The Unified Method in Polygonal Domains via the Explicit Fourier Transform of Legendre Polynomials

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Abstract

The recent numerical implementation by Fornberg and collaborators of the so-called unified method to linear elliptic PDEs in polygonal domains involves the computation of the finite Fourier transform of the Legendre polynomials. A variation of this approach, introduced by two of the authors, also involves the same computation. Here, instead of expressing the finite Fourier transform of the Legendre polynomials in terms of Bessel functions $\mathrm{J}_{n+\frac{1}{2}}$, we employ an explicit formula in terms of exponentials. We illustrate the usefulness of this formula, which is considerably cheaper to use, by implementing the unified method to the modified Helmholtz equation in the interior of a square. For completeness we present an explicit formula for the finite Fourier transform of all Jacobi polynomials.

1 Introduction

A novel method, often called the unified method, for analysing boundary value problems for linear and integrable nonlinear PDEs has been introduced by one of the authors [5]–[7] and used extensively in the literature. Recently, B. Fornberg and collaborators [4],[9] have introduced a powerful numerical implementation of the unified method to linear elliptic PDEs formulated in the interior of a polygon. In this case, the unified method yields two simple algebraic equations, the so-called global relations, which couple the finite Fourier transforms of the given boundary data with the finite Fourier transforms of the unknown boundary values. For the determination of these boundary values one has to choose

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(a) appropriate basis functions and (b) suitable collocation points in the Fourier space. Several such choices have featured in the literature [4], [8]–[16]; it appears that the best choice is the following: (a) the unknown boundary values are expanded in terms of Legendre polynomials and (b) the collocation points are chosen on the rays introduced in [8], [16]. We note that Chebyshev and Legendre polynomials give rise to equivalent finite bases of L^2, and hence either choice will result in the same numerical method (however the conditioning of the resulting linear systems will differ as conditioning is not invariant under matrix column operations).

The above numerical implementation requires the computation of the finite Fourier transform of Legendre polynomials. For this purpose, Fornberg and collaborators have used the fact that the finite Fourier transform of the Legendre polynomials can be expressed in terms of $\mathrm{J}_{n+\frac{1}{2}}$, Bessel functions of order half integer. Here we employ an explicit formula for the Fourier transform of the Legendre polynomials. We have not been able to find such a formula in the literature, so we include the relevant derivation. Furthermore, for completeness, we include the corresponding formula for all Jacobi polynomials.

As an example of the applicability of the analytic formula for the Fourier transform of the Legendre polynomials, we implement the unified method to the modified Helmholtz equation in the simplest possible polygon, namely a square.

Taking into consideration that the finite Fourier transform of the Legendre polynomials can be computed either explicitly or in terms of Bessel functions, the explicit formula presented in Section 3 implies a new explicit expression for Bessel functions of order half integer.

2 The modified Helmholtz equation in the interior of a square

Let $u(x,y)$ satisfy the modified Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \beta^2 u = 0, \quad \beta > 0, \quad (x,\, y) \in \mathbb{R}, \tag{2.1}$$

in the interior of a domain $\Omega \subset \mathbb{R}^2$. It is straightforward to verify that the following differential form W is closed:

$$\begin{aligned} &W(x,y,\lambda) \\ &= \mathrm{e}^{-\mathrm{i}\frac{\beta}{2}\left(\lambda z - \frac{\bar{z}}{\lambda}\right)} \left\{ \left[-u_y + \frac{\beta}{2}\left(\lambda + \frac{1}{\lambda}\right) u \right] \mathrm{d}x + \left[u_x + \frac{\beta}{2}\left(\mathrm{i}\lambda + \frac{1}{\mathrm{i}\lambda}\right) u \right] \mathrm{d}y \right\}, \\ &\qquad\qquad\qquad (x,\, y) \in \Omega, \quad \lambda \in \mathbb{C} \setminus \{0\}. \end{aligned} \tag{2.2}$$

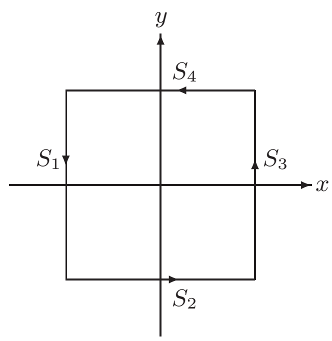

Figure 2.1: The square with sides of length 2.

Hence if $\partial\Omega$ denotes the boundary of Ω then the following equation is valid,

$$\int_{\partial\Omega} W(x, y, \lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \{0\}. \tag{2.3}$$

The above equation, called in [5] *the global relation,* is valid for all λ, thus it provides a *family* of equations which can be used to characterise the Dirichlet to Neumann map. Actually, for elliptic PDEs involving second order derivatives it is necessary to employ *two* global relations. The second global relation is

$$\int_{\partial\Omega} \tilde{W}(x, y, \lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \{0\}, \tag{2.4}$$

where

$$\tilde{W}(x, y, \lambda) = W\left(x, y, \frac{1}{\lambda}\right). \tag{2.5}$$

For polygonal domains, equations (2.3) and (2.4) characterise the Dirichlet to Neumann map. Rigorous aspects of this method, often referred to as the *unified method*, are discussed in [1]–[3].

In what follows we will use the simplest possible polygon, namely a square.

Consider the square from Figure 2.1 with corners at the points

$$(-1, 1), \quad (-1, 1), \quad (1, -1), \quad (1, 1).$$

For the sides S_1, S_2, S_3, S_4, we have respectively

$$z = -1 + \mathrm{i}y, \quad z = x - \mathrm{i}, \quad z = 1 + \mathrm{i}y, \quad z = x + \mathrm{i}. \tag{2.6}$$

In order to analyse the global relations, taking into account the orientations of the sides, we introduce the following expressions:

$$\hat{u}_1(\lambda) = \mathrm{e}^{\frac{\beta}{2}\left(\mathrm{i}\lambda+\frac{1}{\mathrm{i}\lambda}\right)} \int_{+1}^{-1} \mathrm{e}^{\frac{\beta}{2}\left(\lambda+\frac{1}{\lambda}\right)y} \left[u_x^{(1)} + \frac{\beta}{2}\left(\mathrm{i}\lambda + \frac{1}{\mathrm{i}\lambda}\right) u^{(1)}\right] \mathrm{d}y, \tag{2.7a}$$

$$\hat{u}_2(\lambda) = \mathrm{e}^{-\frac{\beta}{2}\left(\lambda+\frac{1}{\lambda}\right)} \int_{-1}^{+1} \mathrm{e}^{\frac{\beta}{2}\left(-\mathrm{i}\lambda-\frac{1}{\mathrm{i}\lambda}\right)x} \left[-u_y^{(2)} + \frac{\beta}{2}\left(\lambda + \frac{1}{\lambda}\right) u^{(2)}\right] \mathrm{d}x, \tag{2.7b}$$

$$\hat{u}_3(\lambda) = \mathrm{e}^{-\frac{\beta}{2}\left(\mathrm{i}\lambda+\frac{1}{\mathrm{i}\lambda}\right)} \int_{-1}^{+1} \mathrm{e}^{\frac{\beta}{2}\left(\lambda+\frac{1}{\lambda}\right)y} \left[u_x^{(3)} + \frac{\beta}{2}\left(\mathrm{i}\lambda + \frac{1}{\mathrm{i}\lambda}\right) u^{(3)}\right] \mathrm{d}y, \tag{2.7c}$$

$$\hat{u}_4(\lambda) = \mathrm{e}^{\frac{\beta}{2}\left(\lambda+\frac{1}{\lambda}\right)} \int_{+1}^{-1} \mathrm{e}^{\frac{\beta}{2}\left(-\mathrm{i}\lambda-\frac{1}{\mathrm{i}\lambda}\right)x} \left[-u_y^{(4)} + \frac{\beta}{2}\left(\lambda + \frac{1}{\lambda}\right) u^{(4)}\right] \mathrm{d}x. \tag{2.7d}$$

Let $\hat{D}_j$ and $\hat{N}_j$ denote the parts of $\hat{u}_j$ corresponding to Dirichlet and Neumann boundary values respectively. Then

$$\hat{u}_1(\lambda) = -\mathrm{e}^{\frac{\beta}{2}\left(\mathrm{i}\lambda+\frac{1}{\mathrm{i}\lambda}\right)} \hat{N}_1(\lambda) - \frac{\beta}{2}\left(\mathrm{i}\lambda + \frac{1}{\mathrm{i}\lambda}\right) \mathrm{e}^{\frac{\beta}{2}\left(\mathrm{i}\lambda+\frac{1}{\mathrm{i}\lambda}\right)} \hat{D}_1(\lambda), \tag{2.8a}$$

$$\hat{u}_2(\lambda) = -\mathrm{e}^{-\frac{\beta}{2}\left(\lambda+\frac{1}{\lambda}\right)} \hat{N}_2(-\mathrm{i}\lambda) + \frac{\beta}{2}\left(\lambda + \frac{1}{\lambda}\right) \mathrm{e}^{-\frac{\beta}{2}\left(\lambda+\frac{1}{\lambda}\right)} \hat{D}_2(-\mathrm{i}\lambda), \tag{2.8b}$$

$$\hat{u}_3(\lambda) = \mathrm{e}^{-\frac{\beta}{2}\left(\mathrm{i}\lambda+\frac{1}{\mathrm{i}\lambda}\right)} \hat{N}_3(\lambda) + \frac{\beta}{2}\left(\mathrm{i}\lambda + \frac{1}{\mathrm{i}\lambda}\right) \mathrm{e}^{-\frac{\beta}{2}\left(\mathrm{i}\lambda+\frac{1}{\mathrm{i}\lambda}\right)} \hat{D}_3(\lambda), \tag{2.8c}$$

$$\hat{u}_4(\lambda) = \mathrm{e}^{\frac{\beta}{2}\left(\lambda+\frac{1}{\lambda}\right)} \hat{N}_4(-\mathrm{i}\lambda) - \frac{\beta}{2}\left(\lambda + \frac{1}{\lambda}\right) \mathrm{e}^{\frac{\beta}{2}\left(\lambda+\frac{1}{\lambda}\right)} \hat{D}_4(-\mathrm{i}\lambda). \tag{2.8d}$$

The first of the global relations (2.3) becomes

$$\sum_{j=1}^{n} \hat{u}_j(\lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \{0\}. \tag{2.9}$$

For simplicity, we consider the following *symmetric* Dirichlet boundary value problem:

$$u^{(1)} = u(-1,y) = \cosh(1)\cosh(\sqrt{3}y) + \cosh(\sqrt{3})\cosh(y), \quad -1 < y < 1; \tag{2.10a}$$

$$u^{(3)} = u(1,y) \quad = \cosh(1)\cosh(\sqrt{3}y) + \cosh(\sqrt{3})\cosh(y), \quad -1 < y < 1; \tag{2.10b}$$

$$u^{(2)} = u(x,-1) = \cosh(1)\cosh(\sqrt{3}x) + \cosh(\sqrt{3})\cosh(x), \quad -1 < x < 1; \tag{2.10c}$$

$$u^{(4)} = u(x,1) \quad = \cosh(1)\cosh(\sqrt{3}x) + \cosh(\sqrt{3})\cosh(x), \quad -1 < x < 1. \tag{2.10d}$$

Symmetry implies

$$u(x,y)=u(-x,y), \quad u(x,y)=u(x,-y), \quad u(x,y)=u(y,x). \tag{2.11}$$

Thus,

$$u_x^{(3)}=-u_x^{(1)}, \quad u_y^{(4)}=-u_y^{(2)}, \quad u_y^{(2)}=u_x^{(1)}\Big|_{(x,y)\leftrightarrow(y,x)}. \tag{2.12}$$

Hence, the first of the global relations, namely equation (2.9), becomes

$$\begin{aligned}
&\cos\left[\frac{\beta}{2}\left(\lambda-\frac{1}{\lambda}\right)\right]\hat{N}_1(\lambda)+\cos\left[\frac{\beta}{2}\left(\mathrm{i}\lambda-\frac{1}{\mathrm{i}\lambda}\right)\right]\hat{N}_1(-\mathrm{i}\lambda)\\
&=\frac{\beta}{2}\left(\lambda-\frac{1}{\lambda}\right)\sin\left[\frac{\beta}{2}\left(\lambda-\frac{1}{\lambda}\right)\right]\hat{D}_1(\lambda)\\
&\quad+\frac{\beta}{2}\left(\mathrm{i}\lambda-\frac{1}{\mathrm{i}\lambda}\right)\sin\left[\frac{\beta}{2}\left(\mathrm{i}\lambda-\frac{1}{\mathrm{i}\lambda}\right)\right]\hat{D}_1(-\mathrm{i}\lambda), \quad \lambda\in\mathbb{C}\setminus\{0\}.
\end{aligned} \tag{2.13a}$$

The simplest way to obtain the second of the global relations, namely equation (2.4), is to take the Schwartz conjugate of equation (2.13a) (i.e. to take the complex conjugate of (2.13a), subsequently replacing $\bar{\lambda}$ with λ). This yields the equation

$$\begin{aligned}
&\cos\left[\frac{\beta}{2}\left(\lambda-\frac{1}{\lambda}\right)\right]\hat{N}_1(\lambda)+\cos\left[\frac{\beta}{2}\left(\mathrm{i}\lambda-\frac{1}{\mathrm{i}\lambda}\right)\right]\hat{N}_1(\mathrm{i}\lambda)\\
&=\frac{\beta}{2}\left(\lambda-\frac{1}{\lambda}\right)\sin\left[\frac{\beta}{2}\left(\lambda-\frac{1}{\lambda}\right)\right]\hat{D}_1(\lambda)\\
&\quad+\frac{\beta}{2}\left(\mathrm{i}\lambda-\frac{1}{\mathrm{i}\lambda}\right)\sin\left[\frac{\beta}{2}\left(\mathrm{i}\lambda-\frac{1}{\mathrm{i}\lambda}\right)\right]\hat{D}_1(\mathrm{i}\lambda), \quad \lambda\in\mathbb{C}\setminus\{0\}.
\end{aligned} \tag{2.13b}$$

Equations (2.13) are coupling the finite Fourier transform of the unknown Neumann boundary value $u_x^{(1)}$, to the finite Fourier transform of the given Dirichlet datum $u^{(1)}$. In order to solve these equations we have to make two choices: (a) expand $u_x^{(1)}$ in terms of appropriate basis functions; (b) evaluate (2.13) at appropriate collocation points $\{\lambda_n\}_{n=1}^M$. Regarding (a), Fourier basis functions [8], [10]–[16], as well as Chebyshev and Legendre polynomials [4], [9], [12], have been used earlier. Regarding (b), in most of the earlier papers the collocation points $\lambda\in\mathbb{C}\setminus\{0\}$ were chosen to lie on the rays in the complex λ-plane which are parallel to the edges of the polygon and its reflection in the imaginary axis. Recently, B. Fornberg and collaborators introduced the use of the so-called Halton nodes [4], [9].

It appears that the most efficient numerical method involves the following [15]: (a) approximating the unknown boundary values in terms of Legendre polynomials (following Fornberg) and (b) using the collocation points employed in our earlier work [16], where ideas of Sifalakis and collaborators [12] for the Laplace equation were extended to the modified

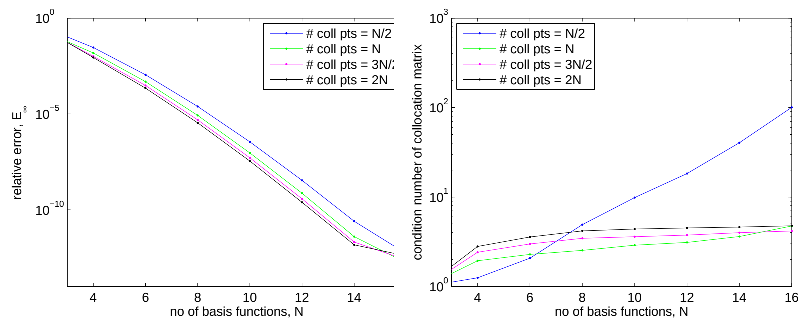

Figure 2.2: Numerical solution of the global relations (2.13) for the symmetric Dirichlet boundary value problem (2.10).

Helmholtz equation. These points are on certain rays of the complex λ-plane, determined by the exponential appearing in the global relation. Furthermore, in order to ensure that the collocation matrix remains well conditioned as the number N of basis functions increases, it is important following Fornberg (i) to normalise each row, as well as each column, of the collocation matrix by its l^1-norm [4] and (ii) to "over-determine" the linear system by choosing the number of collocation points to be about the same as the number of unknowns.

Numerical experiments suggest that Legendre polynomials yield spectral accuracy rather than the algebraic accuracy found with a Fourier basis. Furthermore, as a result of choosing the collocation points to be on the above rays, the semi-block circulant structure of the collocation matrix for regular polygons, demonstrated for the Laplace equation in [11], is preserved in modified Helmholtz equation as well. In addition, it is remarkable that the condition number is *independent* of β [15].

Plots of the relative error E_∞ (defined in [16]), as well as of the matrix condition number as a function of N, for $N/2$, N, $3N/2$ and $2N$ collocation points, are presented in Figure 2.2. The rectangular collocation matrix was inverted by using the "backslash" command in Matlab. It is clear that over-determining the linear system by a factor of 2 is sufficient to achieve very good matrix conditioning.

3 Fourier expansions of polynomials

3.1 A general theory

Let $p_m \in \mathbb{P}_m[x]$, the linear space of mth degree polynomials. We set

$$I[p_m] = \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}\lambda x} p_m(x)\,\mathrm{d}x, \qquad m \in \mathbb{Z}_+. \tag{3.1}$$

Repeatedly integrating by parts we obtain the following:

$$\begin{aligned}
I[p_m] &= -\frac{1}{\mathrm{i}\lambda}\int_{-1}^{1} p_m(x)\frac{\mathrm{d}\mathrm{e}^{-\mathrm{i}\lambda x}}{\mathrm{d}x}\,\mathrm{d}x = -\frac{1}{\mathrm{i}\lambda}[p_m(1)\mathrm{e}^{-\mathrm{i}\lambda} - p_m(-1)\mathrm{e}^{\mathrm{i}\lambda}] \\
&\quad + \frac{1}{\mathrm{i}\lambda}I[p_m'] \\
&= -\frac{1}{\mathrm{i}\lambda}[p_m(1)\mathrm{e}^{-\mathrm{i}\lambda} - p_m(-1)\mathrm{e}^{\mathrm{i}\lambda}] - \frac{1}{(\mathrm{i}\lambda)^2}[p_m'(1)\mathrm{e}^{-\mathrm{i}\lambda} - p_m'(-1)\mathrm{e}^{\mathrm{i}\lambda}] \\
&\quad + \frac{1}{(\mathrm{i}\lambda)^2}I[p_m''] \\
&= \cdots \\
&= -\sum_{n=0}^{k}\frac{1}{(\mathrm{i}\lambda)^{n+1}}[p_m^{(n)}(1)\mathrm{e}^{-\mathrm{i}\lambda} - p_m^{(n)}(-1)\mathrm{e}^{\mathrm{i}\lambda}] + \frac{1}{(\mathrm{i}\lambda)^{k+1}}I[p_m^{(k+1)}], \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k \in \mathbb{Z}_+.
\end{aligned}$$

Using $p_m^{(m+1)} \equiv 0$, we deduce the explicit representation

$$I[p_m] = -\mathrm{e}^{-\mathrm{i}\lambda}\sum_{n=0}^{m}\frac{p_m^{(n)}(1)}{(\mathrm{i}\lambda)^{n+1}} + \mathrm{e}^{\mathrm{i}\lambda}\sum_{n=0}^{m}\frac{p_m^{(n)}(-1)}{(\mathrm{i}\lambda)^{n+1}}, \qquad m \in \mathbb{Z}_+. \tag{3.2}$$

3.2 Jacobi polynomials

Let $p_m = \mathrm{P}_m^{(\alpha,\beta)}$, the mth *Jacobi polynomial,* where $\alpha, \beta > -1$. Then

$$\begin{aligned}
\mathrm{P}_m^{(\alpha,\beta)}(1) &= \frac{(1+\alpha)_m}{m!}, \\
\mathrm{P}_m^{(\alpha,\beta)}(-1) &= (-1)^m\mathrm{P}_m^{(\beta,\alpha)}(1) = (-1)^m\frac{(1+\beta)_m}{m!}
\end{aligned}$$

(this can be easily deduced from `http://dlmf.nist.gov/18.6`). Moreover,

$$\partial_x\mathrm{P}_m^{(\alpha,\beta)}(x) = \tfrac{1}{2}(\alpha+\beta+m+1)\mathbb{P}_{m-1}^{(\alpha+1,\beta+1)}(x)$$

(`http://dlmf.nist.gov/18.9`) and, by induction,

$$\partial_x^m\mathrm{P}_m^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+m+1)_n}{2^n}\mathrm{P}_{m-n}^{(\alpha+n,\beta+n)}(x), \qquad n = 0,\ldots,m.$$

Therefore

$$\begin{aligned}
\partial_x^n\mathrm{P}_m^{(\alpha,\beta)}(1) &= \frac{(\alpha+\beta+m+1)_n}{2^n}\frac{(\alpha+n+1)_{m-n}}{(m-n)!} \\
&= \frac{(\alpha+\beta+1)_{m+n}}{(\alpha+\beta+1)_m}\frac{(\alpha+1)_m}{(\alpha+1)_n}\frac{1}{2^n(m-n)!} \\
&= \frac{(\alpha+1)_m}{(\alpha+\beta+1)_m}\frac{(\alpha+\beta+1)_{m+n}}{2^n(m-n)!(\alpha+1)_n}.
\end{aligned}$$

Similarly

$$\begin{aligned}\partial_x^n \mathrm{P}_m^{(\alpha,\beta)}(-1) &= (-1)^n \partial_x^n \mathrm{P}_m^{(\beta,\alpha)}(1)\\ &= (-1)^{m+n} \frac{(\beta+1)_m}{(\alpha+\beta+1)_m} \frac{(\alpha+\beta+1)_{m+n}}{2^n (m-n)! (\beta+1)_n}.\end{aligned}$$

Substitution in (3.2) implies the following formula for the finite Fourier transform of the mth Jacobi polynomial:

$$\begin{aligned}I[\mathrm{P}_m^{(\alpha,\beta)}] = (-1)^m \frac{(\beta+1)_m}{(\alpha+\beta+1)_m} \mathrm{e}^{\mathrm{i}\lambda} \sum_{n=0}^{m} (-1)^n \frac{(\alpha+\beta+1)_{m+n}}{2^n (\beta+1)_n (m-n)!} \frac{1}{(\mathrm{i}\lambda)^{n+1}}\\ - \frac{(\alpha+1)_m}{(\alpha+\beta+1)_m} \mathrm{e}^{-\mathrm{i}\lambda} \sum_{n=0}^{m} \frac{(\alpha+\beta+1)_{m+n}}{2^n (\alpha+1)_n (m-n)!} \frac{1}{(\mathrm{i}\lambda)^n}. \qquad (3.3)\end{aligned}$$

3.2.1 Legendre polynomials

Since $\mathrm{P}_m = \mathrm{P}_m^{(0,0)}$, (3.3) reduces to

$$\begin{aligned}I[\mathrm{P}_m] = (-1)^m \mathrm{e}^{\mathrm{i}\lambda} \sum_{n=0}^{m} (-1)^n \frac{(m+n)!}{2^n n! (m-n)!} \frac{1}{(\mathrm{i}\lambda)^{n+1}}\\ - \mathrm{e}^{-\mathrm{i}\lambda} \sum_{n=0}^{m} \frac{(m+n)!}{2^n n! (m-n)!} \frac{1}{(\mathrm{i}\lambda)^{n+1}}.\end{aligned}$$

Hence, we obtain the following formula for the finite Fourier transform of Legendre polynomials:

$$I[\mathrm{P}_m] = \sum_{n=0}^{m} \frac{(m+n)!}{2^n n! (m-n)!} \frac{1}{(\mathrm{i}\lambda)^{n+1}} [(-1)^{m+n} \mathrm{e}^{\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}\lambda}], \qquad m \in \mathbb{Z}_+. \qquad (3.4)$$

3.2.2 Chebyshev polynomials

We have

$$\mathrm{T}_m(x) = \frac{m!}{(\frac{1}{2})_m} \mathrm{P}_m^{(-1/2,-1/2)}(x),$$

therefore

$$\mathrm{T}_m^{(n)}(1) = \frac{m!}{(\frac{1}{2})_m} \frac{(m)_n}{2^n} \frac{(n+\frac{1}{2})_{m-n}}{(m-n)!} = \frac{m(m+n-1)!}{2^n (m-n)! (\frac{1}{2})_n}, \qquad m \in \mathbb{N}$$

and

$$\mathrm{T}_m^{(n)}(-1) = (-1)^{m+n} \frac{m(m+n-1)!}{2^n (m-n)! (\frac{1}{2})_n}, \qquad m \in \mathbb{N}.$$

Thus, substituting in (3.2), we obtain the following formula for the finite Fourier transform of the Chebyshev polynomials:

$$I[\mathrm{T}_m] = \sum_{n=0}^{m} \frac{1}{(\mathrm{i}\lambda)^{n+1}} \frac{m(m+n-1)!}{2^n (m-n)! (\frac{1}{2})_n} [(-1)^{m+n} \mathrm{e}^{\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}\lambda}], \qquad m \in \mathbb{N}. \tag{3.5}$$

3.2.3 Computations

It is slightly unwieldy to compute repeatedly Pochhammer symbols, but also unnecessary. Indeed, the following recursive formulæ are valid:

$$I[\mathrm{P}_m] = \sum_{n=0}^{m} p_{m,n} \frac{(-1)^{m-n} \mathrm{e}^{\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}\lambda}}{(\mathrm{i}\lambda)^{n+1}}, \tag{3.6}$$

where

$$p_{m,0} = 1, \quad p_{m,n} = \frac{(m+\frac{1}{2})^2 - (n-\frac{1}{2})^2}{2n} p_{m,n-1}, \; n = 1, \ldots, m. \tag{3.7}$$

Similarly,

$$I[\mathrm{T}_m] = \sum_{n=0}^{m} t_{m,n} \frac{(-1)^{m-n} \mathrm{e}^{\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}\lambda}}{(\mathrm{i}\lambda)^{n+1}}, \tag{3.8}$$

where

$$t_{m,0} = 1, \quad t_{m,n} = \frac{m^2 - (n-1)^2}{2n-1} t_{m,n-1}, \; n = 1, \ldots, m. \tag{3.9}$$

3.2.4 An explicit representation for the half-order Bessel functions

Let $\mathrm{J}_{m+\frac{1}{2}}(\lambda)$ denote the half-order Bessel function, i.e.

$$\begin{aligned} \mathrm{J}_{m+\frac{1}{2}}(\lambda) = {} & \frac{1}{\pi} \int_0^{\pi} \cos\left(\left(m + \frac{1}{2}\right)\tau - \lambda \sin\tau\right) \mathrm{d}\tau \\ & + \frac{(-1)^{m+1}}{\pi} \int_0^{\infty} \exp\left(-\lambda \sinh\tau - \left(m + \frac{1}{2}\right)\tau\right) \mathrm{d}\tau, \\ & \lambda \in \mathbb{C}, \quad m \in \mathbb{Z}_+. \end{aligned} \tag{3.10}$$

Then equation (3.6) together with the formula for the Fourier transform of the Legendre polynomials employed in [9], namely,

$$I[P_m](\lambda) = \frac{1}{\mathrm{i}^m} \sqrt{\frac{2\pi}{\lambda}} \mathrm{J}_{m+\frac{1}{2}}(\lambda), \tag{3.11}$$

gives the following new explicit representation for Bessel functions of order half integer:

$$\mathrm{J}_{m+\frac{1}{2}}(\lambda) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{m+1} p_{m,n-1} \frac{(-1)^{m-n-1}\mathrm{e}^{\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}\lambda}}{\mathrm{i}^{n-m}\lambda^{n-\frac{1}{2}}},$$
$$\lambda \in \mathbb{C} \setminus \{0\}, \quad m = 0, 1, 2, \ldots, \quad (3.12)$$

where $p_{m,n}$ are defined in (3.7) (see also (3.4)). Since Bessel functions of order half integer can be easily converted to *spherical Bessel functions,*

$$\mathrm{J}_{n+\frac{1}{2}}(\lambda) = \sqrt{\frac{2\lambda}{\pi}}\mathrm{j}_n(\lambda),$$

(3.12) is also available for the representation and rapid computation of the latter.

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