$, $p_{\boldsymbol{0},\boldsymbol{e}_1}(0) = -p_{\boldsymbol{e}_1,\boldsymbol{e}_1} = 0$, we have $p_{\boldsymbol{0},\boldsymbol{e}_1} \equiv 0$. For an identical reason, also $p_{\boldsymbol{0},\boldsymbol{e}_2} \equiv 0$. This gives us enough information to truncate the expansion for $r \le 1$: the error e_1 is displayed in the middle row of Fig. 4.2. It is evident that we already obtain a high-quality approximation, with precision of about six significant digits.

Next, to $r = 2$. We now have

$$\begin{aligned} p_{\boldsymbol{e}_1,2\boldsymbol{e}_1} &= \mathrm{i}(p'_{\boldsymbol{e}_1,\boldsymbol{e}_1} - 2\mathrm{i}p_0 p_{\boldsymbol{e}_1,\boldsymbol{e}_1}) = \frac{1 - 3\mathrm{i}t}{1 - \mathrm{i}t}, \\ p_{\boldsymbol{e}_2,2\boldsymbol{e}_2} &= \mathrm{i}(p'_{\boldsymbol{e}_2,\boldsymbol{e}_2} - 2\mathrm{i}p_0 p_{\boldsymbol{e}_2,\boldsymbol{e}_2}) = \frac{2t - 4\mathrm{i}t^2}{1 - \mathrm{i}t} \end{aligned}$$

and

$$\begin{aligned} p'_{\boldsymbol{0},\boldsymbol{e}_1+\boldsymbol{e}_2} &= 2\mathrm{i}p_0 p_{\boldsymbol{0},\boldsymbol{e}_1+\boldsymbol{e}_2} + \mathrm{i}p_{\boldsymbol{0},\boldsymbol{e}_1} p_{\boldsymbol{0},\boldsymbol{e}_2}, \quad t \ge 0, \qquad p_{\boldsymbol{0},\boldsymbol{e}_1+\boldsymbol{e}_2}(0) = 0, \\ p'_{\boldsymbol{0},2\boldsymbol{e}_1} &= 2\mathrm{i}p_0 p_{\boldsymbol{0},2\boldsymbol{e}_1} + \mathrm{i}p^2_{\boldsymbol{0},\boldsymbol{e}_1}, \quad t \ge 0, \qquad p_{\boldsymbol{0},2\boldsymbol{e}_1}(0) = -p_{\boldsymbol{e}_1,2\boldsymbol{e}_1}(0) = -1, \\ p'_{\boldsymbol{0},2\boldsymbol{e}_2} &= 2\mathrm{i}p_0 p_{\boldsymbol{0},2\boldsymbol{e}_2} + \mathrm{i}p^2_{\boldsymbol{0},\boldsymbol{e}_2}, \quad t \ge 0, \qquad p_{\boldsymbol{0},2\boldsymbol{e}_2}(0) = -p_{\boldsymbol{e}_2,2\boldsymbol{e}_2}(0) = 0, \end{aligned}$$

with the solutions

$$p_{\boldsymbol{0},\boldsymbol{e}_1+\boldsymbol{e}_2}, p_{\boldsymbol{0},2\boldsymbol{e}_2} \equiv 0, \qquad p_{\boldsymbol{0},2\boldsymbol{e}_1}(t) = -\frac{1}{(1 - \mathrm{i}t)^2}.$$

These are all the nonzero terms for $r = 2$ and, using them, the asymptotic expansion (as can be observed from the bottom row of Fig. 4.2) produces an error of $\approx 10^{-8}$. This simple example, which has been worked out explicitly, demonstrates the power of our approach.

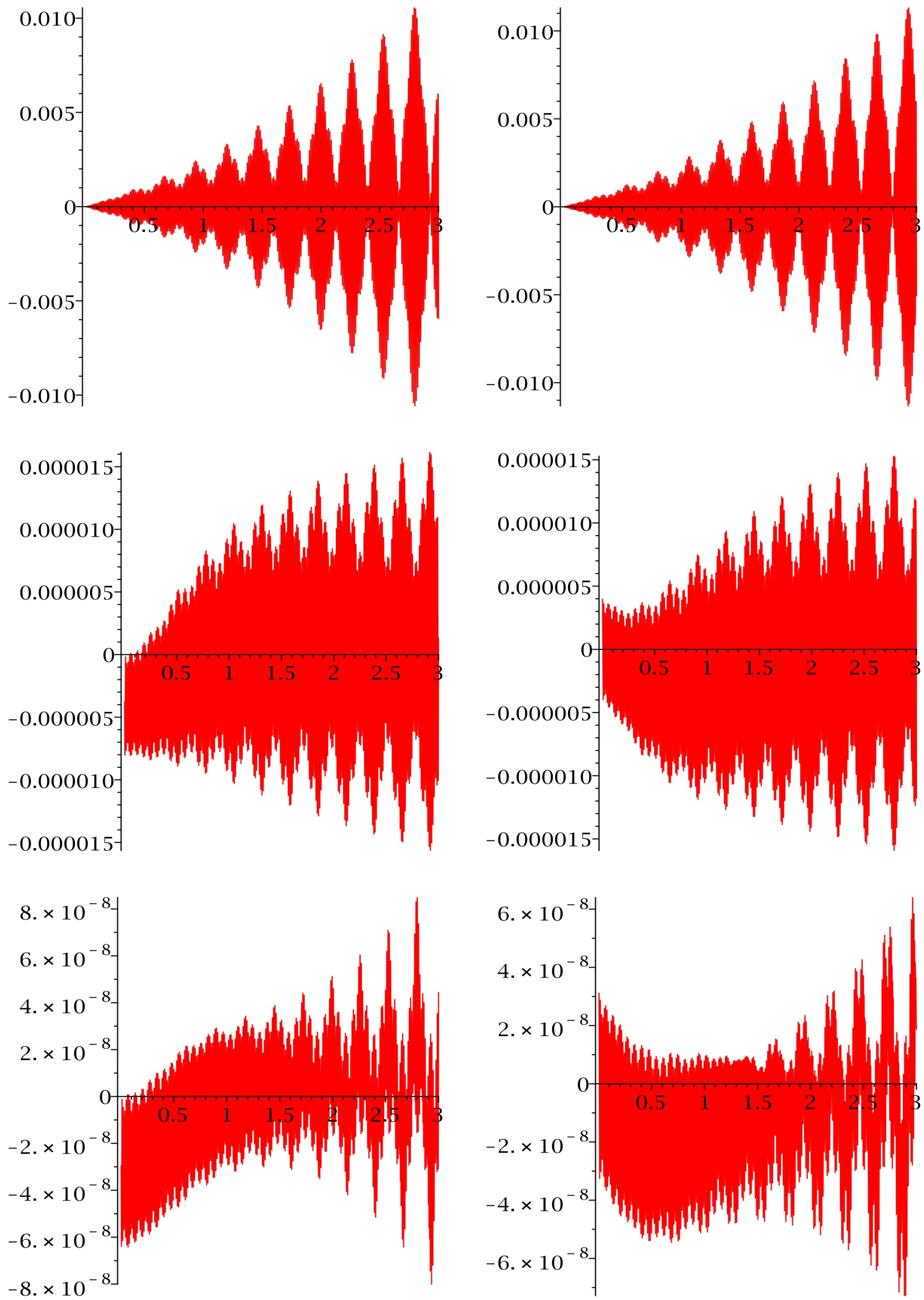

Figure 4.2: Real (on the left) and imaginary parts of the error committed by the asymptotic method. The top row displays e_0, the middle row e_1 and the bottom one e_2.

5 Conclusions

In this paper we have developed a general approach for asymptotic expansions for ODEs forced by highly oscillatory terms with multiple, possibly non-commensurate frequencies. This approach has the twin goals. Firstly, by representing the exact solution of ODEs

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}) + \sum_{m=1}^{\infty} \boldsymbol{a}_m(t)\mathrm{e}^{\mathrm{i}\omega_m t}, \quad t \geq 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0$$

where $\sup \omega_m$ is large as asymptotic series, we render it is a form suitable for analysis. Secondly, the very same asymptotic expansion, once it is truncated, can be used as an exceedingly effective means of numerical simulation, requiring only the solution of non-oscillatory ODEs and simple recursions. The efficacy of such asymptotic-numerical solvers has been illustrated with a number of examples.

The work required to obtain explicitly advanced expansion terms becomes progressively more demanding combinatorially. However, inasmuch as numerical computations are concerned, just a small number of expansion terms suffices in obtaining high-quality solvers once the frequencies ω_m become large.

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