Asymptotic solvers for second-order differential equation systems with multiple frequencies

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Abstract

In this paper, an asymptotic expansion is constructed to solve second-order differential equation systems with highly oscillatory forcing terms involving multiple frequencies. An asymptotic expansion is derived in inverse of powers of the oscillatory parameter and its truncation results in a very effective method of dicretizing the differential equation system in question. Numerical experiments illustrate the effectiveness of the asymptotic method in contrast to the standard Runge–Kutta method.

1 Introduction

The behaviour of signals comprising several non-commensurate frequencies is very important in the design and analysis of electronic circuits. Nonlinearities in circuits can result in such signals giving rise to distortion and resulting in degradation of performance. Hence, the accurate and efficient simulation of circuit behaviour in the presence of such signals is essential. It is to address this issue that the current paper is directed. As an example of nonlinear circuits, the Van der Pol oscillator shall be considered. It has numerous applications in science and engineering, for example, from describing the action potentials of biological neurons [5, 9] to the modelling of resonant tunneling diode circuits [12]. A coupled Van der Pol–Duffing system shall also be considered. Such coupled systems have applications in secure communications, [6, 10]. In addition, these coupled systems can be realised using analog circuitry [1].

Consider a second-order differential equation system of the form

$$\boldsymbol{y}''(t) + \boldsymbol{f}(\boldsymbol{y}(t))\boldsymbol{y}'(t) + \boldsymbol{g}(\boldsymbol{y}(t)) = \boldsymbol{F}_\omega(t), \quad t \geq 0, \tag{1.1}$$

where $\boldsymbol{f} : \mathbb{C}^d \to \mathbb{C}^d$, $\boldsymbol{g} : \mathbb{C}^d \to \mathbb{C}^d$ are two analytical functions,

$$\boldsymbol{f}(\boldsymbol{y}) = \begin{bmatrix} f_{11}(\boldsymbol{y}) & f_{12}(\boldsymbol{y}) & \cdots & f_{1d}(\boldsymbol{y}) \\ f_{21}(\boldsymbol{y}) & f_{22}(\boldsymbol{y}) & \cdots & f_{2d}(\boldsymbol{y}) \\ \vdots & & & \\ f_{d1}(\boldsymbol{y}) & f_{d2}(\boldsymbol{y}) & \cdots & f_{dd}(\boldsymbol{y}) \end{bmatrix}_{d\times d},$$

$$\boldsymbol{g} = \begin{bmatrix} g_1(\boldsymbol{y}) \\ g_2(\boldsymbol{y}) \\ \vdots \\ g_d(\boldsymbol{y}) \end{bmatrix}_{d\times 1},$$

where every entry $f_{jv}(\boldsymbol{y}) : \mathbb{C}^d \to \mathbb{R}$ and $g_j(\boldsymbol{y}) : \mathbb{C}^d \to \mathbb{R}$ is an analytic scalar function for $j, v = 1, 2, \cdots, d$. The initial conditions are $\boldsymbol{y}(0) = \boldsymbol{y}_0 \in \mathbb{C}^d$ and $\boldsymbol{y}'(0) = \boldsymbol{y}'_0 \in \mathbb{C}^d$, and the forcing term $\boldsymbol{F}_\omega(t)$ is

$$\boldsymbol{F}_\omega(t) = \sum_{m=1}^{M} \boldsymbol{a}_m(t) e^{i\omega_m t},$$

in which $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_M \in \mathbb{R}_+ \to \mathbb{C}^d$ are analytic functions. Note that we have assumed that there is a finite set of frequencies $\omega_1, \cdots, \omega_M \in \mathbb{R} \setminus \{0\}$ in the forcing term. At least some of these frequencies are large which results in a highly oscillatory solution and one which is very expensive to obtain with classical discretization methods. Furthermore, assume that the functions $\boldsymbol{f}(\boldsymbol{y}(t))$ and $\boldsymbol{g}(\boldsymbol{y}(t))$ are analytic to ensure the existence and uniqueness of the solution $\boldsymbol{y}(t)$.

When $d = 1$, $\omega_{2m-1} = m\omega$, $\omega_{2m} = -m\omega$, $m = 0, 1, \cdots, \lfloor \frac{M}{2} \rfloor$, $\omega \gg 1$, the above multiple frequency case reduces to a single frequency case

$$y''(t) + f(y(t))y'(t) + g(y(t)) = \sum_{k=-\infty}^{\infty} b_k(t)e^{ik\omega t}, \quad t \geq 0,$$

which has been already analysed in [3].

2 Construction of the asymptotic expansion

We start with a set $\mathcal{U}_0 = \{1, 2, \cdots, M\}$ and $\omega_j = \kappa_j \omega$, $j = 1, 2, \cdots, M$, where ω serves as the oscillatory parameter. Therefore, the original equation (1.1) may be written in the form

$$\begin{aligned} \boldsymbol{y}''(t) + \boldsymbol{f}(\boldsymbol{y}(t))\boldsymbol{y}'(t) + \boldsymbol{g}(\boldsymbol{y}(t)) &= \sum_{m=1}^{M} \boldsymbol{a}_m(t)e^{i\kappa_m\omega t} \qquad (2.1) \\ &= \sum_{m\in\mathcal{U}_0} \boldsymbol{a}_m(t)e^{i\kappa_m\omega t}, \quad t \geq 0. \end{aligned}$$

Our basic ansatz is that the solution $\boldsymbol{y}(t)$ admits an expansion in inverse powers of the oscillatory parameter ω,

$$\boldsymbol{y}(t) \sim \sum_{r=0}^{\infty} \frac{1}{\omega^r} \sum_{m\in\mathcal{U}_r} \boldsymbol{p}_{r,m}(t)e^{i\sigma_m\omega t},$$

The sets $\mathcal{U}_r$, the scalars σ_m and the functions $\boldsymbol{p}_{r,m}(t)$ will be described in the sequel. The important point to note at this stage is that the functions $\boldsymbol{p}_{r,m}(t)$ are independent of ω and can be derived recursively: $\boldsymbol{p}_{r,0}(t)$ by solving a non-oscillatory ODE and $\boldsymbol{p}_{r,m}(t)$ for $m \neq 0$ by recursion.

It is very important, however, to impose $\boldsymbol{p}_{0,m} \equiv \boldsymbol{0}$ and $\boldsymbol{p}_{1,m} \equiv \boldsymbol{0}$ for $m \neq 0$ so that differentiation does not result in the presence of a positive power of ω in the resultant equations for $\boldsymbol{p}_{r,m}(t)$ involved in the asymptotic method.

Therefore, the proposed solution $\boldsymbol{y}(t)$ in inverse powers of the oscillatory parameter ω is

$$\boldsymbol{y}(t) \sim \boldsymbol{p}_{0,0}(t) + \frac{1}{\omega}\boldsymbol{p}_{1,0}(t) + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m\in\mathcal{U}_r} \boldsymbol{p}_{r,m}(t)e^{i\sigma_m\omega t} \qquad (2.2)$$

As just stated it will be assumed in the ansatz that $\boldsymbol{p}_{1,0}$ is the only non-zero $\boldsymbol{p}_{1,m}$ in $\mathcal{U}_1$. So $\mathcal{U}_1 = \{0\}$.

Following the approach in [4] (cf. also [11]), the expression (2.2) for $\boldsymbol{y}(t)$ is substituted into the second-order differential equation (2.1). The first-order derivative of $\boldsymbol{y}(t)$ is

$$\begin{aligned}
\boldsymbol{y}' &\sim \boldsymbol{p}'_{0,0} + \frac{1}{\omega}\boldsymbol{p}'_{1,0} + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m\in\mathcal{U}_r} \left(\boldsymbol{p}'_{r,m} e^{i\sigma_m\omega t} + i\sigma_m\omega\boldsymbol{p}_{r,m} e^{i\sigma_m\omega t}\right) \\
&= \boldsymbol{p}'_{0,0} + \frac{1}{\omega}\boldsymbol{p}'_{1,0} + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m\in\mathcal{U}_r} \boldsymbol{p}'_{r,m} e^{i\sigma_m\omega t} + \sum_{r=2}^{\infty} \frac{1}{\omega^{r-1}} \sum_{m\in\mathcal{U}_r} i\sigma_m \boldsymbol{p}_{r,m} e^{i\sigma_m\omega t} \\
&= \boldsymbol{p}'_{0,0} + \frac{1}{\omega}\boldsymbol{p}'_{1,0} + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m\in\mathcal{U}_r} \boldsymbol{p}'_{r,m} e^{i\sigma_m\omega t} + \sum_{r=1}^{\infty} \frac{1}{\omega^{r}} \sum_{m\in\mathcal{U}_{r+1}} i\sigma_m \boldsymbol{p}_{r+1,m} e^{i\sigma_m\omega t}, \\
&= \boldsymbol{p}'_{0,0} + \frac{1}{\omega}\left[\boldsymbol{p}'_{1,0} + \sum_{m\in\mathcal{U}_2} i\sigma_m \boldsymbol{p}_{2,m} e^{i\sigma_m\omega t}\right] \\
&\qquad + \sum_{r=2}^{\infty} \frac{1}{\omega^r}\left[\sum_{m\in\mathcal{U}_r} \boldsymbol{p}'_{r,m} e^{i\sigma_m\omega t} + \sum_{m\in\mathcal{U}_{r+1}} i\sigma_m \boldsymbol{p}_{r+1,m} e^{i\sigma_m\omega t}\right]
\end{aligned}$$

Likewise, the second-order derivative of $\boldsymbol{y}(t)$ is

$$\begin{aligned}
\boldsymbol{y}''(t) &\sim \boldsymbol{p}''_{0,0} + \frac{1}{\omega}\left[\boldsymbol{p}''_{1,0} + \sum_{m\in\mathcal{U}_2} i\sigma_m \left(\boldsymbol{p}'_{2,m} + (i\sigma_m\omega\boldsymbol{p}_{2,m})\right) e^{i\sigma_m\omega t}\right] \\
&\qquad + \sum_{r=2}^{\infty} \frac{1}{\omega^r}\left[\sum_{m\in\mathcal{U}_r} \left(\boldsymbol{p}''_{r,m} + i\sigma_m\omega\boldsymbol{p}'_{r,m}\right) e^{i\sigma_m\omega t}\right. \\
&\qquad \left. + \sum_{m\in\mathcal{U}_{r+1}} i\sigma_m \left(\boldsymbol{p}'_{r+1,m} + i\sigma_m\omega\boldsymbol{p}_{r+1,m}\right) e^{i\sigma_m\omega t}\right], \\
&= \boldsymbol{p}''_{0,0} + \sum_{m\in\mathcal{U}_2} (i\sigma_m)^2 \boldsymbol{p}_{2,m} e^{i\sigma_m\omega t} + \frac{1}{\omega}\left[\boldsymbol{p}''_{1,0} + 2\sum_{m\in\mathcal{U}_2} (i\sigma_m)\boldsymbol{p}'_{2,m} e^{i\sigma_m\omega t}\right. \\
&\qquad \left. + \sum_{m\in\mathcal{U}_3} (i\sigma_m)^2 \boldsymbol{p}_{3,m} e^{i\sigma_m\omega t}\right] + \sum_{r=2}^{\infty} \frac{1}{\omega^r}\left[\sum_{m\in\mathcal{U}_r} \boldsymbol{p}''_{r,m} e^{i\sigma_m\omega t}\right.
\end{aligned}$$

$$
+ 2 \sum_{m \in \mathcal{U}_{r+1}} (i\sigma_m) \boldsymbol{p}'_{r+1,m} e^{i\sigma_m \omega t} + \sum_{m \in \mathcal{U}_{r+2}} (i\sigma_m)^2 \boldsymbol{p}_{r+2,m} e^{i\sigma_m \omega t} \Bigg]
$$

The function $\boldsymbol{f}(\boldsymbol{y}(t))_{d\times d}$ is analytic and its Taylor expansion about $\boldsymbol{p}_{0,0}(t)$ may be determined. $f_{jv}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{\eta}_1, \cdots, \boldsymbol{\eta}_n] : \mathbb{C}^d \times \overbrace{\mathbb{C}^d \times \cdots \times \mathbb{C}^d}^{n \text{ times}} \to \mathbb{C}$ is the nth derivative operator which is linear in each of $\boldsymbol{\eta}_k$s such that

$$
f_{jv}(\boldsymbol{y}_0 + t\boldsymbol{\epsilon}) = f_{jv}(\boldsymbol{y}_0) + \sum_{n=1}^{\infty} \frac{t^n}{n!} f_{jv}^{(n)}(\boldsymbol{y}_0)[\boldsymbol{\epsilon}, \cdots, \boldsymbol{\epsilon}]
$$

for sufficiently small $|t| > 0$. Hence,

$$
\begin{aligned}
f_{jv}(\boldsymbol{y}) &= f_{jv}\left(\boldsymbol{p}_{0,0}(t) + \frac{1}{\omega}\boldsymbol{p}_{1,0}(t) + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m \in \mathcal{U}_r} \boldsymbol{p}_{r,m}(t) e^{i\sigma_m \omega t}\right) \\
&= f_{jv}(\boldsymbol{p}_{0,0}) + \sum_{n=1}^{\infty} \frac{1}{n!} f_{jv}^{(n)}(\boldsymbol{p}_{0,0}) \left[\sum_{\ell_1=1}^{\infty} \frac{1}{\omega^{\ell_1}} \sum_{k_1 \in \mathcal{U}_{\ell_1}} \boldsymbol{p}_{\ell_1,k_1} e^{i\sigma_{k_1}\omega t}, \cdots, \right. \\
&\qquad \left. \sum_{\ell_n=1}^{\infty} \frac{1}{\omega^{\ell_n}} \sum_{k_n \in \mathcal{U}_{\ell_n}} \boldsymbol{p}_{\ell_n,k_n} e^{i\sigma_{k_n}\omega t}\right] \\
&= f_{jv}(\boldsymbol{p}_{0,0}) + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\ell_1=1}^{\infty} \cdots \sum_{\ell_n=1}^{\infty} \frac{1}{\omega^{\ell_1 + \cdots + \ell_n}} \sum_{k_1 \in \mathcal{U}_{\ell_1}} \cdots \sum_{k_n \in \mathcal{U}_{\ell_n}} \\
&\quad f_{jv}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n})\omega t} \\
&= f_{jv}(\boldsymbol{p}_{0,0}) + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{r=n}^{\infty} \frac{1}{\omega^r} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r}^o} \sum_{k_1 \in \mathcal{U}_{\ell_1}} \cdots \sum_{k_n \in \mathcal{U}_{\ell_n}} \\
&\qquad f_{jv}^{(n)}(p_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n})\omega t} \\
&= f_{jv}(\boldsymbol{p}_{0,0}) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r}^o} \sum_{k_1 \in \mathcal{U}_{\ell_1}} \cdots \sum_{k_n \in \mathcal{U}_{\ell_n}} \\
&\qquad f_{jv}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n})\omega t},
\end{aligned}
$$

where

$$
\mathbb{I}_{n,r}^o = \{\boldsymbol{\ell} = (\ell_1, \cdots, \ell_n)^T \in \mathbb{N}^n : \quad \sum_{j=1}^{n} \ell_j = r\}, \quad 1 \le n \le r.
$$

To avoid redundancy, set

$$\mathbb{I}_{n,r} = \{\boldsymbol{\ell} = (\ell_1, \cdots, \ell_n)^T \in \mathbb{N}^n : \quad \sum_{j=1}^{n} \ell_j = r, \quad \ell_1 \leq \ell_2 \leq \cdots \leq \ell_n\},$$

and the symbol $\theta_{\boldsymbol{\ell}}$ stands for the multiplicity of $\boldsymbol{\ell}$. This is the number of terms in $\mathbb{I}^o_{n,r}$ that can be brought to it by permutation. It follows that

$$f_{jv}(\boldsymbol{y}(t)) = f_{jv}(\boldsymbol{p}_{0,0}) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r}} \theta_{\boldsymbol{\ell}} \sum_{k_1 \in \mathcal{U}_{\ell_1}} \cdots \sum_{k_n \in \mathcal{U}_{\ell_n}} \\ f^{(n)}_{jv}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t}.$$

and the matrix function

$$\boldsymbol{f}(\boldsymbol{y}(t)) = \boldsymbol{f}(\boldsymbol{p}_{0,0}) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r}} \theta_{\boldsymbol{\ell}} \sum_{k_1 \in \mathcal{U}_{\ell_1}} \cdots \sum_{k_n \in \mathcal{U}_{\ell_n}} \\ \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t}.$$

Similar expansions can be applied to the vector function

$$\boldsymbol{g}(\boldsymbol{y}(t)) = \boldsymbol{g}(\boldsymbol{p}_{0,0}) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r}} \theta_{\boldsymbol{\ell}} \sum_{k_1 \in \mathcal{U}_{\ell_1}} \cdots \sum_{k_n \in \mathcal{U}_{\ell_n}} \\ \boldsymbol{g}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t}.$$

Incorporating these expansions into the differential equation (2.1), we have

$$\boldsymbol{p}''_{0,0} + \sum_{m\in\mathcal{U}_2} (i\sigma_m)^2 \boldsymbol{p}_{2,m} e^{i\sigma_m\omega t} + \frac{1}{\omega}\Bigg[\boldsymbol{p}''_{1,0} + 2\sum_{m\in\mathcal{U}_2} (i\sigma_m)\boldsymbol{p}'_{2,m} e^{i\sigma_m\omega t} \\ + \sum_{m\in\mathcal{U}_3} (i\sigma_m)^2 \boldsymbol{p}_{3,m} e^{i\sigma_m\omega t}\Bigg] + \sum_{r=2}^{\infty} \frac{1}{\omega^r}\Bigg[\sum_{m\in\mathcal{U}_r} \boldsymbol{p}''_{r,m} e^{i\sigma_m\omega t} \\ + 2\sum_{m\in\mathcal{U}_{r+1}} (i\sigma_m)\boldsymbol{p}'_{r+1,m} e^{i\sigma_m\omega t} + \sum_{m\in\mathcal{U}_{r+2}} (i\sigma_m)^2 \boldsymbol{p}_{r+2,m} e^{i\sigma_m\omega t}\Bigg] \tag{2.3}$$

$$
\begin{aligned}
&+\begin{bmatrix} f_{11}(\boldsymbol{y}) \; f_{12}(\boldsymbol{y}) \cdots \; f_{1d}(\boldsymbol{y}) \\ f_{21}(\boldsymbol{y}) \; f_{22}(\boldsymbol{y}) \cdots \; f_{2d}(\boldsymbol{y}) \\ \vdots \\ f_{d1}(\boldsymbol{y}) \; f_{d2}(\boldsymbol{y}) \cdots \; f_{dd}(\boldsymbol{y}) \end{bmatrix}_{d\times d} \\
&\qquad \times \left(\boldsymbol{p}'_{0,0} + \frac{1}{\omega}\left(\boldsymbol{p}'_{1,0} + \sum_{m\in\mathcal{U}_2} i\sigma_m \boldsymbol{p}_{2,m} e^{i\sigma_m\omega t} \right) \right. \\
&+ \left. \sum_{r=2}^{\infty} \frac{1}{\omega^r} \left(\sum_{m\in\mathcal{U}_r} \boldsymbol{p}'_{r,m} e^{i\sigma_m\omega t} + \sum_{m\in\mathcal{U}_{r+1}} i\sigma_m \boldsymbol{p}_{r+1,m} e^{i\sigma_m\omega t} \right) \right)_{d\times 1} \\
&+ \boldsymbol{g}(\boldsymbol{p}_{0,0}) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r}} \theta_{\boldsymbol{\ell}} \sum_{k_1\in\mathcal{U}_{\ell_1}} \cdots \sum_{k_n\in\mathcal{U}_{\ell_n}} \\
&\qquad \boldsymbol{g}_n(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n} \right] e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t} = \sum_{m\in\mathcal{U}_0} \boldsymbol{a}_m(t) e^{i\kappa_m\omega t}.
\end{aligned}
$$

3 Construction of the asymptotic expansion

In this section, we derive the constituent terms of the asymptotic expansion. Explicit expressions shall be given for the first r values in the expansion to aid in the understanding of the general expression for $r \geq 0$. The expression for $y(t)$ exhibits two distinct hierarchies of scales – *amplitudes* ω^{-r} for $r \geq 1$ and for each r, *frequencies* $e^{i\sigma_m\omega t}$. Construction involves first separating amplitudes and then separating frequencies. However, before deriving the expansion, the procedure for incorporation of the initial conditions shall be addressed.

3.1 The initial conditions

The initial conditions for the second-order differential equations are

$$\boldsymbol{y}(0) = \boldsymbol{p}_{0,0}(0) = \boldsymbol{y}_0, \quad \boldsymbol{y}'(0) = \boldsymbol{p}'_{0,0}(0) = \boldsymbol{y}'_0.$$

The *ansatz* must satisfy the same initial conditions,

$$\boldsymbol{p}_{0,0}(0) + \frac{1}{\omega}\boldsymbol{p}_{1,0}(0) + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m\in\mathcal{U}_r} \boldsymbol{p}_{r,m}(0) = \boldsymbol{y}_0;$$

$$
\begin{aligned}
&\boldsymbol{p}_{0,0}'(0) + \frac{1}{\omega}\left[\boldsymbol{p}_{1,0}'(0) + \sum_{m\in\mathcal{U}_2} i\sigma_m \boldsymbol{p}_{2,m}(0)\right] \\
&+\sum_{r=2}^{\infty}\frac{1}{\omega^r}\left[\sum_{m\in\mathcal{U}_r}\boldsymbol{p}_{r,m}'(0) + \sum_{m\in\mathcal{U}_{r+1}} i\sigma_m \boldsymbol{p}_{r+1,m}(0)\right] = \boldsymbol{y}_0'.
\end{aligned}
$$

To this end, we set

$$
\begin{aligned}
\boldsymbol{p}_0(0) &= \boldsymbol{y}_0, & \boldsymbol{p}_0'(0) &= \boldsymbol{y}_0; \\
\boldsymbol{p}_{1,0}(0) &= \mathbf{0}, & \boldsymbol{p}_{1,0}'(0) &= -\sum_{m\in\mathcal{U}_2} i\sigma_m \boldsymbol{p}_{2,m}(0); \\
\boldsymbol{p}_{2,0}(0) &= -\sum_{m\in\mathcal{U}_2\setminus\{0\}} \boldsymbol{p}_{2,m}(0), \boldsymbol{p}_{2,0}'(0) \\
&= -\sum_{m\in\mathcal{U}_2\setminus\{0\}} \boldsymbol{p}_{2,m}'(0) - \sum_{m\in\mathcal{U}_3} i\sigma_m \boldsymbol{p}_{3,m}(0); \\
&\vdots \\
\boldsymbol{p}_{\ell,0}(0) &= -\sum_{m\in\mathcal{U}_\ell\setminus\{0\}} \boldsymbol{p}_{\ell,m}(0), \boldsymbol{p}_{\ell,0}'(0) \\
&= -\sum_{m\in\mathcal{U}_\ell\setminus\{0\}} \boldsymbol{p}_{\ell,m}'(0) - \sum_{m\in\mathcal{U}_{\ell+1}} i\sigma_m \boldsymbol{p}_{\ell+1,m}(0); \\
&\vdots\,.
\end{aligned}
$$

3.2 The zeroth term $r = 0$

We extract the $O(1)$ terms from (2.3),

$$
\boldsymbol{p}_{0,0}'' + \sum_{m\in\mathcal{U}_2}(i\sigma_m)^2\boldsymbol{p}_{2,m}e^{i\sigma_m\omega t} + \boldsymbol{f}(\boldsymbol{p}_{0,0})\boldsymbol{p}_{0,0}' + \boldsymbol{g}(\boldsymbol{p}_{0,0}) = \sum_{m\in\mathcal{U}_0}\boldsymbol{a}_m(t)e^{i\kappa_m\omega t},
$$

where

$$
\boldsymbol{f}(\boldsymbol{p}_{0,0}) = \begin{bmatrix} f_{11}(\boldsymbol{p}_{0,0}) & f_{12}(\boldsymbol{p}_{0,0}) & \cdots & f_{1d}(\boldsymbol{p}_{0,0}) \\ f_{21}(\boldsymbol{p}_{0,0}) & f_{22}(\boldsymbol{p}_{0,0}) & \cdots & f_{2d}(\boldsymbol{p}_{0,0}) \\ \vdots & & & \\ f_{d1}(\boldsymbol{p}_{0,0}) & f_{d2}(\boldsymbol{p}_{0,0}) & \cdots & f_{dd}(\boldsymbol{p}_{0,0}) \end{bmatrix}_{d\times d},
$$

$$\boldsymbol{g}(\boldsymbol{p}_{0,0}) = \begin{bmatrix} g_1(\boldsymbol{p}_{0,0}) \\ g_2(\boldsymbol{p}_{0,0}) \\ \vdots \\ g_d(\boldsymbol{p}_{0,0}) \end{bmatrix}_{d\times 1}.$$

Equating the $e^{i\sigma_m\omega t}$ terms results in for $m = 0$

$$\begin{aligned} \boldsymbol{p}''_{0,0}(t) + \boldsymbol{f}(\boldsymbol{p}_{0,0})\boldsymbol{p}'_{0,0}(t) + \boldsymbol{g}(\boldsymbol{p}_{0,0}) = 0, \qquad t \geq 0, \\ \boldsymbol{p}_{0,0}(0) = \boldsymbol{y}_0, \qquad \boldsymbol{p}'_{0,0}(0) = \boldsymbol{y}'_0. \end{aligned}$$

By setting $\mathcal{U}_2 = \mathcal{U}_0 \cup \{0\}$,

$$\sigma_m = \kappa_m, \quad m = 1, \cdots, M; \quad \sigma_0 = \kappa_0 = 0, \quad m \in \mathcal{U}_2,$$

the following expression is obtained for $m \neq 0$

$$\boldsymbol{p}_{2,m}(t) = \frac{\boldsymbol{a}_m(t)}{(i\kappa_m)^2}, \quad m \neq 0.$$

3.3 The $r = 1$ terms

When $r = 1$, note that $n = 1$ and $\mathbb{I}^o_{1,1} = 1$ and $\theta_{\boldsymbol{\ell}} = 1$, Then

$$\begin{aligned} &\sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r}} \theta_{\boldsymbol{\ell}} \sum_{k_1\in\mathcal{U}_{\ell_1}} \cdots \sum_{k_n\in\mathcal{U}_{\ell_n}} f^{(n)}_{jv}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}] e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t} \\ &= \sum_{m\in\mathcal{U}_1} f^{(1)}_{jv}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,m}] e^{i\sigma_m\omega t}. \end{aligned}$$

Based on the assumption that, $\boldsymbol{p}_{1,m} \equiv 0$, $m \neq 0$ and that $\mathcal{U}_1 = \{0\}$, the above term is reduced to $f^{(1)}_{jv}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}]$.

We now consider the $O(\omega^{-1})$ terms

$$\begin{aligned} &\boldsymbol{p}''_{1,0} + 2\sum_{m\in\mathcal{U}_2} (i\sigma_m)\boldsymbol{p}'_{2,m} e^{i\sigma_m\omega t} + \sum_{m\in\mathcal{U}_3} (i\sigma_m)^2 \boldsymbol{p}_{3,m} e^{i\sigma_m\omega t} + \boldsymbol{f}(\boldsymbol{p}_{0,0})\boldsymbol{p}'_{1,0} \\ &+ \boldsymbol{f}(\boldsymbol{p}_{0,0}) \sum_{m\in\mathcal{U}_2} (i\sigma_m)\boldsymbol{p}_{2,m} e^{i\sigma_m\omega t} + \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}]\boldsymbol{p}'_{0,0} + \boldsymbol{g}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] = 0, \end{aligned}$$

where

$$\boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] = \begin{bmatrix} f_{11}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] & f_{12}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] & \cdots & f_{1d}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] \\ f_{21}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] & f_{22}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] & \cdots & f_{2d}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] \\ \vdots & & & \\ f_{d1}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] & f_{d2}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] & \cdots & f_{dd}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] \end{bmatrix}_{d\times d},$$

$$\boldsymbol{g}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] = \begin{bmatrix} g_1^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] \\ g_2^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] \\ \vdots \\ g_d^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] \end{bmatrix}_{d\times 1},$$

in which

$$\begin{aligned} f_{jv}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] &= \nabla(f_{jv}(\boldsymbol{p}_{0,0}))_{1\times d} \times [\boldsymbol{p}_{1,0}]_{d\times 1}, \\ g_j^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] &= \nabla(g_j(\boldsymbol{p}_{0,0}))_{1\times d} \times [\boldsymbol{p}_{1,0}]_{d\times 1}. \end{aligned}$$

If $m = 0$, the unperturbed equation and the associated initial conditions are obtained

$$\begin{aligned} &\boldsymbol{p}_{1,0}'' + \boldsymbol{f}(\boldsymbol{p}_{0,0})\boldsymbol{p}_{1,0}' + \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}]\boldsymbol{p}_{0,0}' + \boldsymbol{g}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] = 0 \\ &\boldsymbol{p}_{1,0}^{(1)}(0) = 0, \qquad\qquad \boldsymbol{p}_{1,0}^{(1)}(0) = -\sum_{m\in\mathcal{U}_2} (i\kappa_m)\boldsymbol{p}_{2,m}^{(1)}(0). \end{aligned}$$

If $m \neq 0$, we extract the terms with $e^{i\sigma_m \omega t}$. Set

$$\mathcal{U}_3 = \mathcal{U}_0 \cup \{0\} = \{0, 1, 2, \cdots, M\} = \mathcal{U}_2.$$

It follows that

$$\sigma_0 = 0, \qquad \sigma_m = \kappa_m, \qquad m \in\ \mathcal{U}_3 = \mathcal{U}_0 \cup \{0\}$$

and

$$\boldsymbol{p}_{3,m} = -\frac{1}{i\kappa_m}\left[2\boldsymbol{p}_{2,m}' + \boldsymbol{f}(\boldsymbol{p}_{0,0})\boldsymbol{p}_{2,m}\right], \quad m \neq 0.$$

3.4 The $r = 2$ terms

When $r = 2$,

$$
\begin{aligned}
&\sum_{m\in\mathcal{U}_2} \boldsymbol{p}''_{2,m} e^{i\sigma_m\omega t} + 2\sum_{m\in\mathcal{U}_3} (i\sigma_m)\boldsymbol{p}'_{3,m} e^{i\sigma_m\omega t} + \sum_{m\in\mathcal{U}_4} (i\sigma_m)^2 \boldsymbol{p}_{4,m} e^{i\sigma_m\omega t} \\
&+ \boldsymbol{f}(\boldsymbol{p}_{0,0}) \left(\sum_{m\in\mathcal{U}_2} \boldsymbol{p}'_{2,m} e^{i\sigma_m\omega t} + \sum_{m\in\mathcal{U}_3} i\sigma_m \boldsymbol{p}_{3,m} e^{i\sigma_m\omega t} \right) + \\
&+ \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}] \left(\boldsymbol{p}'_{1,0} + \sum_{m\in\mathcal{U}_2} i\sigma_m \boldsymbol{p}_{2,m} e^{i\sigma_m\omega t} \right) \\
&+ \sum_{n=1}^{2} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,2}} \theta_{\boldsymbol{\ell}} \sum_{k_1\in\mathcal{U}_{\ell_1}} \cdots \sum_{k_n\in\mathcal{U}_{\ell_n}} \boldsymbol{f}_n(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t} \boldsymbol{p}'_{0,0} \\
&+ \sum_{n=1}^{2} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,2}} \theta_{\boldsymbol{\ell}} \sum_{k_1\in\mathcal{U}_{\ell_1}} \cdots \sum_{k_n\in\mathcal{U}_{\ell_n}} \boldsymbol{g}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t} = 0.
\end{aligned}
$$

where

$$
\begin{aligned}
&\sum_{n=1}^{2} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,2}} \theta_{\boldsymbol{\ell}} \sum_{k_1\in\mathcal{U}_{\ell_1}} \cdots \sum_{k_n\in\mathcal{U}_{\ell_n}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}] e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t} \\
&= \sum_{m\in\mathcal{U}_2} \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] e^{i\kappa_m\omega t} \\
&\ + \frac{1}{2} \sum_{m_1\in\mathcal{U}_1} \sum_{m_2\in\mathcal{U}_1} \boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,m_1}, \boldsymbol{p}_{1,m_2}] e^{i(\kappa_{m_1}+\kappa_{m_2})\omega t} \\
&= \sum_{m\in\mathcal{U}_2} \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] e^{i\kappa_m\omega t} + \frac{1}{2}\boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}, \boldsymbol{p}_{1,0}],
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{n=1}^{2} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,2}} \theta_{\boldsymbol{\ell}} \sum_{k_1\in\mathcal{U}_{\ell_1}} \cdots \sum_{k_n\in\mathcal{U}_{\ell_n}} \boldsymbol{g}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}] e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t} \\
&= \sum_{m\in\mathcal{U}_2} \boldsymbol{g}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] e^{i\kappa_m\omega t} + \frac{1}{2}\boldsymbol{g}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}, \boldsymbol{p}_{1,0}],
\end{aligned}
$$

in which

$$\boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] = \begin{bmatrix} f_{11}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] & f_{12}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] & \cdots & f_{1d}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] \\ f_{21}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] & f_{22}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] & \cdots & f_{2d}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] \\ \vdots & & & \\ f_{d1}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] & f_{d2}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] & \cdots & f_{dd}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] \end{bmatrix}_{d\times d},$$

$$\begin{aligned} &\boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] \\ &= \begin{bmatrix} f_{11}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] & f_{12}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] & \cdots & f_{1d}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] \\ f_{21}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] & f_{22}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] & \cdots & f_{2d}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] \\ \vdots & & & \\ f_{d1}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] & f_{d2}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] & \cdots & f_{dd}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] \end{bmatrix}_{d\times d}, \end{aligned}$$

$$\boldsymbol{g}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] = \begin{bmatrix} g_1^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] \\ g_2^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] \\ \vdots \\ g_d^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] \end{bmatrix}_{d\times 1},$$

$$\boldsymbol{g}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] = \begin{bmatrix} g_1^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] \\ g_2^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] \\ \vdots \\ g_d^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] \end{bmatrix}_{d\times 1},$$

and

$$\begin{aligned} f_{jv}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] &= \nabla(f_{jv}(\boldsymbol{p}_{0,0}))_{1\times d} \times [\boldsymbol{p}_{2,m}]_{d\times 1}, \\ f_{jv}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] &= [\boldsymbol{p}_{1,0}]_{1\times d} \left(\frac{\partial^2 f_{jv}(\boldsymbol{p}_{0,0})}{\partial y_{m_1}\partial y_{m_2}}\right)_{d\times d} [\boldsymbol{p}_{1,0}]_{d\times 1}, \\ g_j^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}] &= \nabla(g_j(\boldsymbol{p}_{0,0}))_{1\times d} \times [\boldsymbol{p}_{2,m}]_{d\times 1}, \\ g_j^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] &= [\boldsymbol{p}_{1,0}]_{1\times d} \left(\frac{\partial^2 f_{jv}(\boldsymbol{p}_{0,0})}{\partial y_{m_1}\partial y_{m_2}}\right)_{d\times d} [\boldsymbol{p}_{1,0}]_{d\times 1} \end{aligned}$$

with $\boldsymbol{p}_{1,m}(t) \equiv 0$ for $m \neq 0$.

In the case $m = 0$,

$$\boldsymbol{p}_{2,0}'' + \boldsymbol{f}(\boldsymbol{p}_{0,0})\boldsymbol{p}_{2,0}' + \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}]\boldsymbol{p}_{1,0}' + \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,0}]\boldsymbol{p}_{0,0}'$$

$$+\frac{1}{2}\boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}]\boldsymbol{p}'_{0,0}+\boldsymbol{g}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,0}]+\frac{1}{2}\boldsymbol{g}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}]=0;$$

$$\boldsymbol{p}_{2,0}(0)=-\sum_{m\in\mathcal{U}_2\backslash\{0\}}\boldsymbol{p}_{2,m}(0);$$
$$\boldsymbol{p}'_{2,0}(0)=-\sum_{m\in\mathcal{U}_2\backslash\{0\}}\boldsymbol{p}'_{2,m}(0)-\sum_{m\in\mathcal{U}_3}i\kappa_m\boldsymbol{p}_{3,m}(0).$$

Set $\mathcal{U}_4=\mathcal{U}_0\cup\{0\}=\{0,1,2,\cdots,M\}$. This means that $\mathcal{U}_2=\mathcal{U}_3=\mathcal{U}_4$ and $\sigma_m=\kappa_m$, $m\in\mathcal{U}_4$. If $m\neq 0$,

$$\begin{aligned}\boldsymbol{p}_{4,m}=-\frac{1}{(i\kappa_m)^2}\Big[&\boldsymbol{p}''_{2,m}+2(i\kappa_m)\boldsymbol{p}'_{3,m}+\boldsymbol{f}(\boldsymbol{p}_{0,0})\left(\boldsymbol{p}'_{2,m}+i\kappa_m\boldsymbol{p}_{3,m}\right)\\&+\boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}]i\kappa_m\boldsymbol{p}_{2,m}+\boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}]\boldsymbol{p}'_{0,0}\\&+\boldsymbol{g}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}]\Big],\qquad m\neq 0.\end{aligned}$$

3.5 $r=3$

When $r=3$, the $O(\frac{1}{\omega^3})$ terms are collected

$$\begin{aligned}&\sum_{m\in\mathcal{U}_3}\boldsymbol{p}''_{3,m}e^{i\sigma_m\omega t}+2\sum_{m\in\mathcal{U}_4}(i\sigma_m)\boldsymbol{p}'_{4,m}e^{i\sigma_m\omega t}+\sum_{m\in\mathcal{U}_5}(i\sigma_m)^2\boldsymbol{p}_{5,m}e^{i\sigma_m\omega t}\\&+\boldsymbol{f}(\boldsymbol{p}_{0,0})\left(\sum_{m\in\mathcal{U}_3}\boldsymbol{p}'_{3,m}e^{i\sigma_m\omega t}+\sum_{m\in\mathcal{U}_4}i\sigma_m\boldsymbol{p}_{4,m}e^{i\sigma_m\omega t}\right)\\&+\boldsymbol{f}_1(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}]\left(\sum_{m\in\mathcal{U}_2}\boldsymbol{p}'_{2,m}e^{i\sigma_m\omega t}+\sum_{m\in\mathcal{U}_3}i\sigma_m\boldsymbol{p}_{3,m}e^{i\sigma_m\omega t}\right)\\&+\left(\sum_{m\in\mathcal{U}_2}\boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}]e^{i\sigma_m\omega t}+\frac{1}{2}\boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}]\right)\\&\qquad\times\left(\boldsymbol{p}'_{1,0}+\sum_{m\in\mathcal{U}_2}i\sigma_m\boldsymbol{p}_{2,m}e^{i\sigma_m\omega t}\right)\\&+\left[\sum_{n=1}^{3}\frac{1}{n!}\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,3}}\theta_{\boldsymbol{\ell}}\sum_{k_1\in\mathcal{U}_{\ell_1}}\cdots\sum_{k_n\in\mathcal{U}_{\ell_n}}\boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0})\left[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}\right]e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t}\right]\boldsymbol{p}'_{0,0}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{3}\frac{1}{n!}\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,3}}\theta_{\boldsymbol{\ell}}\sum_{k_1\in\mathcal{U}_{\ell_1}}\cdots\sum_{k_n\in\mathcal{U}_{\ell_n}}\boldsymbol{g}^{(n)}(\boldsymbol{p}_{0,0})\left[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}\right]e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t}\\
&=0.
\end{aligned}
$$

Since $I_{1,3}=\{3\}$, $\theta_3=1$, $I_{2,3}=\{(1,2),(2,1)\}$, $\theta_{(1,2)}=2$, $I_{3,3}=\{(1,1,1)\}$, $\theta_{(1,1,1)}=1$ and $\boldsymbol{p}_{1,m}\equiv 0$ for $m\neq 0$, we get

$$
\begin{aligned}
&\sum_{n=1}^{3}\frac{1}{n!}\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,3}}\theta_{\boldsymbol{\ell}}\sum_{k_1\in\mathcal{U}_{\ell_1}}\cdots\sum_{k_n\in\mathcal{U}_{\ell_n}}f_n(p_{0,0})\left[p_{\ell_1,k_1},\cdots,p_{\ell_n,k_n}\right]e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t}\\
&=\sum_{m\in\mathcal{U}_3}\boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[p_{3,m}]e^{i\sigma_m\omega t}+\sum_{m\in\mathcal{U}_2}\boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{2,m}]e^{i\sigma_m\omega t}\\
&\quad+\frac{1}{6}\boldsymbol{f}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}].
\end{aligned}
$$

Therefore, the complete equation for $r=3$ becomes

$$
\begin{aligned}
&\sum_{m\in\mathcal{U}_3}\boldsymbol{p}''_{3,m}e^{i\sigma_m\omega t}+2\sum_{m\in\mathcal{U}_4}(i\sigma_m)\boldsymbol{p}'_{4,m}e^{i\sigma_m\omega t}+\sum_{m\in\mathcal{U}_5}(i\sigma_m)^2\boldsymbol{p}_{5,m}e^{i\sigma_m\omega t}\\
&+\boldsymbol{f}(\boldsymbol{p}_{0,0})\left(\sum_{m\in\mathcal{U}_3}\boldsymbol{p}'_{3,m}e^{i\sigma_m\omega t}+\sum_{m\in\mathcal{U}_4}i\sigma_m\boldsymbol{p}_{4,m}e^{i\sigma_m\omega t}\right)\\
&+\boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}]\left(\sum_{m\in\mathcal{U}_2}\boldsymbol{p}'_{2,m}e^{i\sigma_m\omega t}+\sum_{m\in\mathcal{U}_3}i\sigma_m\boldsymbol{p}_{3,m}e^{i\sigma_m\omega t}\right)\\
&+\left(\sum_{m\in\mathcal{U}_2}\boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}]e^{i\sigma_m\omega t}+\frac{1}{2}\boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}]\right)\boldsymbol{p}'_{1,0}\\
&+\frac{1}{2}\boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}]\sum_{m\in\mathcal{U}_2}i\sigma_m\boldsymbol{p}_{2,m}e^{i\sigma_m\omega t}\\
&+\sum_{m_1\in\mathcal{U}_2}\sum_{m_2\in\mathcal{U}_2}\boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m_2}](i\sigma_{m_1})\boldsymbol{p}_{2,m_1}e^{i(\sigma_{m_1}+\sigma_{m_2})\omega t}\\
&+\Bigg[\sum_{m\in\mathcal{U}_3}\boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{3,m}]e^{i\sigma_m\omega t}+\sum_{m\in\mathcal{U}_2}\boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{2,m}]e^{i\sigma_m\omega t}\\
&+\frac{1}{6}\boldsymbol{f}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}]\Bigg]\boldsymbol{p}'_{0,0}\\
&+\sum_{m\in\mathcal{U}_3}\boldsymbol{g}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{3,m}]e^{i\sigma_m\omega t}+\sum_{m\in\mathcal{U}_2}\boldsymbol{g}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{2,m}]e^{i\sigma_m\omega t}
\end{aligned}
$$

$$+\frac{1}{6}\boldsymbol{g}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}]=0,$$

where the terms

$$\boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{3,m}], \qquad \boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{2,m}], \qquad \boldsymbol{g}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{3,m}],$$
$$\boldsymbol{g}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{2,m}]$$

have the same form as the case $r=2$.

$$\boldsymbol{f}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}]=$$
$$\begin{bmatrix} f_{11}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] & f_{12}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] & \cdots & f_{1d}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] \\ f_{21}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] & f_{22}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] & \cdots & f_{2d}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] \\ \vdots & & & \\ f_{d1}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] & f_{d2}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] & \cdots & f_{dd}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] \end{bmatrix}_{d\times d},$$

$$\boldsymbol{g}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}]=\begin{bmatrix} g_1^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] \\ g_2^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] \\ \vdots \\ g_d^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}] \end{bmatrix}_{d\times 1},$$

in which

$$f_{jv}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}]$$
$$=\sum_{j_1=1}^{d}\sum_{j_2=1}^{d}\sum_{j_3=1}^{d}\frac{\partial^3 f_{jv}(\boldsymbol{p}_{0,0})}{\partial y_{j_1}\partial y_{j_2}\partial y_{j_3}}(\boldsymbol{p}_{1,0})_{j_1}(\boldsymbol{p}_{1,0})_{j_2}(\boldsymbol{p}_{1,0})_{j_3},$$

$$g_j^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0},\boldsymbol{p}_{1,0}]=\sum_{j_1=1}^{d}\sum_{j_2=1}^{d}\sum_{j_3=1}^{d}\frac{\partial^3 g_j(\boldsymbol{p}_{0,0})}{\partial y_{j_1}\partial y_{j_2}\partial y_{j_3}}(\boldsymbol{p}_{1,0})_{j_1}(\boldsymbol{p}_{1,0})_{j_2}(\boldsymbol{p}_{1,0})_{j_3}$$

and $(\boldsymbol{p}_{j,k})_s$ denotes the s-th element of the vector $\boldsymbol{p}_{j,k}$.
We now consider the construction of the set $\mathcal{U}_5$. $\mathcal{U}_5$ must include terms with $\kappa_i+\kappa_j$ for $i,j=1,2,\cdots,M$. Therefore, let

$$\mathcal{U}_5=\mathcal{U}_4\cup\{(m_1,m_2):1\le m_1\le m_2\le M\}.$$

Now for $i\neq j$, $\kappa_i+\kappa_j$ may result in a term in $\mathcal{U}_4$. However, it may also result in terms not in $\mathcal{U}_4$. These are the terms included in $\mathcal{U}_5\setminus\mathcal{U}_4$. For

$0 \leq \ell_1 \leq \ell_2 \leq M$, define the multiplicity $\rho^m_{\ell_1,\ell_2}$ as the number of cases when $\kappa_{\pi(\ell_1)} + \kappa_{\pi(\ell_2)} = \kappa_m$, where $\pi(\ell)$ is a permutation of ℓ. Let $\rho^{m_1,m_2}_{\ell_1,\ell_2}$ be the number of permutations such that $\kappa_{\pi(\ell_1)} + \kappa_{\pi(\ell_2)} = \kappa_{m_1} + \kappa_{m_2}$ for $0 \leq \ell_1 \leq \ell_2 \leq M$ and $1 \leq m_1 \leq m_2 \leq M$.

If $m = 0$, the terms with respect to $\sigma_0 = 0$ obey

$$
\begin{aligned}
&\boldsymbol{p}''_{3,0} + \boldsymbol{f}(\boldsymbol{p}_{0,0})\boldsymbol{p}'_{3,0} + \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}]\boldsymbol{p}'_{2,0} + \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,0}]\boldsymbol{p}'_{1,0} \\
&+ \frac{1}{2}\boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}, \boldsymbol{p}_{1,0}]\boldsymbol{p}'_{1,0} \\
&+ \sum_{\ell_1 \in \mathcal{U}_2} \sum_{\ell_2 \in \mathcal{U}_2} \sum_{\substack{\kappa_{\ell_1} + \kappa_{\ell_2} = 0 \\ \ell_1 \leq \ell_2}} \rho^0_{\ell_1,\ell_2} (i\kappa_{\ell_1}) \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,\ell_2}]\boldsymbol{p}_{2,\ell_1} \\
&+ \left[\boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{3,0}] + \boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}, \boldsymbol{p}_{2,0}] + \frac{1}{6}\boldsymbol{f}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}, \boldsymbol{p}_{1,0}, \boldsymbol{p}_{1,0}]\right]\boldsymbol{p}'_{0,0} \\
&+ \boldsymbol{g}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{3,0}] + \boldsymbol{g}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}, \boldsymbol{p}_{2,0}] + \frac{1}{6}\boldsymbol{g}^{(3)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}, \boldsymbol{p}_{1,0}, \boldsymbol{p}_{1,0}] = 0,
\end{aligned}
$$

with

$$
\begin{aligned}
\boldsymbol{p}_{3,0}(0) &= -\sum_{m \in \mathcal{U}_3 \setminus \{0\}} \boldsymbol{p}_{3,m}(0), \\
\boldsymbol{p}'_{3,0}(0) &= -\sum_{m \in \mathcal{U}_3 \setminus \{0\}} \boldsymbol{p}'_{3,m}(0) - \sum_{m \in \mathcal{U}_4} i\kappa_m \boldsymbol{p}_{4,m}(0).
\end{aligned}
$$

Then match all the terms in $\mathcal{U}_4 \setminus \{0\} \subset \mathcal{U}_5$. This yields the recurrence

$$
\begin{aligned}
&-(i\kappa_m)^2 \boldsymbol{p}_{5,m} = \boldsymbol{p}''_{3,m} + 2(i\kappa_m)\boldsymbol{p}'_{4,m} + \boldsymbol{f}(\boldsymbol{p}_{0,0})\left(\boldsymbol{p}'_{3,m} + i\kappa_m \boldsymbol{p}_{4,m}\right) \\
&+ \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}]\left(\boldsymbol{p}'_{2,m} + i\kappa_m \boldsymbol{p}_{3,m}\right) + \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,m}]\boldsymbol{p}'_{1,0} \\
&+ \frac{1}{2}\boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}, \boldsymbol{p}_{1,0}] i\kappa_m \boldsymbol{p}_{2,m} + \sum_{\substack{\kappa_{\ell_1} + \kappa_{\ell_2} = \kappa_m \\ \ell_1 \leq \ell_2}} \rho^m_{\ell_1,\ell_2} (i\kappa_{\ell_1}) \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,\ell_2}]\boldsymbol{p}_{2,\ell_1} \\
&+ \left(\boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{3,m}] + \boldsymbol{f}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}, \boldsymbol{p}_{2,m}]\right)\boldsymbol{p}'_{0,0} \\
&+ \boldsymbol{g}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{3,m}] + \boldsymbol{g}^{(2)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{1,0}, \boldsymbol{p}_{2,m}].
\end{aligned}
$$

Finally, we match the terms in $\mathcal{U}_5 \setminus \mathcal{U}_4$. Define the pairs (ℓ_1, ℓ_2) satisfying

$\ell_1 \leq \ell_2$ and $\kappa_{\ell_1} + \kappa_{\ell_2} \neq \sigma_j$ for $j = 0, 1, \cdots, M$. It follows that

$$(i(\kappa_{m_1} + \kappa_{m_2}))^2 \boldsymbol{p}_{5,m} = - \sum_{\substack{\kappa_{\ell_1}+\kappa_{\ell_2}=\kappa_{m_1}+\kappa_{m_2} \\ \ell_1 \leq \ell_2}} \rho^{m_1,m_2}_{\ell_1,\ell_2} (i\kappa_{\ell_1}) \boldsymbol{f}^{(1)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{2,\ell_2}]\boldsymbol{p}_{2,\ell_1}.$$

These are all terms with respect to $r = 3$.

3.6 The general case $r \geq 1$

The terms in $\mathcal{U}_{r+1}$ are composed of $\kappa_{j_1} + \kappa_{j_2} + \cdots + \kappa_{j_q}$, $q \leq r$ and $j_1 \leq j_2 \leq \cdots \leq j_q$. Set $\rho^{m_1,\cdots,m_q}_{\ell_1,\cdots,\ell_p}$ to be the number of distinct p-tuples $(\ell_1, \cdots, \ell_p)$, where each ℓ_i and m_i lie in $\{0, 1, 2, \cdots, M\}$, $m_1 \leq m_2 \leq \cdots m_q$, such that

$$\sum_{i=1}^{p} \kappa_{\ell_i} = \sum_{i=1}^{q} \kappa_{m_i}.$$

Extract all the terms at the level r

$$\begin{aligned}
&\sum_{m\in\mathcal{U}_r} \boldsymbol{p}''_{r,m} e^{i\sigma_m\omega t} + 2 \sum_{m\in\mathcal{U}_{r+1}} (i\sigma_m)\boldsymbol{p}'_{r+1,m} e^{i\sigma_m\omega t} + \sum_{m\in\mathcal{U}_{r+2}} (i\sigma_m)^2 \boldsymbol{p}_{r+2,m} e^{i\sigma_m\omega t} \\
&+ \boldsymbol{f}(\boldsymbol{p}_{0,0}) \left(\sum_{m\in\mathcal{U}_r} \boldsymbol{p}'_{r,m} e^{i\sigma_m\omega t} + \sum_{m\in\mathcal{U}_{r+1}} i\sigma_m \boldsymbol{p}_{r+1,m} e^{i\sigma_m\omega t} \right) \\
&+ \sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r}} \theta_{\boldsymbol{\ell}} \sum_{k_1\in\mathcal{U}_{\ell_1}} \cdots \sum_{k_n\in\mathcal{U}_{\ell_n}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t} \boldsymbol{p}'_{0,0} \\
&+ \left[\sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r-1}} \theta_{\boldsymbol{\ell}} \sum_{k_1\in\mathcal{U}_{\ell_1}} \cdots \sum_{k_n\in\mathcal{U}_{\ell_n}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t} \right] \\
&\times \left(\boldsymbol{p}'_{1,0} + \sum_{m\in\mathcal{U}_2} i\sigma_m \boldsymbol{p}_{2,m} e^{i\sigma_m\omega t} \right) \\
&+ \sum_{r_1=2}^{r-1} \left[\sum_{n=1}^{r-r_1} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r-r_1}} \theta_{\boldsymbol{\ell}} \sum_{k_1\in\mathcal{U}_{\ell_1}} \cdots \sum_{k_n\in\mathcal{U}_{\ell_n}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t} \right] \\
&\times \left(\sum_{m\in\mathcal{U}_{r_1}} \boldsymbol{p}'_{r_1,m} e^{i\sigma_m\omega t} + \sum_{m\in\mathcal{U}_{r_1+1}} i\sigma_m \boldsymbol{p}_{r_1+1,m} e^{i\sigma_m\omega t} \right)
\end{aligned}$$

$$+\sum_{n=1}^{r}\frac{1}{n!}\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r}}\theta_{\boldsymbol{\ell}}\sum_{k_1\in\mathcal{U}_{\ell_1}}\cdots\sum_{k_n\in\mathcal{U}_{\ell_n}}\boldsymbol{g}^{(n)}(\boldsymbol{p}_{0,0})\left[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}\right]e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t}=0,$$

where

$$\boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0})\left[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}\right]=$$
$$\begin{bmatrix} f_{11}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}] \cdots f_{1d}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}] \\ f_{21}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}] \cdots f_{2d}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}] \\ \vdots \\ f_{d1}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}] \cdots f_{dd}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}] \end{bmatrix}_{d\times d},$$

$$\boldsymbol{g}^{(n)}(\boldsymbol{p}_{0,0})\left[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}\right]=\begin{bmatrix} g_{1}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}] \\ g_{2}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}] \\ \vdots \\ g_{d}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}] \end{bmatrix}_{d\times d}$$

and

$$\begin{aligned}
&f_{jv}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}]\\
&=\sum_{j_1=1}^{d}\cdots\sum_{j_n=1}^{d}\frac{\partial^n f_{jv}(\boldsymbol{p}_{0,0})}{\partial y_{j_1}\cdots\partial y_{j_n}}(\boldsymbol{p}_{\ell_1,k_1})_{j_1}\cdots(\boldsymbol{p}_{\ell_n,k_n})_{j_n},\\
&g_{j}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}]\\
&=\sum_{j_1=1}^{d}\cdots\sum_{j_n=1}^{d}\frac{\partial^n g_{j}(\boldsymbol{p}_{0,0})}{\partial y_{j_1}\cdots\partial y_{j_n}}(\boldsymbol{p}_{\ell_1,k_1})_{j_1}\cdots(\boldsymbol{p}_{\ell_n,k_n})_{j_n}.
\end{aligned}$$

There are two kinds of frequencies that feature above:

$$\begin{aligned}
&e^{i(\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t}=e^{i\eta\omega t},\quad e^{i(\sigma_q+\sigma_{k_1}+\cdots+\sigma_{k_n})\omega t}=e^{i\eta\omega t},\\
&q\in\mathcal{U}_{r_1},\quad \sigma_{k_i}\in\mathcal{U}_{\ell_i},\quad \ell\in\mathbb{I}_{n,r},\quad\text{or}\quad \ell\in\mathbb{I}_{n,r-r_1},
\end{aligned}$$

where $r_1\geq 2$ is fixed. Now let

$$\eta=\sum_{j=1}^{\xi}\kappa_{m_j},$$

for $\xi \in \{1, 2, \cdots, r \quad \text{or} \quad r-1\}$ and $0 \leq m_1 \leq m_2 \leq \cdots \leq m_\xi \leq M$. If $\eta \in \mathcal{U}_{r+1}$, there exists $\sigma_m = \eta$ for $m \in \mathcal{U}_{r+1}$. Otherwise, add to $\mathcal{U}_{r+1}$ the ordered ξ-tuple $(m_1, m_2, \cdots, m_\xi)$ to generate the set $\mathcal{U}_{r+2}$.

We impose the same natural partial ordering on $\mathcal{U}_r$ as in [4]: first the singletons in lexicographic ordering, second the pairs in lexicographic ordering, then the triplets etc. This defines a relation $m_1 \preceq m_2$ for all $m_1, m_2 \in \mathcal{U}_r$,

$$\mathcal{W}^n_{r,m} = \left\{ (\boldsymbol{\ell}, \boldsymbol{k}) : k_i \in \mathcal{U}_{\ell_i}, \boldsymbol{\ell} \in \mathbb{I}_{n,r}, \sum_{i=1}^{n} \sigma_{k_i} = \sigma_m, k_1 \preceq \cdots \preceq k_n \right\}$$

for all $m \in \mathcal{U}_r$ and $n \in \{1, 2, \cdots, r\}$. For the second frequency form, for fixed $r_1 \geq 2$ and $q \in \mathcal{U}_{r_1}$, define the relation

$$\begin{aligned} \mathcal{W}^n_{r,m-q} = \Bigg\{ &(r_1, q, \boldsymbol{\ell}, \boldsymbol{k}) : k_i \in \mathcal{U}_{\ell_i}, \boldsymbol{\ell} \in \mathbb{I}_{n,r}, m \in \mathcal{U}_r, \\ &\sum_{i=1}^{n} \sigma_{k_i} = \sigma_m - \sigma_q, k_1 \preceq \cdots \preceq k_n, q \in \mathcal{U}_{r_1} \Bigg\}. \end{aligned}$$

The corresponding number of distinct p-tuples $(k_1, \cdots, k_p)$ is defined as $\rho^{m-q}_{\boldsymbol{k}}$ such that $k_1 \leq k_2 \leq \cdots \leq k_p$,

$$\sigma_{k_1} + \cdots + \sigma_{k_p} = \sigma_m - \sigma_q$$

for the fixed $q \in \mathcal{U}_{r_1}$, $r_1 \geq 2$.

Firstly, we obtain the terms in $\mathcal{U}_{r+2}$ which lie in the set $\mathcal{U}_r$, i.e. the case $m \in \mathcal{U}_r \subset \mathcal{U}_{r+2}$. Then we get

$$\begin{aligned} &(i\sigma_m)^2 \boldsymbol{p}_{r+2,m} = -\boldsymbol{p}''_{r,m} - 2(i\sigma_m)\boldsymbol{p}'_{r+1,m} - \boldsymbol{f}(\boldsymbol{p}_{0,0}) \left(\boldsymbol{p}'_{r,m} + i\sigma_m \boldsymbol{p}_{r+1,m} \right) \\ &- \sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r}} \theta_{\boldsymbol{\ell}} \sum_{(\boldsymbol{\ell},\boldsymbol{k}) \in \mathcal{W}^n_{r,m}} \rho^m_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n} \right] \boldsymbol{p}'_{0,0} \\ &- \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r-1}} \theta_{\boldsymbol{\ell}} \sum_{(\boldsymbol{\ell},\boldsymbol{k}) \in \mathcal{W}^n_{r,m}} \rho^m_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n} \right] \boldsymbol{p}'_{1,0} \\ &- \sum_{q \in \mathcal{U}_2} \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r-1}} \theta_{\boldsymbol{\ell}} \sum_{(2,q,\boldsymbol{\ell},\boldsymbol{k}) \in \mathcal{W}^n_{r,m-q}} \rho^{m-q}_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n} \right] i\sigma_q \boldsymbol{p}_{2,q} \\ &- \sum_{r_1=2}^{r-1} \left[\sum_{n=1}^{r-r_1} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r-r_1}} \theta_{\boldsymbol{\ell}} \sum_{(r_1,q,\boldsymbol{\ell},\boldsymbol{k}) \in \mathcal{W}^n_{r,m-q}} \rho^{m-q}_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(p_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n} \right] \right] \end{aligned}$$

$$
\begin{aligned}
&\times \left(\sum_{q\in\mathcal{U}_{r_1}} \boldsymbol{p}'_{r_1,q} + \sum_{q\in\mathcal{U}_{r_1+1}} i\sigma_q \boldsymbol{p}_{r_1+1,q} \right) \\
&- \sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r}} \theta_{\boldsymbol{\ell}} \sum_{(\boldsymbol{\ell},\boldsymbol{k})\in\mathcal{W}^n_{r,m}} \rho^m_{\boldsymbol{k}} \boldsymbol{g}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1},\cdots,p_{\ell_n,k_n}\right].
\end{aligned}
$$

When $m = 0$ in the above equation, $\sigma_0 = 0$. Therefore, the term $p_{r,0}$ can be computed as

$$
\begin{aligned}
&\boldsymbol{p}''_{r,0} + \boldsymbol{f}(\boldsymbol{p}_{0,0})\boldsymbol{p}'_{r,0} \\
&+ \sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r}} \theta_{\boldsymbol{\ell}} \sum_{(\boldsymbol{\ell},\boldsymbol{k})\in\mathcal{W}^n_{r,0}} \rho^0_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}\right] \boldsymbol{p}'_{0,0} \\
&+ \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r-1}} \theta_{\boldsymbol{\ell}} \sum_{(\boldsymbol{\ell},\boldsymbol{k})\in\mathcal{W}^n_{r,0}} \rho^0_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}\right] \boldsymbol{p}'_{1,0} \\
&+ \sum_{q\in\mathcal{U}_2} \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r-1}} \theta_{\boldsymbol{\ell}} \sum_{(2,q,\boldsymbol{\ell},\boldsymbol{k})\in\mathcal{W}^n_{r,-q}} \rho^{-q}_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0})[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_1,k_1}] i\sigma_q \boldsymbol{p}_{2,q} \\
&+ \sum_{r_1=2}^{r-1} \left[\sum_{n=1}^{r-r_1} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r-r_1}} \theta_{\boldsymbol{\ell}} \sum_{(r_1,q,\boldsymbol{\ell},\boldsymbol{k})\in\mathcal{W}^n_{r,-q}} \rho^{-q}_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}\right] \right] \\
&\times \left(\sum_{q\in\mathcal{U}_{r_1}} \boldsymbol{p}'_{r_1,q} + \sum_{q\in\mathcal{U}_{r_1+1}} i\sigma_q \boldsymbol{p}_{r_1+1,q} \right) \\
&+ \sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r}} \theta_{\boldsymbol{\ell}} \sum_{(\boldsymbol{\ell},\boldsymbol{k})\in\mathcal{W}^n_{r,0}} \rho^0_{\boldsymbol{k}} \boldsymbol{g}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}\right] = 0.
\end{aligned}
$$

The corresponding initial conditions are

$$
\begin{aligned}
\boldsymbol{p}_{r,0}(0) &= -\sum_{m\in\mathcal{U}_r\setminus\{0\}} \boldsymbol{p}_{r,m}(0), \\
\boldsymbol{p}'_{r,0}(0) &= -\sum_{m\in\mathcal{U}_r\setminus\{0\}} \boldsymbol{p}'_{r,m}(0) - \sum_{m\in\mathcal{U}_{r+1}} i\sigma_m \boldsymbol{p}_{r+1,m}(0).
\end{aligned}
$$

Secondly, consider $m \in \mathcal{U}_{r+1} \setminus \mathcal{U}_r$ which belong to $\mathcal{U}_{r+2}$,

$$
(i\sigma_m)^2 \boldsymbol{p}_{r+2,m} = -2(i\sigma_m)\boldsymbol{p}'_{r+1,m} - \boldsymbol{f}(\boldsymbol{p}_{0,0}) \left(i\sigma_m \boldsymbol{p}_{r+1,m}\right)
$$

$$
\begin{aligned}
&- \sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r}} \theta_{\boldsymbol{\ell}} \sum_{(\boldsymbol{\ell},\boldsymbol{k}) \in \mathcal{W}^n_{r+1,m}} \rho^m_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] \boldsymbol{p}'_{0,0} \\
&- \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r-1}} \theta_{\boldsymbol{\ell}} \sum_{(\boldsymbol{\ell},\boldsymbol{k}) \in \mathcal{W}^n_{r+1,m}} \rho^m_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] \boldsymbol{p}'_{1,0} \\
&- \sum_{q \in \mathcal{U}_2} \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r-1}} \theta_{\boldsymbol{\ell}} \sum_{(2,q,\boldsymbol{\ell},\boldsymbol{k}) \in \mathcal{W}^n_{r+1,m-q}} \rho^{m-q}_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] i\sigma_q \boldsymbol{p}_{2,q} \\
&- \sum_{r_1=2}^{r-1} \left[\sum_{n=1}^{r-r_1} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r-r_1}} \theta_{\boldsymbol{\ell}} \sum_{(r_1,q,\boldsymbol{\ell},\boldsymbol{k}) \in \mathcal{W}^n_{r+1,m-q}} \rho^{m-q}_{\boldsymbol{k}} \boldsymbol{f}_n(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] \right] \\
&\qquad \times \left(\sum_{q \in \mathcal{U}_{r_1}} \boldsymbol{p}'_{r_1,q} + \sum_{q \in \mathcal{U}_{r_1+1}} i\sigma_q \boldsymbol{p}_{r_1+1,q} \right) \\
&- \sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r}} \theta_{\boldsymbol{\ell}} \sum_{(\boldsymbol{\ell},\boldsymbol{k}) \in \mathcal{W}^n_{r+1,m}} \rho^m_{\boldsymbol{k}} \boldsymbol{g}_n(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right].
\end{aligned}
$$

From this equation, the term $\boldsymbol{p}_{r+2,m}$, $m \in \mathcal{U}_{r+1} \setminus \mathcal{U}_r \subseteq \mathcal{U}_{r+2}$ is derived.

Finally, extract the terms $\boldsymbol{p}_{r+2,m}$, $m \in \mathcal{U}_{r+2} \setminus \mathcal{U}_{r+1}$.

$$
\begin{aligned}
&(i\sigma_m)^2 \boldsymbol{p}_{r+2,m} \\
&= -\boldsymbol{p}'_{0,0} \sum_{n=1}^{r} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r}} \theta_{\boldsymbol{\ell}} \sum_{(\boldsymbol{\ell},\boldsymbol{k}) \in \mathcal{W}^n_{r+2,m}} \rho^m_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] \\
&\quad - \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r-1}} \theta_{\boldsymbol{\ell}} \sum_{(\boldsymbol{\ell},\boldsymbol{k}) \in \mathcal{W}^n_{r+2,m}} \rho^m_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] \boldsymbol{p}'_{1,0} \\
&\quad - \sum_{q \in \mathcal{U}_2} \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r-1}} \theta_{\boldsymbol{\ell}} \sum_{(2,q,\boldsymbol{\ell},\boldsymbol{k}) \in \mathcal{W}^n_{r+2,m-q}} \rho^{m-q}_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] i\sigma_q \boldsymbol{p}_{2,q} \\
&\quad - \sum_{r_1=2}^{r-1} \left[\sum_{n=1}^{r-r_1} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{I}_{n,r-r_1}} \theta_{\boldsymbol{\ell}} \sum_{(r_1,q,\boldsymbol{\ell},\boldsymbol{k}) \in \mathcal{W}^n_{r+2,m-q}} \rho^{m-q}_{\boldsymbol{k}} \boldsymbol{f}^{(n)}(\boldsymbol{p}_{0,0}) \left[\boldsymbol{p}_{\ell_1,k_1}, \cdots, \boldsymbol{p}_{\ell_n,k_n}\right] \right] \\
&\qquad \times \left(\sum_{q \in \mathcal{U}_{r_1}} \boldsymbol{p}'_{r_1,q} + \sum_{q \in \mathcal{U}_{r_1+1}} i\sigma_q \boldsymbol{p}_{r_1+1,q} \right)
\end{aligned}
$$

$$-\sum_{n=1}^{r}\frac{1}{n!}\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r}}\theta_{\boldsymbol{\ell}}\sum_{(\boldsymbol{\ell},\boldsymbol{k})\in\mathcal{W}^n_{r+2,m}}\rho^m_{\boldsymbol{k}}\boldsymbol{g}_n(\boldsymbol{p}_{0,0})\left[\boldsymbol{p}_{\ell_1,k_1},\cdots,\boldsymbol{p}_{\ell_n,k_n}\right].$$

We can compute the term $\boldsymbol{p}_{r+2,m}$, $m\in\mathcal{U}_{r+2}\setminus\mathcal{U}_{r+1}$ directly from this equation.

4 Numerical experiments

In this section, we present some examples that illustrate the construction and properties of the expansion given in Section 3. In all cases, we will compare the approximation given by the first few terms of the asymptotic-numerical solver with the exact solution (which will be computed numerically with standard Maple routines up to prescribed accuracy.) We use the notation

$$e_s=\left|y(t)-\sum_{r=0}^{s}\frac{1}{\omega^r}\sum_{m\in\mathcal{U}_r}\boldsymbol{p}_{r,m}(t)e^{i\sigma_m\omega t}\right|,\quad s\geq 0$$

for the error.

4.1 Example 1

We consider first the scalar Van der Pol oscillator. Its governing equation is

$$y''(t)-\mu(1-y^2)y'(t)+\varepsilon y(t)=F_\omega(t),\quad y(0)=1,\quad y'(0)=0,\quad (4.1)$$

with

$$F_\omega(t)=\sum_{m\in\mathcal{U}_0}a_m(t)e^{i\kappa_m\omega t},$$

where $\mathcal{U}_0=\{1,2\}$, $\kappa_1=1$, $\kappa_2=\sqrt{2}$, $\mu=0.744313$, $\varepsilon=0.983299$, $a_1(t)=1$ and $a_2(t)=t$.

We will work out the first four terms of the asymptotic expansion ($r=0,1,2,3$)

$$y(t)\sim p_{0,0}(t)+\frac{1}{\omega}p_{1,0}(t)+\frac{1}{\omega^2}\left[p_{2,0}(t)+\sum_{m\in\mathcal{U}_2\setminus\{0\}}p_{2,m}(t)e^{i\sigma_m\omega t}\right]$$

$$+\frac{1}{\omega^3}\left[p_{3,0}(t)+\sum_{m\in\mathcal{U}_3\setminus\{0\}}p_{3,m}(t)e^{i\sigma_m\omega t}\right].$$

Taking $r=0$, we obtain the two equations

$$p_{0,0}''+\mu(p_0^2-1)p_{0,0}'+\varepsilon p_{0,0},\quad p_0(0)=1,\quad p_0'(0)=0.$$

and

$$p_{2,m}(t)=\frac{a_m(t)}{(i\kappa_m)^2},\quad m=1,2,$$

where

$$p_{2,1}(t)=\frac{1}{(i\kappa_1)^2},\qquad p_{2,2}(t)=\frac{t}{(i\kappa_2)^2};$$

When $r=1$, the resulting equations are

$$\begin{aligned}&p_{1,0}''+\mu(p_{0,0}^2-1)p_{1,0}'+2\mu p_{0,0}'p_{0,0}p_{1,0}+\varepsilon p_{1,0}=0,\\&p_{1,0}(0)=0,\qquad p_{1,0}'(0)=-\frac{1}{i\kappa_1}.\end{aligned}$$

and

$$p_{3,m}(t)=-\frac{1}{i\kappa_m}\left[2p_{2,m}'+\mu(p_{0,0}^2-1)p_{2,m}\right],\quad m=1,2,$$

Thus

$$p_{3,1}(t)=-\frac{\mu(p_{0,0}^2-1)}{(i\kappa_1)^3},\qquad p_{3,2}(t)=-\frac{1}{(i\kappa_2)^3}\left(2+\mu t(p_{0,0}^2-1)\right);$$

With $r=2$, the term $p_{2,0}''$ satisfies the differential equation

$$\begin{aligned}&p_{2,0}''+\mu(p_{0,0}^2-1)p_{2,0}'+2\mu p_{0,0}p_{1,0}'p_{1,0}+2\mu p_{0,0}'p_{0,0}p_{2,0}+\mu p_{0,0}'p_{1,0}p_{1,0}\\&\qquad+\varepsilon p_{2,0}=0,\\&p_{2,0}(0)=-\frac{1}{(i\kappa_1)^2},\qquad p_{2,0}'(0)=\frac{1}{(i\kappa_2)^2}.\end{aligned}$$

In addition, $p_{4,m}$ for $m \neq 0$

$$p_{4,1}(t) = -\frac{1}{(i\kappa_1)^4}\left[-2\mu p_{0,0}p'_{0,0} - \mu^2(p_{0,0}^2-1)^2 + 2\mu(i\kappa_1)p_{0,0}p_{1,0} + \varepsilon\right],$$

$$p_{4,2}(t) = -\frac{1}{(i\kappa_2)^4}\left[-3\mu(p_{0,0}^2-1) - 2\mu t p_{0,0}p'_{0,0} - \mu^2 t(p_{0,0}^2-1)^2 \right.$$
$$\left. + 2\mu t(i\kappa_2)p_{0,0}p_{1,0} + \varepsilon t\right];$$

When $r = 3$, the term $p_{3,0}$ obeys

$$p''_{3,0} + \mu(p_{0,0}^2 - 1)p'_{3,0} + 2\mu p_{0,0}p_{1,0}p'_{2,0} + 2\mu p'_{1,0}p_{0,0}p_{2,0} + \mu p'_{1,0}p_{1,0}p_{1,0}$$
$$+2\mu p'_{0,0}(p_{0,0}p_{3,0} + p_{1,0}p_{2,0}) + \varepsilon p_{3,0} = 0$$

subject to the initial conditions

$$p_{3,0}(0) = \frac{2}{(i\kappa_2)^3}, \qquad p'_{3,0}(0) = \frac{\varepsilon}{(i\kappa_1)^3}.$$

Figs 4.1 and 4.2 show the asymptotic error with $r = 0, 1, 2, 3$ for $\omega = 100$ and $\omega = 1000$, respectively. As evident from these figures, the asymptotic error decreases for increasing r. Furthermore, the accuracy of the asymptotic method increases greatly for the same number of r levels for higher values of ω. This feature makes the method most suitable for simulation of modern electronic systems where ever-rising frequencies are present.

Focussing on the CPU time, some impressive results are obtained when the asymptotic method is compared to the Runge–Kutta method. While the asymptotic method takes about 5 seconds to compute the solution for $\omega = 500$ and $\omega = 5000$, the `rkf45` method takes about 207 seconds for $\omega = 500$. Moreover, this increases to 2432 seconds for $\omega = 5000$. Again, this marks the significant value of the method for computing results involving very high frequencies.

4.2 Example 2

To illustrate the use of the asymptotic method for a second-order differential equation system, a coupled Van der Pol–Duffing oscillator is chosen from [1],

$$\ddot{x} - \mu_1(1 - x^2)\dot{x} + vx = \alpha_1\dot{y} + \alpha_2 y + \iota(t)$$

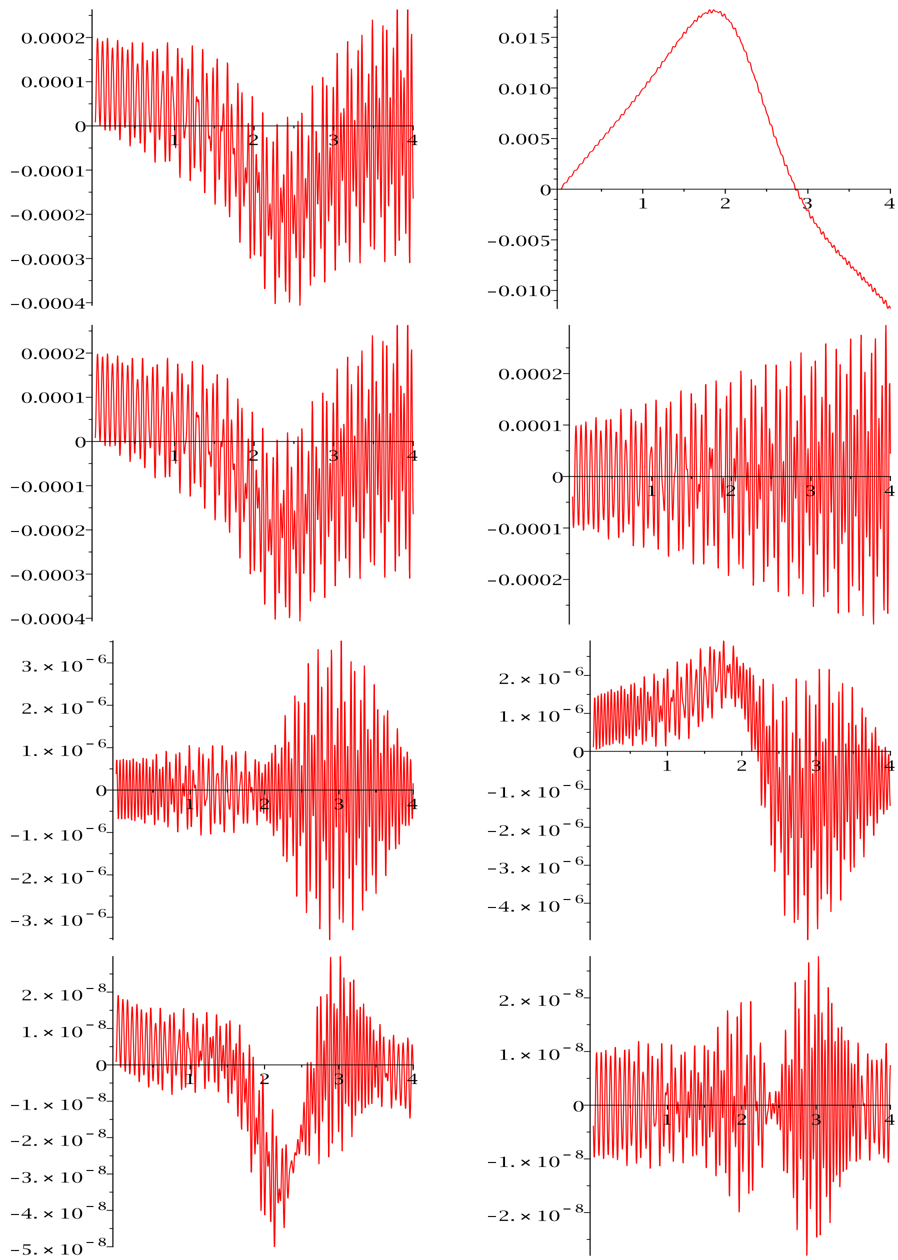

Figure 4.1: Equation (4.1). The top row: real (the left) imaginary (the right) parts of e_0. The middle row: real and imaginary parts of e_1. The third row: real and imaginary parts of e_2. The bottom row: real and imaginary parts of e_3 for $\omega = 100$.

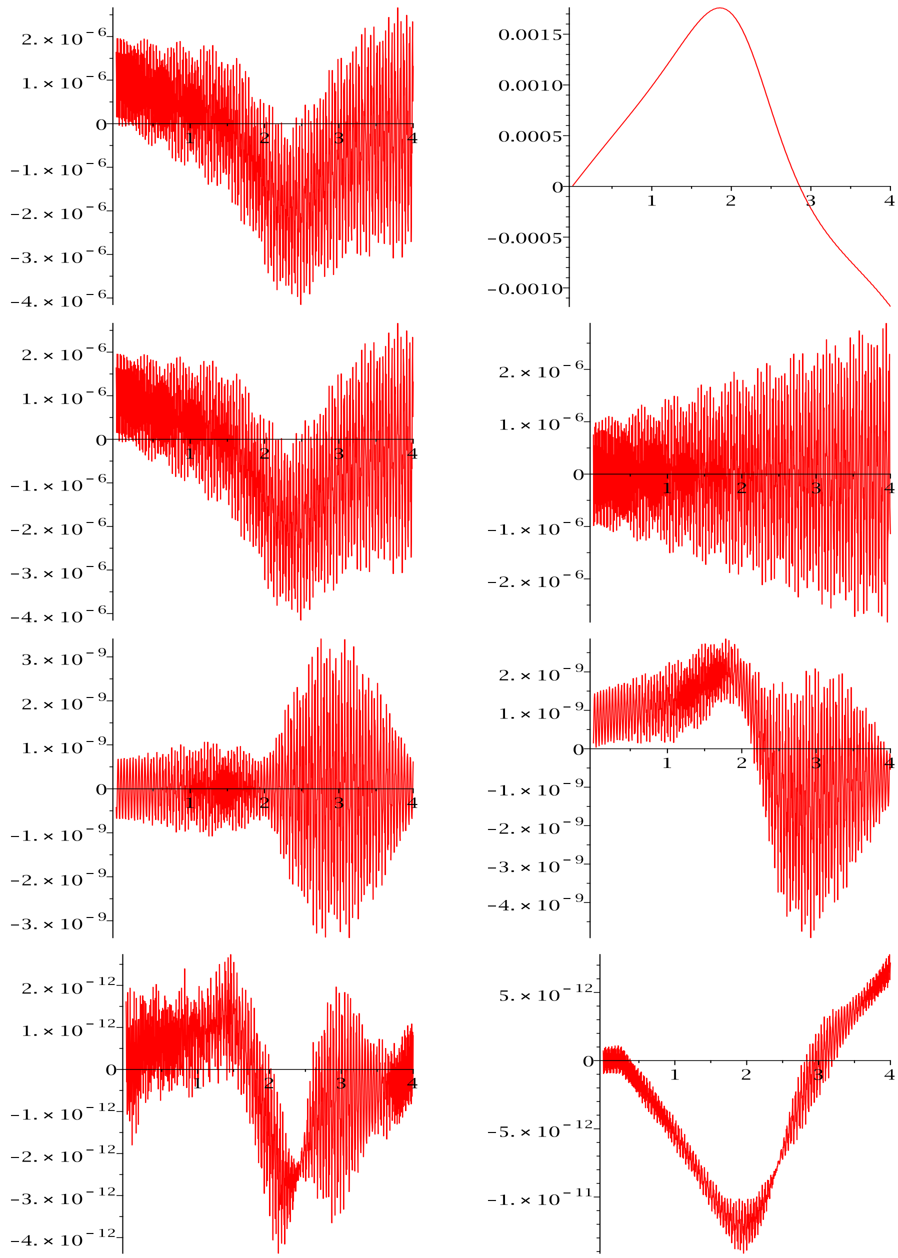

Figure 4.2: Equation (4.1). The top row: real (the left) imaginary (the right) parts of e_0. The middle row: real and imaginary parts of e_1. The third row: real and imaginary parts of e_2. The bottom row: real and imaginary parts of e_3 for $\omega = 1000$.

$$\ddot{y}+\mu_2\dot{y}+\gamma y+c_0y^3=\alpha_3\dot{x}+\alpha_4 x \tag{4.2}$$

The forcing term $\iota(t)$ is composed of two non-commensurate frequencies.

$$\iota(t)=\sum_{m=1}^{M}\boldsymbol{a}_m(t)e^{i\omega_m t},$$

In equation 4.2, $\mu_1 = 0.744313$, $\mu_2 = 0.668083$, $\alpha_1 = 0.235191$, $\alpha_2 = 0$, $\alpha_3 = 0.981204$, $\alpha_4 = 0$, $\gamma = -1.6$, $c_0 = 0.222375$, $v = 0.983299$, $\kappa_1 = \sqrt{2}$, $\kappa_2 = 2$,

$$\boldsymbol{a}_1(t)=\begin{bmatrix}1\\0\end{bmatrix},\quad \boldsymbol{a}_1(t)=\begin{bmatrix}\sin t\\0\end{bmatrix}$$

and the initial conditions are $x(0) = 1$, $x'(0) = 0$, $y(0) = 0$, $y'(0) = 0$. Let $\boldsymbol{p}_{\ell,k}=\begin{bmatrix}(\boldsymbol{p}_{\ell,k})_1\\(\boldsymbol{p}_{\ell,k})_2\end{bmatrix}$, where $(\boldsymbol{p}_{\ell,k})_j$ is the j-th elements of the vector $\boldsymbol{p}_{\ell,k}$. The corresponding representation functions in this equation are

$$\boldsymbol{f}(\boldsymbol{y})=\begin{bmatrix}-\mu_1(1-x^2) & -\alpha_1\\ -\alpha_3 & \mu_2\end{bmatrix},\quad \boldsymbol{f}^{(1)}(\boldsymbol{y})=\begin{bmatrix}(2\mu_1 x,0) & (0,0)\\ (0,0) & (0,0)\end{bmatrix},$$

$$\boldsymbol{f}^{(2)}(\boldsymbol{y})[\boldsymbol{p}_{\ell_1,k_1},\boldsymbol{p}_{\ell_2,k_2}]=\begin{bmatrix}2\mu_1(\boldsymbol{p}_{\ell_1,k_1})_1(\boldsymbol{p}_{\ell_2,k_2})_1 & 0\\ 0 & 0\end{bmatrix},$$

$$\boldsymbol{g}(\boldsymbol{y})=\begin{bmatrix}vx\\ \gamma y+c_0y^3\end{bmatrix},\quad \boldsymbol{g}^{(1)}(\boldsymbol{y})=\begin{bmatrix}(v,0)\\ (0,\gamma+3c_0y^2)\end{bmatrix},$$

$$\boldsymbol{g}^{(2)}(\boldsymbol{y})[\boldsymbol{p}_{\ell_1,k_1},\boldsymbol{p}_{\ell_2,k_2}]=\begin{bmatrix}0\\ 6c_0y(\boldsymbol{p}_{\ell_1,k_1})_2(\boldsymbol{p}_{\ell_2,k_2})_2\end{bmatrix},$$

$$\boldsymbol{g}^{(3)}(\boldsymbol{y})[\boldsymbol{p}_{\ell_1,k_1},\boldsymbol{p}_{\ell_2,k_2},\boldsymbol{p}_{\ell_3,k_3},]=\begin{bmatrix}0\\ 6c_0(\boldsymbol{p}_{\ell_1,k_1})_2(\boldsymbol{p}_{\ell_2,k_2})_2(\boldsymbol{p}_{\ell_3,k_3})_2\end{bmatrix}$$

The remaining terms are zero.

Assume that the the unknown functions $x(t)$ and $y(t)$ are

$$\begin{aligned}\begin{bmatrix}x(t)\\y(t)\end{bmatrix}&\sim\boldsymbol{p}_{0,0}+\frac{1}{\omega}\boldsymbol{p}_{1,0}+\frac{1}{\omega^2}\left[\boldsymbol{p}_{2,0}+\boldsymbol{p}_{2,1}e^{i\kappa_1\omega t}+\boldsymbol{p}_{2,2}e^{i\kappa_2\omega t}\right]\\&+\frac{1}{\omega^3}\left[\boldsymbol{p}_{3,0}+\boldsymbol{p}_{3,1}e^{i\kappa_1\omega t}+\boldsymbol{p}_{3,2}e^{i\kappa_2\omega t}\right].\end{aligned}$$

The initial conditions are,

$$\boldsymbol{p}_{0,0}(0)=\begin{bmatrix}1\\0\end{bmatrix},\quad \boldsymbol{p}'_{0,0}(0)=\begin{bmatrix}0\\0\end{bmatrix}$$

$$
\begin{aligned}
&\boldsymbol{p}_{1,0}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{p}'_{1,0}(0) = -\left[(i\kappa_1)\boldsymbol{p}_{2,1}(0) + (i\kappa_2)\boldsymbol{p}_{2,2}(0)\right] \\
&\boldsymbol{p}_{r,0}(0) = -\left[\boldsymbol{p}_{r,1}(0) + \boldsymbol{p}_{r,2}(0)\right], \\
&\boldsymbol{p}'_{r,0}(0) = -\left[\boldsymbol{p}'_{r,1}(0) + \boldsymbol{p}'_{r,2}(0)\right] - \left[(i\kappa_1)\boldsymbol{p}_{r+1,1}(0) + (i\kappa_2)\boldsymbol{p}_{r+1,2}(0)\right], \quad r \geq 2.
\end{aligned}
$$

The basic equation is

$$
\begin{bmatrix} (\boldsymbol{p}_{0,0})''_1 \\ (\boldsymbol{p}_{0,0})''_2 \end{bmatrix} + \begin{bmatrix} \mu_1\left((\boldsymbol{p}_{0,0})_1^2 - 1\right) & -\alpha_1 \\ -\alpha_3 & \mu_2 \end{bmatrix} \begin{bmatrix} (\boldsymbol{p}_{0,0})'_1 \\ (\boldsymbol{p}_{0,0})'_2 \end{bmatrix} + \begin{bmatrix} v(\boldsymbol{p}_{0,0})_1 \\ \gamma(\boldsymbol{p}_{0,0})_2 + c_0(\boldsymbol{p}_{0,0})_2^3 \end{bmatrix} = 0
$$

with

$$
\boldsymbol{p}_{0,0}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{p}'_{0,0}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

Moreover, we have

$$
\boldsymbol{p}_{2,1}(t) = \frac{1}{(i\kappa_1)^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{p}_{2,2}(t) = \frac{1}{(i\kappa_2)^2} \begin{bmatrix} \sin t \\ 0 \end{bmatrix}.
$$

For $r = 1$, the equation for $\boldsymbol{p}_{1,0}$ is

$$
\begin{aligned}
&\begin{bmatrix} (\boldsymbol{p}_{1,0})''_1 \\ (\boldsymbol{p}_{1,0})''_2 \end{bmatrix} + \begin{bmatrix} \mu_1((\boldsymbol{p}_{0,0})_1^2 - 1)(\boldsymbol{p}_{1,0})'_1 - \alpha_1(\boldsymbol{p}_{1,0})'_2 \\ -\alpha_3(\boldsymbol{p}_{1,0})'_1 + \mu_2(\boldsymbol{p}_{1,0})'_2 \end{bmatrix} \\
&+ \begin{bmatrix} 2\mu_1(\boldsymbol{p}_{0,0})_1(\boldsymbol{p}_{1,0})_1(\boldsymbol{p}_{0,0})'_1 \\ 0 \end{bmatrix} + \begin{bmatrix} v(\boldsymbol{p}_{1,0})_1 \\ (\gamma + 3c_0(\boldsymbol{p}_{0,0})_2^2)(\boldsymbol{p}_{1,0})_2 \end{bmatrix} = 0
\end{aligned}
$$

with the initial conditions

$$
\boldsymbol{p}_{1,0}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{p}'_{1,0}(0) = -\frac{1}{i\kappa_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

and

$$
\boldsymbol{p}_{3,1}(t) = -\frac{1}{(i\kappa_1)^3} \begin{bmatrix} \mu_1((\boldsymbol{p}_{0,0})_1^2 - 1) \\ -\alpha_3 \end{bmatrix},
$$

$$\boldsymbol{p}_{3,2}(t) = -\frac{1}{(i\kappa_2)^3}\begin{bmatrix}\mu_1((\boldsymbol{p}_{0,0})_1^2 - 1)\sin t + 2\cos t\\ -\alpha_3 \sin t\end{bmatrix}.$$

The $r = 2$ terms are

$$\begin{aligned}
&\begin{bmatrix}(\boldsymbol{p}_{2,0})_1''\\(\boldsymbol{p}_{2,0})_2''\end{bmatrix} + \begin{bmatrix}\mu_1((\boldsymbol{p}_{0,0})_1^2 - 1)(\boldsymbol{p}_{2,0})_1' - \alpha_1(\boldsymbol{p}_{2,0})_2'\\ -\alpha_3(\boldsymbol{p}_{2,0})_1' + \mu_2(\boldsymbol{p}_{2,0})_2'\end{bmatrix}\\
&+\begin{bmatrix}2\mu_1(\boldsymbol{p}_{0,0})_1(\boldsymbol{p}_{1,0})_1(\boldsymbol{p}_{1,0})_1'\\0\end{bmatrix} + \begin{bmatrix}2\mu_1(\boldsymbol{p}_{0,0})_1(\boldsymbol{p}_{2,0})_1(\boldsymbol{p}_{0,0})_1'\\0\end{bmatrix}\\
&+\begin{bmatrix}\mu_1(\boldsymbol{p}_{1,0})_1^2(\boldsymbol{p}_{0,0})_1'\\0\end{bmatrix} + \begin{bmatrix}v(\boldsymbol{p}_{2,0})_1\\(\gamma + 3c_0(\boldsymbol{p}_{0,0})_2^2)(\boldsymbol{p}_{2,0})_2\end{bmatrix}\\
&+\begin{bmatrix}0\\3c_0(\boldsymbol{p}_{0,0})_2(\boldsymbol{p}_{1,0})_2^2\end{bmatrix} = 0
\end{aligned}$$

with the initial conditions

$$\boldsymbol{p}_{2,0}(0) = -\frac{1}{(i\kappa_1)^2}\begin{bmatrix}1\\0\end{bmatrix}, \quad \boldsymbol{p}_{2,0}'(0) = \begin{bmatrix}\frac{1}{(i\kappa_2)^2}\\ \frac{-\alpha_3}{(i\kappa_1)^2}\end{bmatrix},$$

and

$$\begin{aligned}
\boldsymbol{p}_{4,1}(t) = -\frac{1}{(i\kappa_1)^4}&\left\{\begin{bmatrix}-4\mu_1(\boldsymbol{p}_{0,0})_1(\boldsymbol{p}_{0,0})_1'\\0\end{bmatrix} - \begin{bmatrix}\mu_1^2\left((\boldsymbol{p}_{0,0})_1^2 - 1\right)^2 + \alpha_1\alpha_3\\ -\alpha_3\mu_1\left((\boldsymbol{p}_{0,0})_1^2 - 1\right) - \mu_2\alpha_3\end{bmatrix}\right.\\
&\left.+ (i\kappa_1)\begin{bmatrix}2\mu_1(\boldsymbol{p}_{0,0})_1(\boldsymbol{p}_{1,0})_1\\0\end{bmatrix} + \begin{bmatrix}2\mu_1(\boldsymbol{p}_{0,0})_1(\boldsymbol{p}_{0,0})_1'\\0\end{bmatrix} + \begin{bmatrix}v\\0\end{bmatrix}\right\},\\
\boldsymbol{p}_{4,2}(t) = -\frac{1}{(i\kappa_2)^4}&\left\{\begin{bmatrix}-\sin t\\0\end{bmatrix} + \begin{bmatrix}v\sin t\\0\end{bmatrix}\right.\\
&- 2\begin{bmatrix}\mu_1\cos t\left((\boldsymbol{p}_{0,0})_1^2 - 1\right) + 2\mu_1\sin t(\boldsymbol{p}_{0,0})_1(\boldsymbol{p}_{0,0})_1' - 2\sin t\\ -\alpha_3\cos t\end{bmatrix}\\
&+ \begin{bmatrix}\mu_1\left((\boldsymbol{p}_{0,0})_1^2 - 1\right) & -\alpha_1\\ -\alpha_3 & \mu_2\end{bmatrix}\begin{bmatrix}-\mu_1\left((\boldsymbol{p}_{0,0})_1^2 - 1\right)\sin t - \cos t\\ \alpha_3\sin t\end{bmatrix}\\
&\left.+ (i\kappa_2)\begin{bmatrix}2\mu_1(\boldsymbol{p}_{0,0})_1(\boldsymbol{p}_{1,0})_1\sin t\\0\end{bmatrix} + \begin{bmatrix}2\mu_1(\boldsymbol{p}_{0,0})_1(\boldsymbol{p}_{0,0})_1'\sin t\\0\end{bmatrix}\right\}.
\end{aligned}$$

Similarly, the term $\boldsymbol{p}_{3,0}$ satisfies

$$\begin{bmatrix}(\boldsymbol{p}_{3,0})_1''\\(\boldsymbol{p}_{3,0})_2''\end{bmatrix} + \begin{bmatrix}\mu_1((\boldsymbol{p}_{0,0})_1^2 - 1) & -\alpha_1\\ -\alpha_3 & \mu_2\end{bmatrix}\begin{bmatrix}(\boldsymbol{p}_{3,0})_1'\\(\boldsymbol{p}_{3,0})_2'\end{bmatrix}$$

$$
\begin{aligned}
&+\begin{bmatrix} 2\mu_1(\boldsymbol{p}_{0,0})_1(\boldsymbol{p}_{1,0})_1(\boldsymbol{p}_{2,0})_1' \\ 0 \end{bmatrix} + \begin{bmatrix} 2\mu_1(\boldsymbol{p}_{0,0})_1(\boldsymbol{p}_{2,0})_1(\boldsymbol{p}_{1,0})_1' \\ 0 \end{bmatrix} \\
&+\begin{bmatrix} \mu_1(\boldsymbol{p}_{1,0})_1^2(\boldsymbol{p}_{1,0})_1' \\ 0 \end{bmatrix} + \begin{bmatrix} 2\mu_1(\boldsymbol{p}_{0,0})_1(\boldsymbol{p}_{3,0})_1(\boldsymbol{p}_{0,0})_1' \\ 0 \end{bmatrix} \\
&+\begin{bmatrix} 2\mu_1(\boldsymbol{p}_{2,0})_1(\boldsymbol{p}_{1,0})_1(\boldsymbol{p}_{0,0})_1' \\ 0 \end{bmatrix} + \begin{bmatrix} v(\boldsymbol{p}_{3,0})_1 \\ (\gamma + 3c_0(\boldsymbol{p}_{0,0})_2^2)(\boldsymbol{p}_{3,0})_2 \end{bmatrix} \\
&+\begin{bmatrix} 0 \\ 6c_0(\boldsymbol{p}_{0,0})_2(\boldsymbol{p}_{1,0})_2(\boldsymbol{p}_{2,0})_2 \end{bmatrix} = 0.
\end{aligned}
$$

with the initial conditions

$$
\boldsymbol{p}_{3,0}(0) = \begin{bmatrix} \frac{2}{(i\kappa_2)^3} \\ \frac{-\alpha_3}{(i\kappa_1)^3} \end{bmatrix}, \quad \boldsymbol{p}_{3,0}'(0) = \begin{bmatrix} \frac{v-\alpha_1\alpha_3}{(i\kappa_1)^3} \\ \frac{2\alpha_3}{(i\kappa_2)^3} + \frac{\mu_2\alpha_3}{(i\kappa_1)^3} \end{bmatrix}.
$$

For $r = 0, 1, 2, 3$ in the expansion, Fig. 4.3 illustrates the real and imaginary part of the errors for the first variable $x(t)$ with $\omega = 100$, compared with the Maple routine of Runge–Kutta method. Fig. 4.4 illustrates the results for $y(t)$.

In Figs 4.5–6 we have displayed the real and imaginary parts of the errors when the oscillatory parameter is $\omega = 1000$.

It can be seen that the error of the asymptotic method reduces greatly with increasing ω. In terms of the CPU time, the asymptotic method takes 9.5 seconds, compared to the Runge–Kutta method which takes 375 seconds for $\omega = 500$ and 5145 seconds for $\omega = 5000$.

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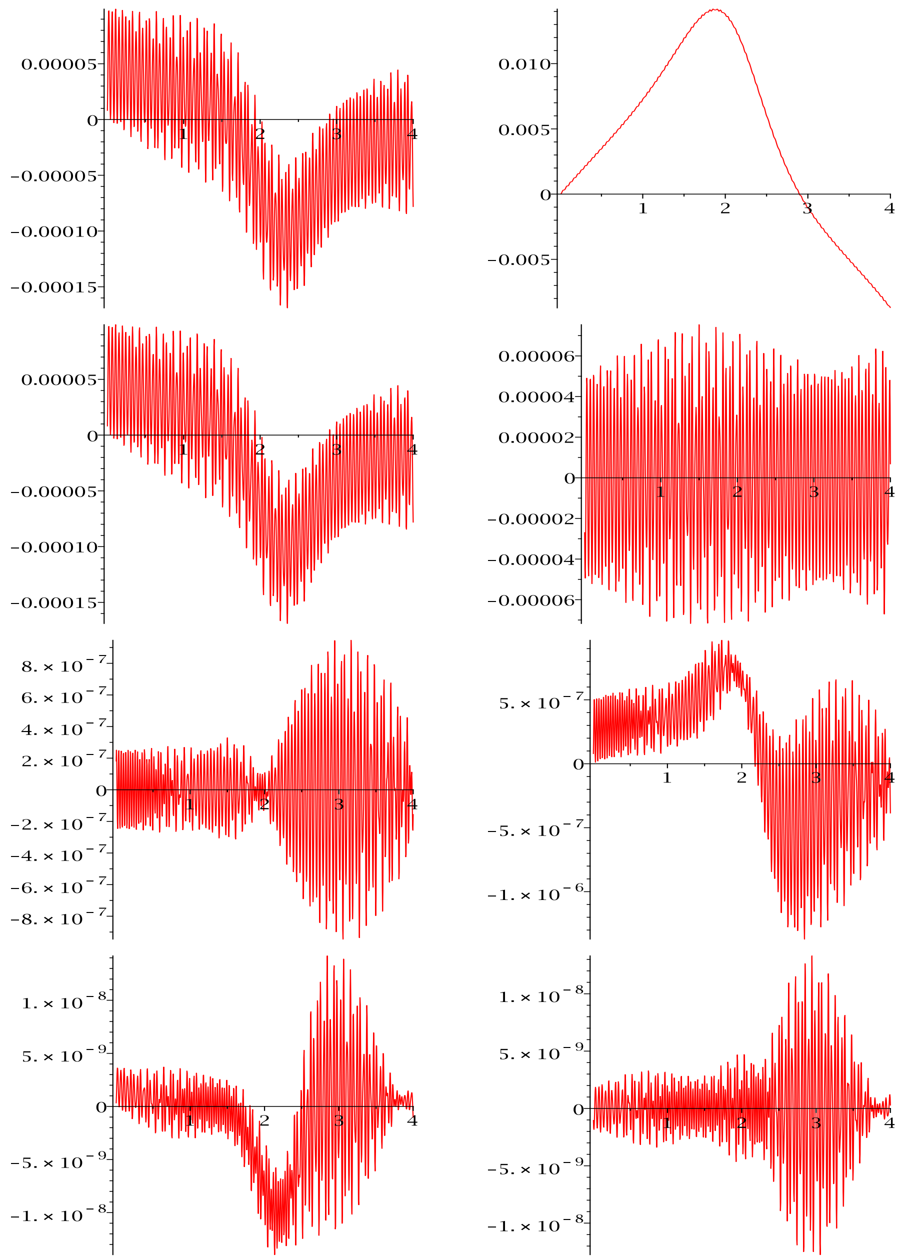

Figure 4.3: Equation (4.2), $x(t)$. The top row: real (the left) imaginary (the right) parts of e_0. The middle row: real and imaginary parts of e_1. The third row: real and imaginary parts of e_2. The bottom row: real and imaginary parts of e_3 for $\omega = 100$.

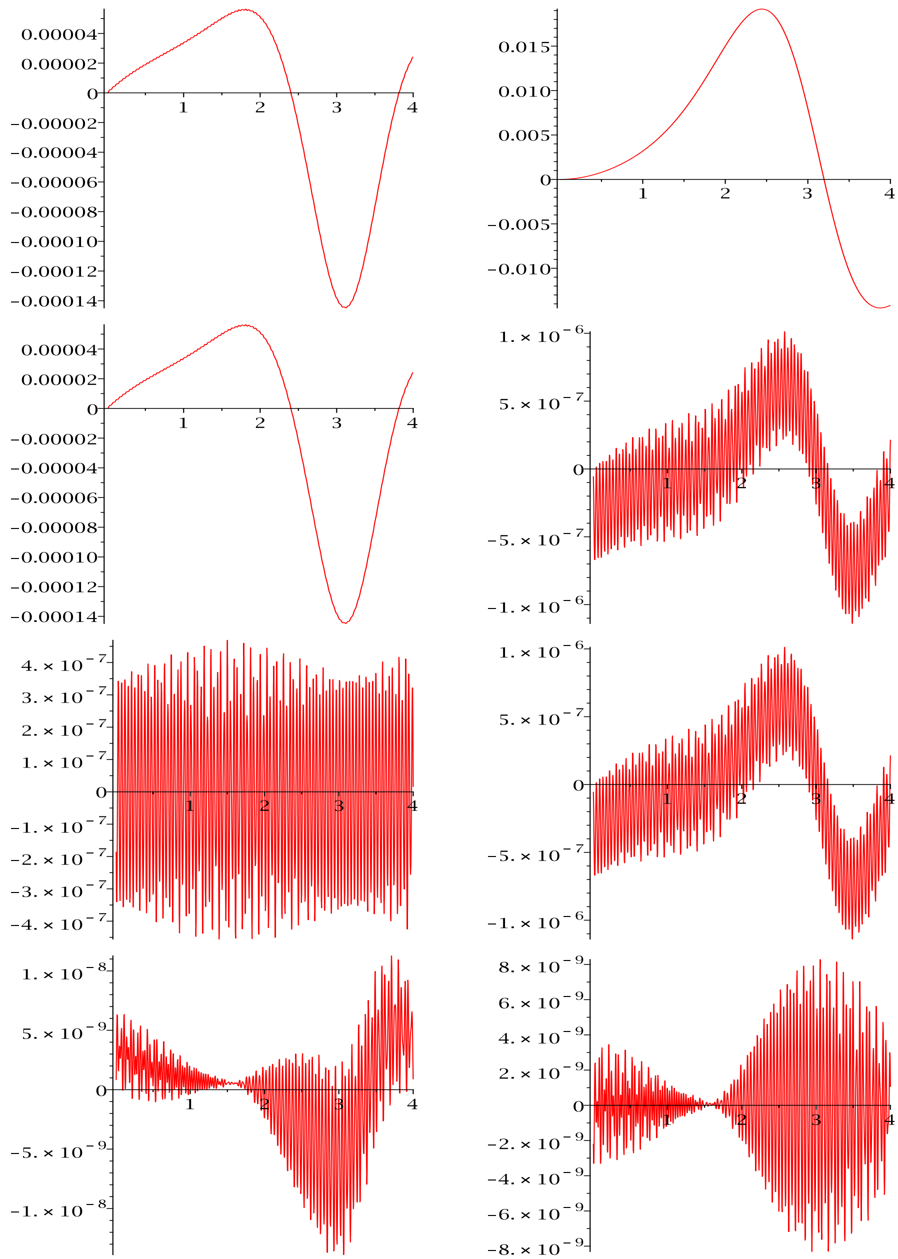

Figure 4.4: Equation (4.2), $y(t)$. The top row: real (the left) imaginary (the right) parts of e_0. The middle row: real and imaginary parts of e_1. The third row: real and imaginary parts of e_2. The bottom row: real and imaginary parts of e_3 for $\omega = 100$.

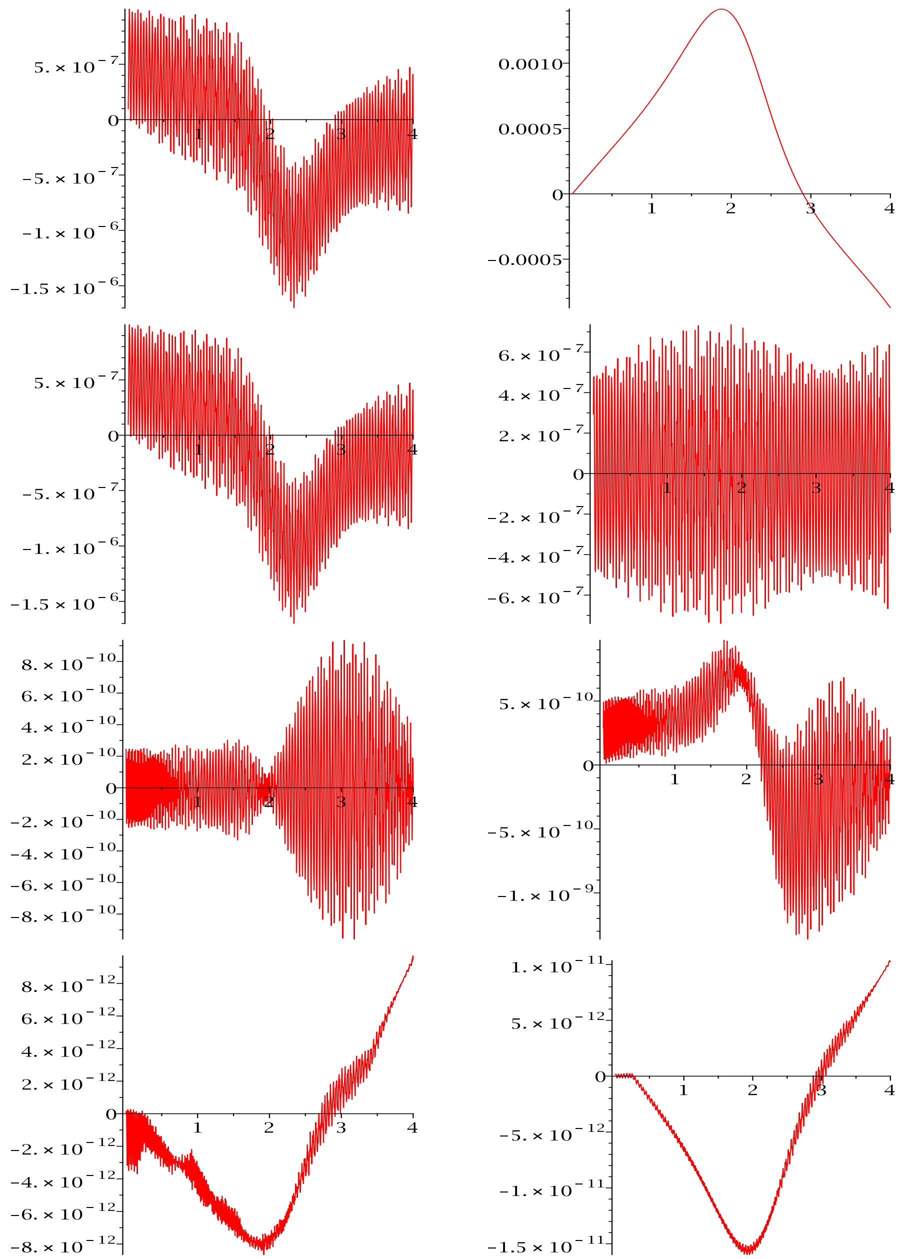

Figure 4.5: Equation (4.2), $x(t)$. The top row: real (the left) imaginary (the right) parts of e_0. The middle row: real and imaginary parts of e_1. The third row: real and imaginary parts of e_2. The bottom row: real and imaginary parts of e_3 for $\omega = 1000$.

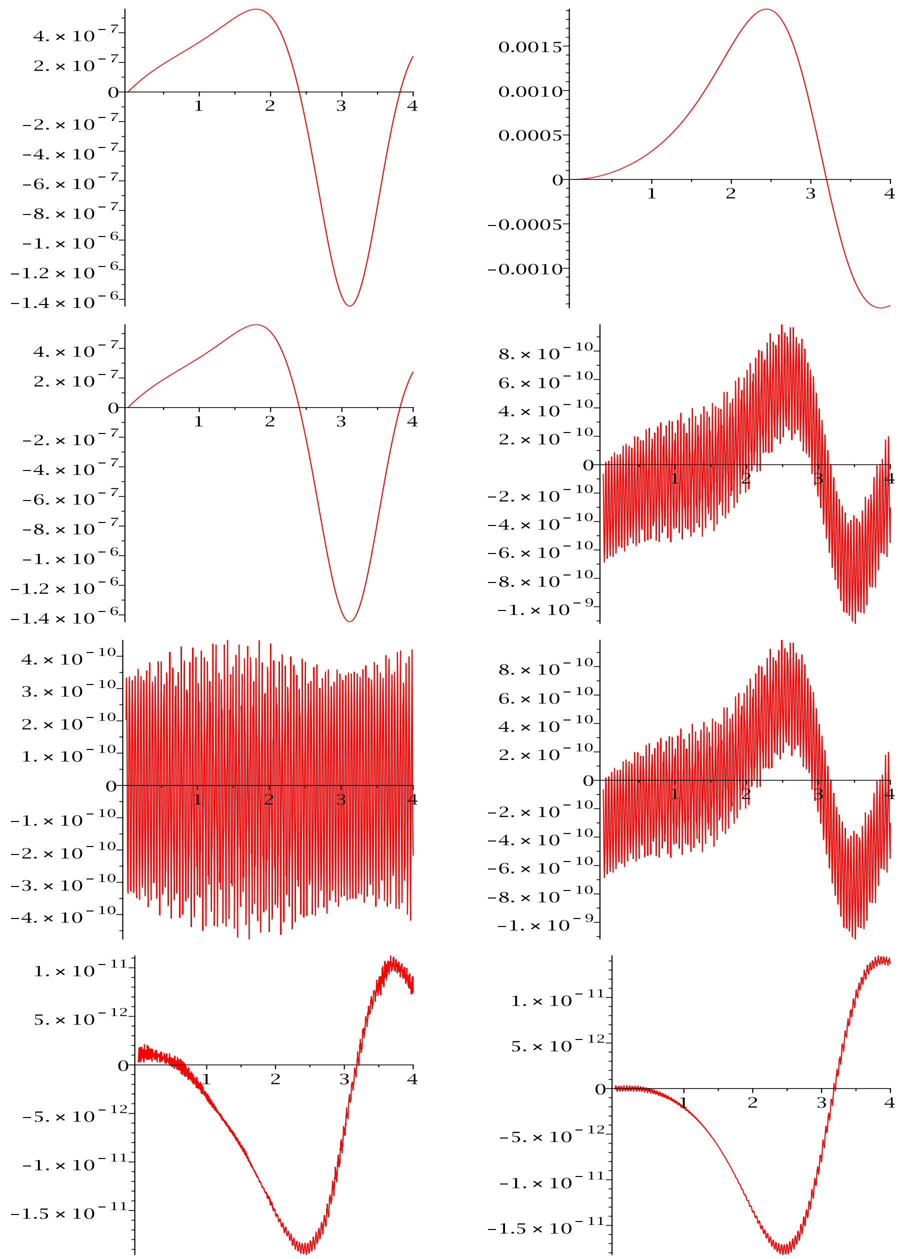

Figure 4.6: Equation (4.2), $y(t)$. The top row: real (the left) imaginary (the right) parts of e_0. The middle row: real and imaginary parts of e_1. The third row: real and imaginary parts of e_2. The bottom row: real and imaginary parts of e_3 for $\omega = 1000$.

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