From orthogonal polynomials on the unit circle to functional equations via generating functions

María José Cantero
Departamento de Matemática Aplicada and IUMA
Escuela de Ingeniería y Arquitectura
Universidad de Zaragoza
Spain
&
Arieh Iserles
Department of Applied Mathematics and Theoretical Physics
Centre for Mathematical Sciences
University of Cambridge
United Kingdom

June 27, 2013

Abstract

We explore orthogonal polynomials on the unit circle whose Schur parameters are $\{c\alpha^n\}_{n=1}^{\infty}$, where $0 < |\alpha|, |c| < 1$. Specifically, we derive two different generating functions. The first can be represented explicitly in terms of sums of a q-hypergeometric type and used to derive explicitly the underlying orthogonal polynomials, while the second obeys a functional differential equation and can be used to determine the asymptotic behaviour of these polynomials. Extending these constructs to orthogonal polynomials of the second kind, we are able to construct the Carathèodoty function and examine the underlying orthogonality measure.

1 Introduction

Let $\{\phi_n\}_{n\in\mathbb{Z}_+}$ be the set of monic polynomials, orthogonal on the complex unit circle $\mathcal{T}$ with respect to the measure $\mathrm{d}\mu$,

$$\int_{\mathcal{T}} \phi_n(z)\bar{\phi}_m(z)\,\mathrm{d}\mu(z) = 0, \qquad m \neq n.$$

It is known that the ϕ_ns obey a three-term recurrence relation of the form

$$\phi_n(z) = z\phi_{n-1}(z) + a_n\phi_{n-1}^*(z), \qquad n \in \mathbb{N},$$

where $a_n = \phi_n(0)$, $|a_n| < 1$ and $p^*(z) = z^n \bar{p}(z^{-1})$, $p \in \mathbb{P}_n$, as well as the difference equation

$$(a_{n+1} + a_n z)\phi_n(z) = a_n \phi_{n+1}(z) + (1 - |a_n|^2) a_{n+1} z \phi_{n-1}(z), \qquad n \in \mathbb{N}, \tag{1.1}$$

with the initial conditions

$$\phi_0(z) \equiv 1, \qquad \phi_1(z) = z + a_1$$

(Simon 2005). Note therefore that any sequence $\{a_n\}_{n\in\mathbb{Z}_+}$ of this kind uniquely defines a set of orthogonal polynomials on the unit circle (OPUC) and is known as a sequence of *Schur parameters*. Let $\alpha, c \in \mathbb{C}$ where $0 < |c|, |\alpha| < 1$. In the present paper we are interested in the orthogonal polynomials corresponding to the sequence

$$a_n = \begin{cases} 1, & n = 0, \\ c\alpha^n, & n \in \mathbb{N}. \end{cases} \tag{1.2}$$

In that case the recurrence (1.1) assumes the form

$$(\alpha + z)\phi_n(z) = \phi_{n+1}(z) + \alpha(1 - |c|^2|\alpha|^{2n}) z \phi_{n-1}(z). \tag{1.3}$$

Note that $\alpha = 1$ in (1.2) corresponds to *Geronimus polynomials*, while $c = 1$ and $\alpha = -q^{1/2}$, $|q| < 1$, to *Rogers–Szegő polynomials*. Moreover, $\alpha = c = 0$ correspond to the standard Lebesgue measure on the unit circle, with $\phi_n(z) = z^n$ (in the context of this paper, for brevity, we call these "Lebesgue polynomials"). Thus, our concern here is to investigate a family of OPUCs whose extreme cases are these three important families.

In Section 2 we explore the generating function

$$\Xi(z,t) = \sum_{n=0}^{\infty} \phi_n(z) t^n. \tag{1.4}$$

We represent Ξ as a sum of two q-hypergeometric functions and explore the extremal cases of Geronimus and Rogers–Szegő polynomials. Section 3 is devoted to another generating function,

$$\Phi(z,t) = \sum_{n=0}^{\infty} \frac{\phi_n(z)}{n!} t^n. \tag{1.5}$$

We show that Φ obeys a functional-differential equation of the pantograph type, hence can be expanded in Dirichlet series. This leads to its explicit representation in terms of q-Bessel functions. We also explore the extremal cases. In Section 4 we derive, using similar methodology, generating functions for orthogonal polynomials of the second kind.

The generating functions (1.4) and (1.5) are, in a deep sense, complementary. While Ξ lends itself more easily to the explicit representation of the polynomials ϕ_n as sums incorporating Gauss–Heine symbols, Φ allows for the derivation of the asymptotic behaviour of the sequence $\{\phi_n\}_{n=0}^{\infty}$. This, in turn, is fundamental to the derivation of the underlying Carathéodory function and of the orthogonality measure: this we accomplish in Section 5.

The construction of orthogonal polynomials in this paper is purely formal: we commence from Schur parameters and use them in the recurrence relation (1.1), rather than commencing from an orthogonality measure. Thus, there is nothing *per se* in our construction to guarantee that the underlying measure exists. Using the Carathéodory function, we demonstrate in Section 5 that a signed measure always exists and present extensive numerical evidence that it is a proper measure for all pairs (α, c) except for a small domains corresponding to large $|\alpha| < 1$.

2 Generating functions and functional equations

We consider first the generating function

$$\Xi(z,t) = \sum_{n=0}^{\infty} \phi_n(z) t^n, \qquad t, z \in \mathbb{C}.$$

Note that it follows at once from the Poincaré criterion that Ξ converges for all $|t| < 1$. Multiplying (1.3) by t^n and summing up results in

$$\sum_{1}^{\infty} \phi_{n+1} t^n = (\alpha + z) \sum_{1}^{\infty} \phi_n t^n - \sum_{1}^{\infty} (1 - |c|^2 q^n) \alpha z \phi_{n-1} t^n$$

$$\Rightarrow \quad t^{-1}[\Xi(z,t) - 1 - (c\alpha + z)t] = (\alpha + z)[\Xi(z,t) - 1] - \alpha z t \Xi(z,t) + \alpha z t q |c|^2 \Xi(z, qt),$$

where $q = |\alpha|^2 \in [0,1]$. We thus deduce that

$$\Xi(z,t) = \frac{1}{(1-\alpha t)(1-zt)} [1 + \alpha(c-1)t + q\alpha z t^2 |c|^2 \Xi(z, qt)]. \tag{2.1}$$

Let $(z,q)_n = (1-z)(1-qz)\cdots(1-q^{n-1}z)$ be the *Gauss–Heine symbol* (Gasper & Rahman 2004). Iterating (2.1),

$$\begin{aligned}
\Xi(z,t) &= \frac{1+\alpha(c-1)t}{(\alpha t, q)_1 (zt, q)_1} + \frac{q(\alpha z t^2 |c|^2)}{(\alpha t, q)_1 (zt, q)_1} \Xi(z, qt) \\
&= \frac{1+\alpha(c-1)t}{(\alpha t, q)_1 (zt, q)_1} + \frac{q(\alpha z t^2 |c|^2)}{(\alpha t, q)_2 (zt, q)_2} [1 + \alpha(c-1)qt + q^3(\alpha z t^2 |c|^2) \Xi(z, q^2 t)] \\
&= \frac{1+\alpha(c-1)t}{(\alpha t, q)_1 (zt, q)_1} + \frac{1+\alpha(c-1)qt}{(\alpha t, q)_2 (zt, q)_2} q(\alpha z t^2 |c|^2) \\
&\quad + \frac{q^4(\alpha z t^2 |c|^2)^2}{(\alpha t, q)_3 (zt, q)_3} [1 + \alpha(c-1)q^2 t + q^5(\alpha z t^2 |c|^2) \Xi(z, q^3 t)] \\
&= \cdots = \sum_{m=0}^{s-1} \frac{1+\alpha(c-1)q^m t}{(\alpha t, q)_{m+1} (zt, q)_{m+1}} q^{m^2} (\alpha z t^2 |c|^2)^m + \frac{q^{s^2} (\alpha z t^2 |c|^2)^s}{(\alpha t, q)_s (zt, q)_s} \Xi(z, q^s t)
\end{aligned}$$

for all $s \in \mathbb{N}$. Letting $s \to \infty$, we obtain

$$\Xi(z,t) = \sum_{m=0}^{\infty} \frac{q^{m^2} (\alpha z t^2 |c|^2)^m}{(\alpha t, q)_{m+1} (zt, q)_{m+1}} + \alpha(c-1)t \sum_{m=0}^{\infty} \frac{q^{m(m+1)} (\alpha z t^2 |c|^2)^m}{(\alpha t, q)_{m+1} (zt, q)_{m+1}}. \tag{2.2}$$

Note that

$$\begin{aligned}\Xi(z,t) &= \frac{1}{(1-\alpha t)(1-zt)}\sum_{m=0}^{\infty}\frac{q^{m(m-1)}(\alpha z q t^2|c|^2)^m}{(q\alpha t,q)_m(qzt,q)_m}\\ &\quad+\frac{\alpha(c-1)t}{(1-\alpha t)(1-zt)}\sum_{m=0}^{\infty}\frac{q^{m(m-1)}(\alpha z q^2 t^2|c|^2)^m}{(q\alpha t,q)_m(qzt,q)_m}\\ &= \frac{1}{(1-\alpha t)(1-zt)}{}_1\phi_2\left[\begin{matrix}q;\\ q\alpha t, qzt,;\end{matrix} q, \alpha z q t^2|c|^2\right]\\ &\quad+\frac{\alpha(c-1)t}{(1-\alpha t)(1-zt)}{}_1\phi_2\left[\begin{matrix}q;\\ q\alpha t, qzt,;\end{matrix} q, \alpha z q^2 t^2|c|^2\right],\end{aligned}$$

where ${}_m\phi_n$, $m,n\in\mathbb{Z}_+$, are *q-hypergeometric* (also known as *basic hypergeometric*) *functions* (Gasper & Rahman 2004).

Let us examine the consequences of (2.2). The main step is the identity

$$\frac{1}{(t,q)_{m+1}} = \sum_{k=0}^{\infty}\left[\begin{matrix}m+k\\ m\end{matrix}\right]_q t^k, \tag{2.3}$$

where the *q-binomial coefficient* is

$$\left[\begin{matrix}n\\ m\end{matrix}\right]_q = \frac{(q,q)_n}{(q,q)_m(q,q)_{n-m}}, \qquad 0\le m\le n.$$

(2.3) must be known and at any rate can be easily proved by induction. Consequently

$$\begin{aligned}F(X) &:= \sum_{m=0}^{\infty}\frac{q^{m^2}X^m t^{2m}}{(\alpha t,q)_{m+1}(zt,q)_{m+1}}\\ &= \sum_{k=0}^{\infty}q^{k^2}X^k\sum_{\ell=0}^{\infty}\left[\begin{matrix}k+\ell\\ k\end{matrix}\right]_q\alpha^\ell\sum_{m=0}^{\infty}\left[\begin{matrix}k+m\\ k\end{matrix}\right]_q z^m t^{\ell+m+2k}\\ &= \sum_{k=0}^{\infty}q^{k^2}X^k\sum_{\ell=0}^{\infty}\left[\begin{matrix}k+\ell\\ k\end{matrix}\right]_q\alpha^\ell\sum_{m=2k+\ell}^{\infty}\left[\begin{matrix}m-k-\ell\\ k\end{matrix}\right]_q z^{m-\ell-2k}t^m\\ &= \sum_{k=0}^{\infty}\sum_{m=2k}^{\infty}\sum_{\ell=0}^{m-2k}\left[\begin{matrix}k+\ell\\ k\end{matrix}\right]_q\left[\begin{matrix}m-k-\ell\\ k\end{matrix}\right]_q q^{k^2}X^k\alpha^\ell z^{m-\ell-2k}t^m\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\lfloor m/2\rfloor}\sum_{\ell=0}^{m-2k}\left[\begin{matrix}k+\ell\\ k\end{matrix}\right]_q\left[\begin{matrix}m-k-\ell\\ k\end{matrix}\right]_q q^{k^2}X^k\alpha^\ell z^{m-\ell-2k}t^m\\ &= \sum_{m=0}^{\infty}f_m(X)t^m,\end{aligned}$$

where

$$f_m(X) = \sum_{k=0}^{\lfloor m/2\rfloor}\sum_{\ell=0}^{m-2k}\left[\begin{matrix}k+\ell\\ k\end{matrix}\right]_q\left[\begin{matrix}m-k-\ell\\ k\end{matrix}\right]_q q^{k^2}X^k\alpha^\ell z^{m-\ell-2k}$$

$$= \sum_{k=0}^{\lfloor m/2 \rfloor} \sum_{\ell=0}^{m-2k} \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q \begin{bmatrix} m-k-\ell \\ k \end{bmatrix}_q q^{k^2} X^k \alpha^{m-2k-\ell} z^\ell$$
$$= \sum_{\ell=0}^{m} \sum_{k=0}^{\lfloor (m-\ell)/2 \rfloor} \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q \begin{bmatrix} m-k-\ell \\ k \end{bmatrix}_q q^{k^2} X^k \alpha^{m-2k-\ell} z^\ell.$$

It follows from the definition of Ξ and from (2.2) that

$$\phi_m(z) = f_m(\alpha z|c|^2)) + \alpha(c-1) f_{m-1}(\alpha z q|c|^2), \qquad m \in \mathbb{N}. \tag{2.4}$$

But

$$f_m(\alpha z|c|^2) = \sum_{\ell=0}^{m} \sum_{k=0}^{\lfloor (m-\ell)/2 \rfloor} \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q \begin{bmatrix} m-k-\ell \\ k \end{bmatrix}_q q^{k^2} |c|^{2k} \alpha^{m-k-\ell} z^{k+\ell}$$
$$= \sum_{\ell=0}^{m} \sum_{k=\ell}^{\lfloor (m-\ell)/2 \rfloor + \ell} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} m-k \\ k-\ell \end{bmatrix}_q q^{(k-\ell)^2} |c|^{2(k-\ell)} \alpha^{m-k} z^k$$
$$= \sum_{k=0}^{m} \sum_{\ell=(2k-m)_+}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} m-k \\ k-\ell \end{bmatrix}_q q^{(k-\ell)^2} |c|^{2(k-\ell)} \alpha^{m-k} z^k$$
$$= \sum_{k=0}^{m} \sum_{l=0}^{\min\{k,m-k\}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} m-k \\ \ell \end{bmatrix}_q q^{\ell^2} |c|^{2\ell} \alpha^{m-k} z^k$$

and, likewise,

$$\alpha f_{m-1}(\alpha z q|c|^2) = \sum_{k=0}^{m-1} \sum_{\ell=0}^{\min\{k,m-k-1\}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} m-k-1 \\ \ell \end{bmatrix}_q q^{\ell(\ell+1)} |c|^{2\ell} \alpha^{m-k} z^k.$$

We thus deduce that

$$\phi_m(z) = \sum_{k=0}^{m} \sum_{\ell=0}^{\min\{k,m-k\}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} m-k \\ \ell \end{bmatrix}_q q^{\ell^2} |c|^{2\ell} \alpha^{m-k} z^k \tag{2.5}$$
$$+ (c-1) \sum_{k=0}^{m-1} \sum_{\ell=0}^{\min\{k,m-k-1\}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} m-k-1 \\ \ell \end{bmatrix}_q q^{\ell(\ell+1)} |c|^{2\ell} \alpha^{m-k} z^k.$$

However, since

$$\begin{bmatrix} m-k \\ \ell \end{bmatrix}_q - q^\ell \begin{bmatrix} m-k-1 \\ \ell \end{bmatrix}_q = \begin{cases} 0, & \ell = 0, \\ \begin{bmatrix} m-k-1 \\ \ell-1 \end{bmatrix}_q, & \ell = 1, \ldots, m-k, \end{cases}$$

we can rewrite (2.5) for $m \in \mathbb{N}$ in the form

$$\phi_m(z) = \sum_{k=1}^{m-1} \sum_{\ell=1}^{\min\{k,m-k-1\}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} m-k-1 \\ \ell-1 \end{bmatrix}_q q^{\ell^2} |c|^{2\ell} \alpha^{m-k} z^k \tag{2.6}$$

$$
+\sum_{k=\lfloor (m+1)/2\rfloor}^{m} \begin{bmatrix} k \\ m-k \end{bmatrix}_q q^{(m-k)^2}|c|^{2(m-k)}\alpha^{m-k}z^k
$$
$$
+c\sum_{k=0}^{m-1}\sum_{\ell=0}^{\min\{k,m-k-1\}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} m-k-1 \\ \ell \end{bmatrix}_q q^{\ell(\ell+1)}|c|^{2\ell}\alpha^{m-k}z^k.
$$

Note that, as a reality check, it follows at once from (2.6) that $\phi_m(0) = c\alpha^m$ for $m \in \mathbb{N}$.

It is easy to consider the extramal cases. We commence by allowing $\alpha \to 1$ (Geronimus polynomials):

$$
\lim_{q\to 1} \begin{bmatrix} n \\ m \end{bmatrix}_q = \binom{n}{m}
$$

implies

$$
\begin{aligned}
\phi_m(z) = &\sum_{k=0}^{m}\sum_{\ell=0}^{\min\{k,m-k\}} \binom{k}{\ell}\binom{m-k}{\ell}|c|^{2\ell}z^k \\
&+ (c-1)\sum_{k=0}^{m-1}\sum_{\ell=0}^{\min\{k,m-k-1\}} \binom{k}{\ell}\binom{m-k-1}{\ell}|c|^{2\ell}z^k \\
= &\sum_{k=1}^{m-1}\sum_{\ell=1}^{\min\{k,m-k-1\}} \binom{k}{\ell}\binom{m-k-1}{\ell-1}|c|^{2\ell}z^k + \sum_{k=\lfloor (m+1)/2\rfloor} \binom{k}{m-k}|c|^{2(m-k)}z^k \\
&+ c\sum_{k=0}^{m-1}\sum_{\ell=0}^{\min\{k,m-k-1\}} \binom{k}{\ell}\binom{m-k-1}{\ell}|c|^{2\ell}z^k.
\end{aligned}
$$

We next prove directly that the above representation obeys the recurrence

$$
\phi_{m+1}(z) = (1+z)\phi_m(z) - (1-|c|^2)z\phi_{m-1}(z).
$$

Substituting our expressions on the left, we obtain after laborious algebra

$$
\begin{aligned}
&(1+z)\phi_m(z) - (1-|c|^2)z\phi_{m-1}(z) \\
= &\sum_{k=0}^{m}\sum_{\ell=0}^{\min\{k,m-k\}} \binom{k}{\ell}\binom{m-k}{\ell}|c|^{2\ell}z^k \\
&+ \sum_{k=1}^{m+1}\sum_{\ell=0}^{\min\{k-1,m-k+1\}} \binom{k-1}{\ell}\binom{m-k+1}{\ell}|c|^{2\ell}z^k \\
&- \sum_{k=1}^{m}\sum_{\ell=0}^{\min\{k-1,m-k\}} \binom{k-1}{\ell}\binom{m-k}{\ell}|c|^{2\ell}z^k \\
&+ \sum_{k=1}^{m}\sum_{\ell=1}^{\min\{k,m-k+1\}} \binom{k-1}{\ell-1}\binom{m-k}{\ell-1}|c|^{2\ell}z^k
\end{aligned}
$$

$$
\begin{aligned}
&+ (c-1) \sum_{k=0}^{m-1} \sum_{\ell=0}^{\min\{k,m-k-1\}} \binom{k}{\ell}\binom{m-k-1}{\ell} |c|^{2\ell} z^k \\
&+ (c-1) \sum_{k=1}^{m} \sum_{\ell=0}^{\min\{k-1,m-k\}} \binom{k-1}{\ell}\binom{m-k}{\ell} |c|^{2\ell} z^k \\
&- (c-1) \sum_{k=1}^{m-1} \sum_{\ell=0}^{\min\{k-1,m-k-1\}} \binom{k-1}{\ell}\binom{m-k-1}{\ell} |c|^{2\ell} z^k \\
&+ (c-1) \sum_{k=1}^{m-1} \sum_{\ell=1}^{\min\{k,m-k\}} \binom{k-1}{\ell-1}\binom{m-k-1}{\ell-1} |c|^{2\ell} z^k .
\end{aligned}
$$

Assuming the convention that $\binom{n}{m} = 0$ for $m \le -1$ and $m \ge n+1$, we have for the term multiplied by $(c-1)$

$$
\begin{aligned}
&\sum_{k=0}^{m-1} \sum_{\ell=0}^{m-1} \binom{k}{\ell}\binom{m-k-1}{\ell} |c|^{2\ell} z^k + \sum_{k=1}^{m} \sum_{\ell=0}^{m} \binom{k-1}{\ell}\binom{m-k}{\ell} |c|^{2\ell} z^k \\
&\quad - \sum_{k=1}^{m-1} \sum_{\ell=0}^{m-1} \binom{k-1}{\ell}\binom{m-k-1}{\ell} |c|^{2\ell} z^k + \sum_{k=1}^{m-1} \sum_{\ell=1}^{m} \binom{k-1}{\ell-1}\binom{m-k-1}{\ell-1} |c|^{2\ell} z^k \\
&= \sum_{k=0}^{m} \sum_{\ell=0}^{m} \binom{k}{\ell}\binom{m-k}{\ell} |c|^{2\ell} z^k,
\end{aligned}
$$

because easy calculation confirms that

$$
\begin{aligned}
&\binom{k}{\ell}\binom{m-k-1}{\ell} + \binom{k-1}{\ell}\binom{m-k}{\ell} - \binom{k-1}{\ell}\binom{m-k-1}{\ell} \\
&+ \binom{k-1}{\ell-1}\binom{m-k-1}{\ell-1} = \binom{k}{\ell}\binom{m-k}{\ell}.
\end{aligned}
$$

This gives the correct multiple of $(c-1)$ in $\phi_{m+1}(z)$. Similar calculation applies to the remaining terms and confirms that our polynomial obeys the recurrence relation. Since $\phi_0(z) \equiv 1$ and $\phi_1(z) = z + c$, it follows that we have recovered Geronimus polynomials.

Further, note that

$$
\sum_{\ell=0}^{\min\{k,m-k\}} \binom{k}{\ell}\binom{m-k}{\ell} x^\ell = \sum_{\ell=0}^{\infty} \frac{(-k)_\ell(-m+k)_\ell}{\ell!^2} x^\ell = {}_2F_1\left[\begin{matrix} -k, -m+k; \\ 1; \end{matrix} \; x\right].
$$

Assume first that $m \le 2k$. According to the Euler identity for hypergeometric functions

$$
{}_2F_1\left[\begin{matrix} -k, -m+k; \\ 1; \end{matrix} \; x\right] = (1-x)^2 {}_2F_1\left[\begin{matrix} k+1, -m+k; \\ 1; \end{matrix} \; \frac{x}{x-1}\right],
$$

(Rainville 1960, p. 60), while a classical representation of Jacobi polynomials is

$$
\mathrm{P}_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!} {}_2F_1\left[\begin{matrix} -n, n+\alpha+\beta+1; \\ \alpha+1; \end{matrix} \; \frac{1-z}{2}\right]
$$

(Rainville 1960, p. 254). Letting $n = m - k \geq 0$, $\alpha = 0$, $\beta = 2k - m \geq 0$ and $z = (1+x)/(1-x)$, we thus have

$$\sum_{\ell=0}^{\min\{k,m-k\}} \binom{k}{\ell}\binom{m-k}{\ell} x^\ell = (1-x)^{m-k} \mathrm{P}_{m-k}^{(0,2k-m)}\left(\frac{1+x}{1-x}\right),$$

while in the case $m \geq 2k$, by symmetry,

$$\sum_{\ell=0}^{\min\{k,m-k\}} \binom{k}{\ell}\binom{m-k}{\ell} x^\ell = (1-x)^{k} \mathrm{P}_{k}^{(0,m-2k)}\left(\frac{1+x}{1-x}\right).$$

In other words,

$$\sum_{\ell=0}^{\min\{k,m-k\}} \binom{k}{\ell}\binom{m-k}{\ell} x^\ell = (1-x)^{\min\{k,m-k\}} \mathrm{P}_{\min\{k,m-k\}}^{(0,|m-2k|)}\left(\frac{1+x}{1-x}\right)$$

for $k = 0, \ldots, m$. We thus deduce that Geronimus polynomials can be written explicitly in the form

$$\begin{aligned}\phi_m(z) = &\sum_{k=0}^{m} (1-|c|^2)^{\min\{k,m-k\}} \mathrm{P}_{\min\{k,m-k\}}^{(0,|m-2k|)}\left(\frac{1+|c|^2}{1-|c|^2}\right) z^k \\ &+ (c-1)\sum_{k=0}^{m-1} (1-|c|^2)^{\min\{k,m-1-k\}} \mathrm{P}_{\min\{k,m-1-k\}}^{(0,|m-1-2k|)}\left(\frac{1+|c|^2}{1-|c|^2}\right) z^k\end{aligned} \tag{2.7}$$

for all $m \in \mathbb{Z}_+$. Letting

$$\varphi_m(z,x) = \sum_{k=0}^{m} (1-x)^{\min\{k,m-2k\}} \mathrm{P}_{\min\{k,m-k\}}^{(0,|m-2k|)}\left(\frac{1+x}{1-x}\right) z^k, \qquad m \in \mathbb{Z}_+,$$

we deduce from (2.7) that

$$\phi_m(z) = \varphi_m(z,|c|^2) + (c-1)z\varphi_{m-1}(z,|c|^2), \qquad m \in \mathbb{Z}_+. \tag{2.8}$$

Since the coefficients of $\varphi_m(z,|c|^2)$ are real and palindromic (the $(m-\ell)$th coefficient is the same as the ℓth coefficient), we deduce at once that

$$\varphi_m(z,x)^* = \varphi_m(z,x), \qquad m \in \mathbb{Z}_+, \quad x \in \mathbb{R},$$

therefore

$$\phi_m^*(z) = \varphi_m(z,|c|^2) + (c-1)z\varphi_m(z,|c|^2).$$

For $c = 1$ (Rogers–Szegő polynomials) it is more convenient to use (2.5), whereby

$$\phi_m(z) = \sum_{k=0}^{m} \sum_{\ell=0}^{\min\{k,m-k\}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} m-k \\ \ell \end{bmatrix}_q q^{\ell^2} \alpha^{m-k} z^k. \tag{2.9}$$

It is known that, given $\alpha = -q^{1/2}$, Rogers–Szegő polynomials can be represented in the form

$$\phi_m(z) = \sum_{k=0}^{m} (-1)^{m-k} \left[\begin{array}{c} m \\ k \end{array} \right]_q q^{(m-k)/2} z^k$$

(Simon 2005). To confirm that (2.9) reduces to this form, we need to prove that

$$\sum_{\ell=0}^{\min\{k,m-k\}} \left[\begin{array}{c} k \\ \ell \end{array} \right]_q \left[\begin{array}{c} m-k \\ \ell \end{array} \right]_q q^{\ell^2} = \left[\begin{array}{c} m \\ k \end{array} \right]_q, \qquad k = 0, \ldots, m. \tag{2.10}$$

Lemma 2.1 *The identity (2.10) is true for all $k, m \in \mathbb{Z}_+$.*

Proof Since

$$\frac{(q,q)_n}{(q,q)_{n-\ell}} = (-1)^\ell q^{n\ell - \frac{1}{2}\ell(\ell-1)} (q^{-n}, q)_\ell, \qquad \ell = 0, \ldots, n,$$

and bearing in mind that $\ell \geq \max\{k, m-k\} + 1$ implies $(q^{-k}, q)_l = 0$, we deduce that

$$\begin{aligned} \sum_{\ell=0}^{\min\{k,m-k\}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} m-k \\ \ell \end{bmatrix}_q q^{\ell^2} &= \sum_{\ell=0}^{\min\{k,m-k\}} \frac{(q^{-k},q)_\ell (q^{-m+k},q)_\ell}{[(q,q)_\ell]^2} q^{(m+1)\ell} \\ &= \sum_{\ell=0}^{\infty} \frac{(q^{-k},q)_\ell (q^{-m+k},q)_\ell}{[(q,q)_\ell]^2} q^{(m+1)\ell} \\ &= {}_2\phi_1 \left[\begin{array}{l} q^{-k}, q^{-m+k}; \\ q; \end{array} q, q^{m+1} \right]. \end{aligned}$$

We use now the q-Gauss sum

$${}_2\phi_1 \left[\begin{array}{l} a, b; \\ c; \end{array} q, \frac{c}{ab} \right] = \frac{(c/a, q)_\infty (c/b)_\infty}{(c, q)_\infty (c/(ab), q)_\infty}$$

(Gasper & Rahman 2004, p. 236) with $a = q^{-k}$, $b = q^{-m+k}$, $c = q$ and the outcome is

$$\sum_{\ell=0}^{\min\{k,m-k\}} \left[\begin{array}{c} k \\ \ell \end{array} \right]_q \left[\begin{array}{c} m-k \\ \ell \end{array} \right]_q q^{\ell^2} = \frac{(q^{k+1},q)_\infty (q^{m-k+1},q)_\infty}{(q,q)_\infty (q^{m+1},q)_\infty}.$$

The identity (2.10) follows from the identities

$$\frac{(q^{k+1},q)_\infty}{(q,q)_\infty} = \frac{1}{(q,q)_k}, \qquad \frac{(q^{m-k+1},q)}{(q^{m+1},q)_\infty} = \frac{(q,q)_m}{(q,q)_{m-k}},$$

□

We deduce that the Rogers–Szegő case is recovered asymptotically, as required.

To complete our analysis of extremal cases, we finally consider $c = 0$, i.e. Lebesgue polynomials. The only surviving term from (2.5) is $k = m$ in the second sum, hence $\phi_m(z) = z^m$. This complete the analysis.

3 Generating functions and functional-differential equations

In this section we seek an alternative generating function, of the form

$$\Phi(t,z)=\sum_{n=0}^{\infty}\frac{\phi_n(z)}{n!}t^n.$$

For brevity we will often suppress the dependence of Φ upon z, in which case we write $\Phi=\Phi(t)$.

Multiplying (1.3) by $t^n/n!$ and summing up for $n=1,2,\ldots$ results in

$$(\alpha+z)\sum_{n=1}^{\infty}\frac{\phi_n}{n!}t^n=\sum_{n=1}^{\infty}\frac{\phi_{n+1}}{n!}t^n+\alpha z\sum_{n=1}^{\infty}\frac{\phi_{n-1}}{n!}t^n-\alpha|c|^2z\sum_{n=1}^{\infty}\frac{\phi_{n-1}}{n!}|\alpha|^{2n}t^n.$$

However,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\phi_n}{n!}t^n&=\Phi(t)-\Phi(0),\\ \sum_{n=1}^{\infty}\frac{\phi_{n+1}}{n!}t^n&=\frac{\partial}{\partial t}\sum_{n=2}^{\infty}\frac{\phi_n}{n!}t^n=\frac{\partial}{\partial t}[\Phi(t)-\Phi(0)-\Phi'(0)t]=\Phi'(t)-\Phi'(0),\\ \sum_{n=1}^{\infty}\frac{\phi_{n-1}}{n!}t^n&=\sum_{n=0}^{\infty}\frac{\phi_n}{(n+1)!}t^{n+1}=\int_0^t\Phi(x)\,\mathrm{d}x,\\ \sum_{n=1}^{\infty}\frac{\phi_{n-1}}{n!}|\alpha|^{2n}t^n&=\sum_{n=0}^{\infty}\frac{\phi_n}{(n+1)!}(|\alpha|^2t)^{n+1}=\int_0^{|\alpha|^2t}\Phi(x)\,\mathrm{d}x\end{aligned}$$

and, putting all this together,

$$(\alpha+z)[\Phi(t)-\Phi(0)]=\Phi'(t)-\Phi'(0)+\alpha z\int_o^t\Phi(x)dx-\alpha z|c|^2\int_0^{|\alpha|^2t}\Phi(x)\,\mathrm{d}x.$$

We differentiate this expression with respect to t, whence

$$(\alpha+z)\Phi'(t)=\Phi''(t)+\alpha z\Phi(t)-\alpha|\alpha|^2|c|^2z\Phi(|\alpha|^2t).$$

We rewrite this *functional differential equation* in the form

$$\Phi''(t)=(\alpha+z)\Phi'(t)-\alpha z\Phi(t)+\alpha\tau z\Phi(qt), \tag{3.1}$$

where $q=|\alpha|^2$, $\tau=q|c|^2$ are both in $(0,1)$, with the initial conditions $\Phi(0)=\phi_0(z)\equiv 1$, $\Phi'(0)=\phi_1(z)=z+c\alpha$.

The equation (3.1) is a special instance of the *pantograph equation*

$$\boldsymbol{y}'(t)=A\boldsymbol{y}(t)+B\boldsymbol{y}(qt),\qquad t\ge 0,\qquad \boldsymbol{y}(0)=\boldsymbol{y}_0\in\mathbb{C}^d, \tag{3.2}$$

where A and B are $d\times d$ complex matrices and $q\in(0,1)$ (Iserles 1993). It is known that (3.2) has a unique solution for all $t\in[0,\infty)$ and that, as long as the eigenvalues of A reside in the open left complex half-plane and the eigenvalues of $A^{-1}B$ in the open complex unit disc, it is true that $\lim_{t\to\infty} \boldsymbol{y}(t) = \boldsymbol{0}$. Moreover, as long as A is nonsingular and the spectral radius of $A^{-1}B$ is less that one, the solution of (3.2) can be expanded into Dirichlet series (Iserles 1993). This has a profound implications to our study of the generating function Φ.

Just to verify that we are on the right track, we note that for $z = 0$ the pantograph reduces to the linear differential equation

$$\Phi''(t) = \alpha\Phi'(t), \qquad t \geq 0, \qquad \Phi(0) = 1, \qquad \Phi'(0) = c\alpha,$$

witht the solution $\Phi(t) = c\mathrm{e}^{\alpha t} + 1 - c$. Therefore

$$a_n = \Phi^{(n)}(0) = \begin{cases} 1, & n = 0, \\ c\alpha^n, & n \in \mathbb{N}, \end{cases}$$

as required.

It is instructive to examine (3.1) in the three important special cases already mentioned. For the Lebesgue case $\alpha = c = 0$ the equation reduces to $\Phi''(t) = z\Phi'(t)$, with the initial conditions $\Phi(0) = 1$, $\Phi'(0) = z$, therefore $\Phi(z,t) = \mathrm{e}^{tz}$ and we recover $\phi_n(z) = z^n$, $n \in \mathbb{Z}_+$. For Geronimus polynomials $\alpha = 1$, hence $q = 1$, $\tau = |c|^2$, and (3.2) reduces to an ordinary differential equation

$$\Phi'' - (1+z)\Phi' + (1-|c|^2)z\Phi = 0, \qquad t \geq 0,$$

whose general solution is $\Phi(t) = \beta_+\mathrm{e}^{t\varrho_+} + \beta_-\mathrm{e}^{t\varrho_-}$ where

$$\varrho_\pm = \frac{1 + z \pm \sqrt{(1-z)^2 + 4|c|^2 z}}{2}$$

are the roots of the quadratic $\varrho^2 - (1+z)\varrho + (1-|c|^2)z = 0$. Fitting the initial conditions $\Phi(0) = 1$, $\Phi'(0) = z + c$, we have

$$\beta_\pm = \frac{1}{2} \pm \frac{(1-z) - 2c}{\sqrt{(1-z)^2 + 4|c|^2 z}}.$$

This results in the known representation of Geronimus polynomials, namely

$$\begin{aligned} \phi_n(z) = {} & \left[\frac{1}{2} - \frac{(1-z)-2c}{\sqrt{(1-z)^2+4|c|^2z}}\right]\left[\frac{1+z+\sqrt{(1-z)^2+4|c|^2z}}{2}\right]^n \\ & + \left[\frac{1}{2} + \frac{(1-z)-2c}{\sqrt{(1-z)^2+4|c|^2z}}\right]\left[\frac{1+z-\sqrt{(1-z)^2+4|c|^2z}}{2}\right]^n, \qquad n \in \mathbb{Z}_+ \end{aligned} \tag{3.3}$$

(Simon 2005, p. 87). This representation of an orthogonal polynomials system using 'non-polynomial' building blocks is similar in this sense to the familiar formula for Chebyshev polynomials of first and second kind (Rainville 1960, p. 301).

Finally, in the Rogers–Szegő case $c = 1$, $\alpha = -q^{1/2}$, we stay with a pantograph equation, specifically

$$\Phi''(t) = (z - q^{1/2})\Phi'(t) + q^{1/2}z\Phi(t) - q^{3/2}z\Phi(qt), \qquad t \geq 0 \tag{3.4}$$

with $\Phi(0) = 1$, $\Phi'(0) = z - q^{1/2}$.

Our next step is to study the solution of (3.1) in order to obtain a general expression for the ϕ_ns. This is the subject matter of the next section. In the sequel we apply our results and analyse the limiting cases of Geronimus and Rogers–Szegő polynomials from the point of view of this section, i.e. commencing from the pantograph equation (3.1).

3.1 Study of the solutions through Dirichlet series

Provided that the pantograph equation (3.2) is in a stable regime, its solution can be expanded into Dirichlet series,

$$\boldsymbol{y}(t) = \sum_{m=0}^{\infty} \mathrm{e}^{tq^m A} v_m, \qquad t \geq 0, \tag{3.5}$$

(Iserles 1993).

The general pantograph equation

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + B\boldsymbol{y}(qt), \qquad t \geq 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0 \in \mathbb{C}^d,$$

has a Dirichlet solution provided that A is invertible and $\|A^{-1}B\|_2 < 1$. In our case, we have

$$\boldsymbol{y} = \begin{bmatrix} \Phi \\ \Phi' \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ -\alpha z & \alpha + z \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ \alpha z \tau & 0 \end{bmatrix},$$

therefore

$$A^{-1}B = \tau \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of A are α and z, both nonzero, hence the matrix is nonsingular, while the spectral radius of $A^{-1}B$ is $\tau \in [0, 1)$. Consequently, the solution of (3.1) can be expanded into a Dirichlet series of the form (3.5). Specifically, Φ possesses the expansion

$$\Phi(t) = \sum_{m=0}^{\infty} v_m \mathrm{e}^{\lambda q^m t}, \qquad v_0 \neq 0,$$

where λ and $\{v_m\}_{m \in \mathbb{Z}_+}$ are independent of t (the variable z is treated as a parameter). Substituting this into (3.1), we have

$$\lambda^2 \sum_{m=0}^{\infty} v_m q^{2m} \mathrm{e}^{\lambda q^m t} = (\alpha+z)\lambda \sum_{m=0}^{\infty} v_m q^m \mathrm{e}^{\lambda q^m t} - \alpha z \sum_{m=0}^{\infty} v_m \mathrm{e}^{\lambda q^m t} + \alpha\tau z \sum_{m=0}^{\infty} v_{m-1} \mathrm{e}^{\lambda q^m t}.$$

Assuming $\lambda \neq 0$, the functions $\mathrm{e}^{\lambda q^m t}$ are linearly independent for all $m \in \mathbb{Z}$, therefore it follows that

$$[\lambda^2 q^{2m} - (\alpha + z)q^m + \alpha z]v_m = \begin{cases} 0, & m = 0, \\ \alpha z \tau v_{m-1}, & m \in \mathbb{N}. \end{cases} \tag{3.6}$$

An immediate consequence of (3.6) is that, letting $m=0$, $v_0 \neq 0$ implies

$$\lambda^2-(\alpha+z)\lambda+\alpha z=(\lambda-\alpha)(\lambda-z)=0$$

and we deduce that there exist two admissible values of λ, namely $\lambda=\alpha$ and $\lambda=z$.

Next, we consider the case $m\in\mathbb{Z}$. Now

$$(\alpha-q^m\lambda)(z-q^m\lambda)v_m=\alpha z\tau v_{m-1},$$

therefore

$$\begin{aligned}\lambda=\alpha\ : &\qquad (1-q^m)\left(1-q^m\frac{\alpha}{z}\right)v_m=\tau v_{m-1},\\ \lambda=z\ : &\qquad (1-q^m)\left(1-q^m\frac{z}{\alpha}\right)v_m=\tau v_{m-1}.\end{aligned}$$

Using easy induction, we have

$$v_m=\frac{\tau^m}{(q,q)_m(\alpha/z,q)_m}v_0 \qquad \text{and} \qquad v_m=\frac{\tau^m}{(q,q)_m(z/\alpha,q)_m}v_0$$

respectively, where, as before, $(\kappa,q)_m$ is the Gauss–Heine symbol,

$$(\kappa,q)_m=\prod_{j=0}^{m-1}(1-\kappa q^j)$$

(Gasper & Rahman 2004). This argument has led us to a Dirichlet-series representation of Φ, formulated in the following theorem.

Theorem 3.1 *The generating function $\Phi(t,z)=\sum_{m=0}^{\infty}\phi_m(z)t^m/m!$ of the OPUC with respect to the Schur parameters (1.2) can be expressed explicitly in the form*

$$\Phi(t,z)=\beta_1(z)\sum_{m=0}^{\infty}\frac{\tau^m}{(q,q)_m(\alpha/z,q)_m}\mathrm{e}^{\alpha q^m t}+\beta_2(z)\sum_{m=0}^{\infty}\frac{\tau^m}{(q,q)_m(z/\alpha,q)_m}\mathrm{e}^{zq^m t}, \quad (3.7)$$

where β_1 and β_2 are determined by the conditions $\Phi(0,z)\equiv 1$, $\partial\Phi(0,z)/\partial t=z+c\alpha$.

Corollary 3.2 *The monic OPUC with respect to the Schur parameters (1.2) is*

$$\phi_m(z)=\alpha^m\beta_1(z)F(\alpha z^{-1},q^m\tau,q)+z^m\beta_2(z)F(\alpha^{-1}z,q^m\tau,q), \qquad m\in\mathbb{Z}_+, \quad (3.8)$$

where

$$F(\zeta,\tau,q)=\sum_{m=0}^{\infty}\frac{\tau^m}{(q,q)_m(\zeta,q)_m}. \quad (3.9)$$

Proof Repeatedly differentiating the Dirichlet series (3.7) term-by-term, a procedure which is justified by its absolute convergence. □

Note that $F(0,\tau,q)$ is the so-called "little q-exponential function",

$$F(0,\tau,q)=\sum_{m=0}^{\infty}\frac{\tau^m}{(q,q)_m}=\mathrm{e}_q(\tau)=\frac{1}{(\tau,q)_\infty}$$

(Gasper & Rahman 2004, p. 236), while

$$\lim_{|\zeta|\to\infty} F(\zeta, q^m\tau, q) \equiv 1, \qquad \lim_{|\zeta|\to 0} \zeta^m F(\zeta, q^m\tau, q) = \begin{cases} 1, & m = 0, \\ 0, & m \in \mathbb{N}, \end{cases}$$

where $|\zeta| \to \infty$ in a sector of the form $|\arg\zeta| > \delta$ for some $\delta > 0$. Therefore

$$\phi_0(0) = \beta_1(0) + \beta_2(0), \qquad \phi_n(0) = \beta_1(0)\alpha^n, \qquad n \in \mathbb{N},$$

where we recall that β_1 and β_2 are determined by the initial conditions,

$$\phi_0(z) \equiv 1, \qquad \phi_1(z) = z + c\alpha.$$

Thus, $\beta_1(0) = c$, $\beta_2(0) = 1 - c$, and we verify from (3.8) the explicit form of Schur parameters,

$$\phi_n(0) = c\alpha^n, \qquad n \in \mathbb{Z}_+.$$

It is convenient to reformulate (3.8) somewhat. Thus, we let

$$\eta_1(z) = \beta_1(z)F(\alpha z^{-1}, \tau, q), \qquad \eta_2(z) = \beta_2(z)F(\alpha^{-1}z, \tau, q)$$

and

$$H_m(\zeta, \tau, q) = \frac{F(\zeta, q^m\tau, q)}{F(\zeta, \tau, q)}, \qquad m \in \mathbb{Z}_+. \tag{3.10}$$

Then (3.8) can be rewritten in the form

$$\phi_m(z) = \alpha^m\eta_1(z)H_m(\alpha z^{-1}, \tau, q) + z^m\eta_2(z)H_m(\alpha^{-1}z, \tau, q), \qquad m \in \mathbb{Z}_+. \tag{3.11}$$

The initial conditions being

$$\begin{aligned} \eta_1 + \eta_2 &= 1, \\ \alpha H_1(\alpha z^{-1}, \tau, q)\eta_1 + zH_1(\alpha^{-1}z, \tau, q)\eta_2 &= z + c\alpha, \end{aligned}$$

we obtain

$$\begin{aligned} \eta_1(z) &= \frac{z + c\alpha - zH_1(\alpha^{-1}z, \tau, q)}{\alpha H_1(\alpha z^{-1}, \tau, q) - zH_1(\alpha^{-1}z, \tau, q)}, \\ \eta_2(z) &= \frac{\alpha H_1(\alpha z^{-1}, \tau, q) - z - c\alpha}{\alpha H_1(\alpha z^{-1}, \tau, q) - zH_1(\alpha^{-1}z, \tau, q)})). \end{aligned} \tag{3.12}$$

The representation (3.11) is not the final form in which we can cast the OPUC $\{\phi_n\}_{n\in\mathbb{Z}_+}$.

Theorem 3.3 *The explicit form of the OPUC with respect to the Schur parameters (1.2) is*

$$\phi_m(z) = \alpha^m\eta_1(z)\prod_{\ell=1}^{m} H_1(\alpha z^{-1}, q^\ell\tau, q) + z^m\eta_2(z)\prod_{\ell=1}^{m} H_1(\alpha^{-1}z, q^\ell\tau, q), \qquad m \in \mathbb{Z}_+, \tag{3.13}$$

where η_1 and η_2 have been given in (3.12).

Proof Follows at once from (3.11), noting that

$$H_m(\zeta,\tau,q) = \frac{F(\zeta,q\tau,q)}{F(\zeta,\tau,q)} \times \frac{F(\zeta,q^2\tau,q)}{F(\zeta,q\tau,q)} \times \cdots \times \frac{F(\zeta,q^m\tau,q)}{F(\zeta,q^{m-1}\tau,q)} = \prod_{\ell=1}^{m} H_1(\zeta,q^\ell\tau,q).$$

□

The representation (3.13) has an important advantage in comparison with the seemingly simpler form (3.11): we need to deal with just a single function H_1, rather than with H_m for all $m \in \mathbb{N}$. It is also reminiscent of the representation (3.3) of Geronimus polynomials and indeed we will prove in the sequel that (3.13) reduces to (3.3) as $\alpha \to 1$.

3.2 The function H_1

3.2.1 Analiticity

It will be proved later in this section that the function

$$F(\zeta,\tau,q) = \sum_{m=0}^{\infty} \frac{\tau^m}{(q,q)_m(\zeta,q)_m}, \qquad \tau \in (0,q),$$

is meromorphic in $\zeta \in \mathbb{C}$. Specifically, it is analytic except for simple polar singularityes at $q^{-\ell}$ for all $\ell \in \mathbb{Z}_+$, because $(q^{-\ell},q)_m = 0$ for $m \geq \ell+1$. Our interest is, however, not in the function F *per se* but in the ratios $H_m(\alpha z^{-1},\tau,q)$ and $H_m(\alpha^{-1}z,\tau,q)$ for $m \in \mathbb{N}$.

Let $\zeta = q^{-\ell} + \varepsilon$ for some $\ell \in \mathbb{Z}_+$ and $0 < |\varepsilon| \ll 1$. It is an easy calculation that

$$(\zeta,q)_m = (-1)^m q^{(m-1-\ell)m} \frac{(q,q)_\ell}{(q,q)_{\ell-m}} + \mathcal{O}(\varepsilon), \qquad m \leq \ell,$$
$$(\zeta,q)_m = (-1)^{\ell+1} \varepsilon q^{\frac{1}{2}(\ell-1)\ell} (q,q)_\ell (q,q)_{m-\ell-1} + \mathcal{O}(\varepsilon^2), \qquad m \geq \ell+1.$$

Therefore, after further algebra,

$$F(q^{-\ell}+\varepsilon,\tau,q) = \frac{1}{\varepsilon} \frac{(-1)^{\ell+1} q^{\frac{1}{2}(\ell-1)\ell} \tau^{\ell+1}}{(q,q)_\ell (q,q)_{\ell+1}} F(q^{\ell+2},\tau,q) + \mathcal{O}(1).$$

We deduce that

$$H_1(q^{-\ell}+\varepsilon,\tau,q) = q^{\ell+1} H_1(q^{\ell+2},\tau,q) + \mathcal{O}(\varepsilon).$$

Therefore the singularity at $q^{-\ell}$ is removable. This, however, does not mean that H_1, unlike F, is an entire function, because it has polar singularities at the zeros of $F(\cdot,\tau,q)$. Indeed, we demonstrate in the sequel that H_1 is meromorphic, with a countable number of isolated poles accumulating at infinity.

Note that, according to Section 3.1, $\lim_{|\zeta|\to\infty} F(\zeta,\tau,q) = 1$ *as long as* ζ *is restricted to a sector of the form* $|\arg\zeta| > \delta > 0$. Hence, subject to this restriction, $\lim_{|\zeta|\to\infty} H_1(\zeta,\tau,q) = 1$, while $\lim_{M\to\infty} H_1(q^{-M},\tau,q) = 0$.

3.2.2 An expansion in ζ

The function F has been given as a power series in τ. However, and given its analyticity in ζ, it is instructive to expand it in power series in the latter variable. We commence by observing that

$$\begin{aligned} F(\zeta,\tau,q) - F(q\zeta,\tau,q) &= \sum_{m=1}^{\infty} \frac{\tau^m}{(q,q)_m}\left[\frac{1}{(\zeta,q)_m} - \frac{1}{(q\zeta,q)_m}\right] \\ &= \frac{\zeta}{1-\zeta}\sum_{m=0}^{\infty}\frac{(1-q^m)\tau^m}{(q,q)_m(q\zeta,q)_m} \\ &= \frac{\zeta}{1-\zeta}[F(q\zeta,\tau,q) - F(q\zeta,q\tau,q)]. \end{aligned}$$

This results in the recurrence relation

$$F(\zeta,\tau,q) = \frac{1}{1-\zeta}[F(q\zeta,\tau,q) - \zeta F(q\zeta,q\tau,q)]. \tag{3.14}$$

Before we advance any further, it is useful to recall the definition of an ${}_r\phi_s$ *basic hypergeometric function*: given $r,s \in \mathbb{Z}_+$ and $q, a_1,\ldots,a_r,b_1,\ldots,b_s \in \mathbb{C}$, $|q|<1$,

$${}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r; \\ b_1,\ldots,b_s; \end{matrix}\; q,z\right] = \sum_{m=0}^{\infty}\frac{(a_1,q)_m(a_2,q)_m\cdots(a_r,q)_m}{(q,q)_m(b_1,q)_m(b_2,q)_m\cdots(b_s,q)_m}\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-r} z^m$$

(Gasper & Rahman 2004, p. 4).

Proposition 3.4 *The function F can be expressed in the form*

$$\begin{aligned} F(\zeta,\tau,q) &= \frac{1}{(\zeta,q)_\infty(\tau,q)_\infty}\sum_{m=0}^{\infty}\frac{(\tau,q)_m}{(q,q)_m}q^{\frac12(m-1)m}(-\zeta)^m \\ &= \frac{1}{(\zeta,q)_\infty(\tau,q)_\infty}\,{}_1\phi_1\left[\begin{matrix}\tau; \\ 0;\end{matrix}\, q,\zeta\right]. \end{aligned} \tag{3.15}$$

Proof We commence by proving that, for any $r \in \mathbb{Z}_+$,

$$F(\zeta,\tau,q) = \frac{1}{(\zeta,q)_r}\sum_{m=0}^{r}\left[\begin{matrix} r \\ m\end{matrix}\right]_q q^{\frac12(m-1)m}(-\zeta)^m F(q^r\zeta,q^m\tau,q),$$

where we recall that

$$\left[\begin{matrix} n \\ m\end{matrix}\right]_q = \frac{(q,q)_n}{(q,q)_m(q,q)_{n-m}}, \qquad 0 \le m \le n,$$

is the *q- binomial symbol* (Gasper & Rahman 2004, P. 235).

This is certainly true for $r = 0$ and, because of (3.14), for $r = 1$. Moreover, using induction on r and applying (3.14) on the right-hand side,

$$\begin{aligned} &F(\zeta, \tau, q) \\ &= \frac{1}{(\zeta, q)_r} \sum_{m=0}^{r} \begin{bmatrix} r \\ m \end{bmatrix}_q q^{\frac{1}{2}(m-1)} \frac{(-\zeta)^m}{(1 - q^r\zeta)} [F(q^{r+1}\zeta, q^m\tau, q) - q^r\zeta F(q^{r+1}\zeta, q^{m+1}\tau, q)] \\ &= \frac{1}{(\zeta, q)_{r+1}} \sum_{m=0}^{r} \begin{bmatrix} r \\ m \end{bmatrix}_q q^{\frac{1}{2}(m-1)m} (-\zeta)^m F(q^{r+1}\zeta, q^m\tau, q) \\ &\quad + \frac{1}{(\zeta, q)_{r+1}} \sum_{m=1}^{r+1} \begin{bmatrix} r \\ m-1 \end{bmatrix}_q q^{\frac{1}{2}(m-2)(m-1)+r} (-\zeta)^m F(q^{r+1}\zeta, q^m\tau, q) \end{aligned}$$

and the desired expression follows from the identity

$$\begin{bmatrix} r \\ m \end{bmatrix}_q + q^{r-m+1} \begin{bmatrix} r \\ m-1 \end{bmatrix}_q = \begin{bmatrix} r+1 \\ m \end{bmatrix}_q$$

(Gasper & Rahman 2004, p. 235).

To prove (3.15), we let $r \to \infty$, noting that for every fixed m

$$\lim_{r\to\infty} \begin{bmatrix} r \\ m \end{bmatrix}_q = \frac{1}{(q, q)_m}$$

and that

$$\lim_{r\to\infty} F(q^r\zeta, \tau, q) = F(0, \tau, q) = \sum_{m=0}^{\infty} \frac{\tau^m}{(q, q)_m} = \mathrm{e}_q(\tau) = \frac{1}{(\tau, q)_\infty}$$

(cf. Section 2). □

Using (3.15), we investigate the analyticity of H_1. Our point of departure is the observation that

$$F(\zeta, \tau, q) = \frac{G(\zeta, \tau, q)}{(\tau, q)_\infty (\zeta, q)_\infty},$$

where

$$G(\zeta, \tau, q) = \sum_{m=0}^{\infty} (-1)^m \frac{(\tau, q)_m}{(q, q)_m} q^{\frac{1}{2}(m-1)m} \zeta^m.$$

It is obvious that G is an entire function of ζ.

Given an entire function $f(\zeta) = \sum_{m=0}^{\infty} f_m \zeta^m$, its *order* is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log \max_{-\pi\le\theta\le\pi} |f(r\mathrm{e}^{i\theta})|}{\log r}$$

cite[p. 182]hille62aft and, while at the first instance it describes the behaviour near the singularity at ∞, it can be used to reveal many other interesting features. An alternative expression for $\rho(f)$ is

$$\rho(G) = \limsup_{m\to\infty} \frac{m \log m}{\log |f_m|^{-1}}$$

(Hille 1962, p. 186). Therefore

$$\rho(G) = \limsup_{m\to\infty} \frac{m\log m}{\log(q,q)_m - \log(\tau,q)_m - \frac{1}{2}(m-1)m|\log q|} = 0.$$

Since $\rho(G) = 0$, $G(0,\tau,q) = 1$ and G is clearly not a polynomial in ζ (recall that $\tau < 1$), we use the *Hadamard factorization theorem* (Hille 1962) to argue that it can be represented in the form

$$G(\zeta,\tau,q) = \prod_{n=1}^{\infty}\left(1-\frac{\zeta}{\sigma_n}\right), \qquad \zeta\in\mathbb{C},$$

where $\sigma_n = \sigma_n(\tau,q) \in \mathbb{C}$ accumulate at ∞. We deduce that

$$H_1(\zeta,\tau,q) = \frac{(\tau,q)_\infty}{(q\tau,q)_\infty}\frac{G(\zeta,q\tau,q)}{G(\zeta,\tau,q)} = (1-\tau)\prod_{n=1}^{\infty}\frac{1-\zeta/\sigma_n(q\tau,q)}{1-\zeta/\sigma_n(\tau,q)}. \tag{3.16}$$

In particular, this indeed proves that H_1 is meromorphic.

The only possible impediment to the analyticity of H_1 are the poles, i.e., the zeros of $G(\cdot,\tau,q)$. However,

$$G(\zeta,0,q) = \sum_{m=0}^{\infty}(-1)^m\frac{q^{\frac{1}{2}(m-1)m}}{(q,q)_m}\zeta^m = \mathrm{E}_q(-\zeta) = (\zeta,q)_\infty$$

where E_q is the 'big q exponential function' (Gasper & Rahman 2004, p. 236). Therefore, for $\tau = 0$ the only zeros of $G(\cdot,0,q)$ are $q^{-\ell}$, $\ell\in\mathbb{Z}_+$, all positive, distinct and cancelling each other in the quotient H_1.

Next, we compute $G(q^{-\ell},\tau,q)$ for $\ell\in\mathbb{Z}_+$ and $\tau > 0$. To this end we utilise the identity

$$(\tau,q)_m = \sum_{k=0}^{m}(-1)^k\begin{bmatrix} m\\ k\end{bmatrix}_q q^{\frac{1}{2}(k-1)k}\tau^k, \qquad m\in\mathbb{Z}_+,$$

whose inductive proof is trivial and left to the reader. Thus,

$$\begin{aligned}
G(q^{-\ell},\tau,q) &= \sum_{m=0}^{\infty}(-1)^m\frac{q^{\frac{1}{2}(m-1)m}}{(q,q)_m}\sum_{k=0}^{m}(-1)^k\begin{bmatrix} m\\ k\end{bmatrix}_q q^{\frac{1}{2}(k-1)k-m\ell}\tau^k\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{(q,q)_k}q^{\frac{1}{2}(k-1)k}\tau^k\sum_{m=k}^{\infty}(-1)^m\frac{q^{\frac{1}{2}(m-1)m-m\ell}}{(q,q)_{m-k}}\\
&= \sum_{k=0}^{\infty}\frac{q^{(k-1)k-k\ell}}{(q,q)_k}\tau^k\sum_{m=0}^{\infty}(-1)^m\frac{q^{\frac{1}{2}(m-1)m+(k-\ell)m}}{(q,q)_m}\\
&= \sum_{k=0}^{\infty}\frac{q^{(k-1)k-k\ell}}{(q,q)_k}\tau^k\mathrm{e}_q(-q^{k-\ell}) = \sum_{k=0}^{\infty}\frac{q^{(k-1)k-k\ell}}{(q,q)_k}\tau^k(q^{k-\ell},q)_\infty\\
&= (q,q)_\infty\sum_{k=\ell+1}^{\infty}\frac{q^{(k-1)k-k\ell}}{(q,q)_k(q,q)_{k-\ell-1}}\tau^k > 0
\end{aligned}$$

(note that all the series above converge).

Likewise, given $0 < \delta \ll 1$, an identical calculation yields

$$G(q^{-\ell+\delta}, \tau, q) = \sum_{k=0}^{\infty} q^{(k-1)k-k(\ell-\delta)} (q^{k-\ell+\delta}, q)_\infty.$$

Now, while for $k \geq \ell + 1$ it is true that

$$(q^{k-\ell+\delta}, q)_\infty = \frac{(q, q)_\infty}{(q, q)_{k-\ell-1}},$$

for $k = 0, 1, \ldots, \ell$ we obtain

$$(q^{k-\ell+\delta}, q)_\infty = (1 - q^\delta)(-1)^{\ell-k} q^{-\frac{1}{2}(\ell-k-1)(\ell-k)} (q, q)_{\ell-k} (q, q)_\infty [1 + \mathcal{O}(\delta)].$$

Therefore

$$\begin{aligned}(q^{k-\ell+\delta}, q)_\infty = (-1)^{\ell-k}(1 - q^\delta) q^{-\frac{1}{2}(\ell-1)\ell} (q, q)_\infty \sum_{k=0}^{\ell} (-1)^k \tau^k \frac{(q, q)_{\ell-k}}{(q, q)_k} q^{\frac{1}{2}(k-1)k+\delta k} \\ \times [1 + \mathcal{O}(\delta)] + G(q^{-\ell}, \tau, q)[1 + \mathcal{O}(\delta)].\end{aligned}$$

While $G(q^{-\ell}, \tau, q) > 0$, we can render the first sum negative by choosing $\delta > 0$ when ℓ is even, $\delta < 0$ otherwise. Moreover, $G(q^{-\ell}, \tau, q) = \mathcal{O}(\tau^\ell)$, hence small ($\tau \in (0, q)$), while the first sum is $\mathcal{O}(1)$ in τ. We deduce that for every suficiently small $\tau > 0$ and $\ell \in \mathbb{N}$ it is true that $\sigma_{2\ell-1}(\tau) < \sigma_{2\ell}(\tau)$ lie in the interval $(q^{-2\ell+1}, q^{-2\ell})$, the first very near the left endpoint and the second very near right endpoint.

Moreover, while $q^{-\frac{1}{2}(\ell-1)\ell}$ increases very rapidly with ℓ, the others terms depend on ℓ in a fairly weak manner. Therefore we can expect $|\sigma_\ell - q^\ell|$ to decrease very rapidly as ℓ grows, and this is confirmed by numerical computations. On the other hand, the interval $(1, q^{-1})$ is the obvious place where thing are more interesting. For $0 < \tau \ll 1$ two zeros emerge from the endpoints, 'sliding' inwards: numerical calculations confirm that after a short while they may coalesce into a double zero, which subsequently bifurcates into the complex plane as a conjugate pair of zeros.

Fig. 3.1 displays G for two values of q and several values of τ in the first two intervals of the form $[q^{-2\ell}, q^{-2\ell-1}]$. In the first interval in the case $q = \frac{9}{16}$ (the left column), at $\tau = 0$ two zeros emerge at the endpoints of the interval and they travel inwards: for $\tau = \frac{1}{20}$ they have hardly moved but they coalesce very near $\tau = \frac{1}{10}$ (actually, at $\tau \approx 0.09992063019$) and, having moved to complex plane, G is positive throughout the first interval for increasing τ. In the second interval not much happen: again, two zeros emerge from the endpoints and travel inwards, but they do it ever-so-slowly and we are already in the asymptotic regime. For $q = \frac{3}{10}$, however, double zeros persist in the first interval for all $\tau \in (0, q]$, while the situation in the second interval hardly changes.

Although the analysis of the function G is valuable in understanding the behaviour of H_1, it is of interest to convert F into a 'proper' power series in ζ, thereby representing H_1 as a quotient of two power series. To this end we replace $\mathrm{E}_q(\zeta) = 1/(\zeta, q)_\infty$

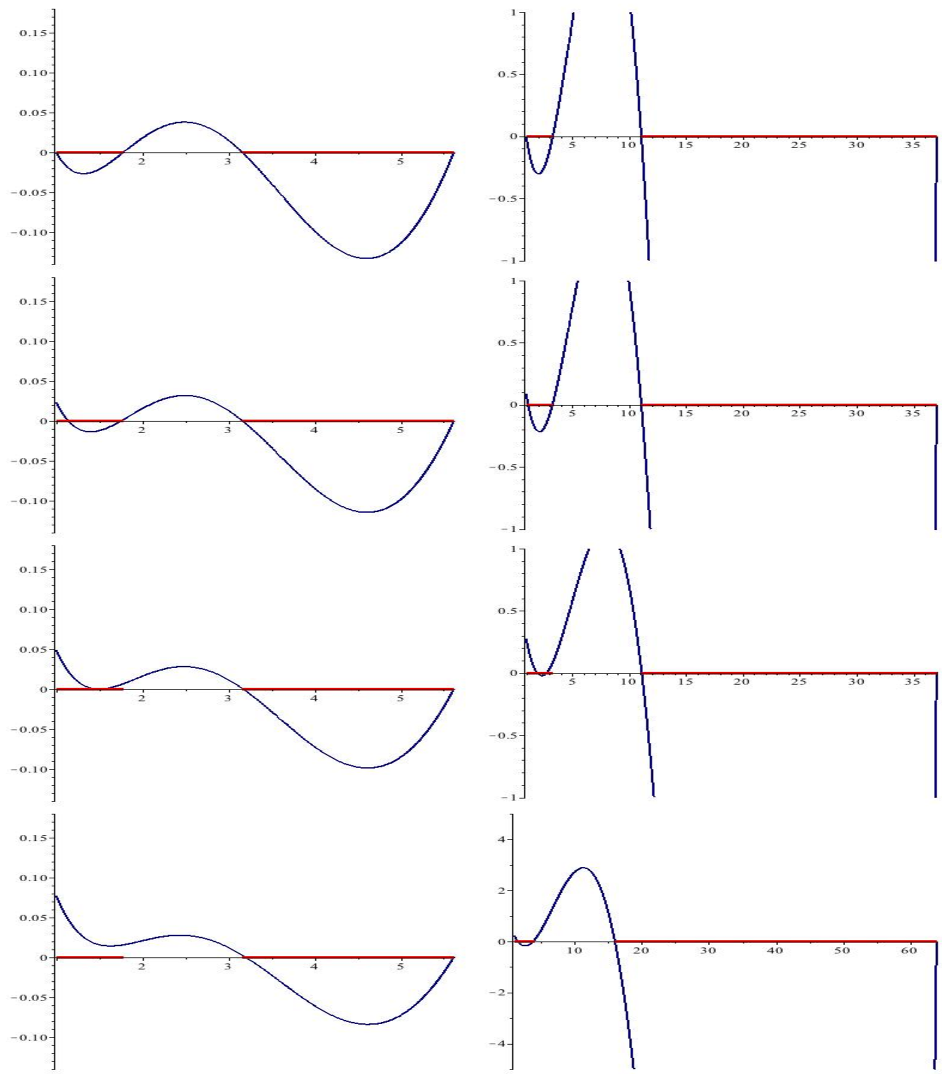

Figure 3.1: The function G in the first two intervals of the form $[q^{-2\ell}, q^{-2\ell-1}]$ (denoted by thick lines) for $q = \frac{9}{16}$ and $\tau = \frac{j}{20}$, $j = 0, 1, 2, 3$ (left column) and $q = \frac{3}{10}$, $\tau = \frac{j}{10}$, $j = 0, 1, 2, 3$.

by its expansion $\sum_{n=0}^{\infty} \zeta^n/(q,q)_n$. It then follows from (3.15) that

$$(\tau, q)_\infty F(\zeta, \tau, q) = \sum_{n=0}^{\infty} \frac{\zeta^n}{(q,q)_n} \sum_{m=0}^{\infty} \frac{(\tau, q)_m}{(q,q)_m} q^{\frac{1}{2}(m-1)m} (-1)^m \zeta^m$$
$$= \sum_{m=0}^{\infty} \frac{(\tau, q)_m}{(q,q)_m} (-1)^m q^{\frac{1}{2}(m-1)m} \sum_{n=m}^{\infty} \frac{\zeta^n}{(q,q)_{n-m}}$$

$$= \sum_{n=0}^{\infty} \frac{\zeta^n}{(q,q)_n} \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{1}{2}(m-1)m} (\tau, q)_m.$$

The outcome is a power-series representation of F,

$$F(\zeta, \tau, q) = \frac{1}{(\tau, q)_\infty} \sum_{n=0}^{\infty} d_n(\tau) \zeta^n,$$

where

$$d_n(\tau) = \frac{1}{(q,q)_n} \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{1}{2}(m-1)m} (\tau, q)_m, \qquad m \in \mathbb{Z}_+. \tag{3.17}$$

Proposition 3.5 *The above coefficients $d_n(\tau)$ satisfy $d_0 \equiv 1$ and*

$$d_n(\tau) = \sum_{m=1}^{n} \frac{q^{(m-1)m}}{(q,q)_m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q \tau^m, \qquad n \in \mathbb{N}. \tag{3.18}$$

Proof

The expressions (3.17) and (3.18) match for $n = 0, 1$. We continue by induction on $n \in \mathbb{N}$. Firstly, using identity I.45 from (Gasper & Rahman 2004, p. 235), we deduce from (3.17) that

$$\begin{aligned} d_n(\tau) &= \frac{1}{(q,q)_n} \sum_{m=0}^{n} (-1)^m \left\{ \begin{bmatrix} n-1 \\ m \end{bmatrix}_q + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q \right\} q^{\frac{1}{2}(m-1)m} (\tau, q)_m \\ &= \frac{d_{n-1}(\tau)}{1-q^n} - q^{n-1} \frac{1-\tau}{(q,q)_n} \sum_{m=0}^{n-1} (-1)^m \begin{bmatrix} n-1 \\ m \end{bmatrix}_q q^{\frac{1}{2}(m-1)m} (q\tau, q)_m \\ &= \frac{1}{1-q^n} [d_{n-1}(\tau) - q^{n-1}(1-\tau) d_{n-1}(q\tau)], \qquad n \in \mathbb{N}. \end{aligned}$$

Likewise, for $n \geq 2$ (3.18) yields

$$\begin{aligned} &\frac{1}{1-q^n} [d_{n-1}(\tau) - q^{n-1}(1-\tau) d_{n-1}(q\tau) \\ &= \frac{1}{1-q^n} \sum_{m=1}^{n-1} \frac{q^{(m-1)m}}{(q,q)_m} \begin{bmatrix} n-2 \\ m-1 \end{bmatrix}_q \tau^m - \frac{q^{n-1}}{1-q^n} (1-\tau) \sum_{m=1}^{n-1} \frac{q^{(m-1)m}}{(q,q)_m} \begin{bmatrix} n-2 \\ m-1 \end{bmatrix}_q (q\tau)^m \\ &= \frac{1}{1-q^n} \sum_{m=1}^{n-1} \frac{q^{(m-1)m}}{(q,q)_m} \begin{bmatrix} n-2 \\ m-1 \end{bmatrix}_q (1-q^{n+m-1}) \tau^m \\ &\quad + \frac{1}{1-q^n} \sum_{m=1}^{n-1} \frac{q^{(m-1)m}}{(q,q)_m} \begin{bmatrix} n-2 \\ m-1 \end{bmatrix}_q q^{n+m-1} \tau^{m+1} \\ &= \frac{1}{1-q^n} \left\{ \sum_{m=1}^{n-1} \frac{q^{(m-1)m}}{(q,q)_m} \begin{bmatrix} n-2 \\ m-1 \end{bmatrix}_q (1-q^{n+m-1}) \tau^m + \sum_{m=2}^{n} \frac{q^{(m-2)m+n}}{(q,q)_{m-1}} \begin{bmatrix} n-2 \\ m-2 \end{bmatrix}_q \tau^m \right\} \end{aligned}$$

$$= \frac{1}{1-q^n} \sum_{m=1}^{n} \frac{q^{(m-1)m}}{(q,q)_m} \left\{ \begin{bmatrix} n-2 \\ m-1 \end{bmatrix}_q (1 - q^{n+m-1} + q^{n-m} \begin{bmatrix} n-2 \\ m-2 \end{bmatrix}_q (1-q^m) \right\} \tau^m$$

$$= \sum_{m=1}^{n} \frac{q^{(m-1)m}}{(q,q)_m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q \tau^m,$$

as can be confirmed by straightforward calculation. We thus obtained the left-hand side of (3.18) . In other words, the functions d_n in (3.17) and (3.18) obey the same recurrence relation. Since they match for $n = 1$, an inductive proof follows. □

Since $(\tau, q)_\infty/(q\tau, q)_\infty = 1-\tau$, the outcome of our analysis is the rational expansion

$$H_1(\zeta, \tau, q) = (1-\tau) \frac{\sum_{n=0}^{\infty} d_n(q\tau)\zeta^n}{\sum_{n=0}^{\infty} d_n(\tau)\zeta^n}, \tag{3.19}$$

where alternative expressions for d_n have been given in (3.17) and (3.18).

Bearing in mind the representation (3.13), combining the values of H_1 at z/α and at α/z, it is perhaps more illuminating to consider (3.19) not as a rational expansion in ζ about the origin but as a Fourier expansion on circles of radii $|\alpha| = q^{1/2}$ and $|\alpha|^{-1} = q^{-1/2}$.

3.2.3 An expansion of the generating function

The above expressions of the function F provides an expansion of the generating function in z and z^{-1}. It is enough to take account of the expression (3.7) and recall that

$$F(\alpha^{-1}z, |c|^2q^N, q) = \frac{(|c|^2, q)_N}{(|c|^2, q)_\infty} \left[1 + \sum_{n=1}^{\infty} d_n(|c|^2q^N) \left(\frac{z}{\alpha}\right)^n \right],$$

$$F(\alpha z^{-1}, |c|^2q^N, q) = \frac{(|c|^2, q)_N}{(|c|^2, q)_\infty} \left[1 + \sum_{n=1}^{\infty} d_n(|c|^2q^N) \left(\frac{\alpha}{z}\right)^n \right],$$

where

$$d_n(x) = \sum_{m=1}^{n} \frac{q^{(m-1)m}}{(q,q)_m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q x^m, \qquad n \in \mathbb{N}.$$

Using the function F, we can express the underlying sequence of OPUC as a expansion in z and z^{-1},(3.8)

$$\phi_N(z) = \alpha^N \beta_1(z) F(\alpha z^{-1}, |c|^2q^N, q) + z^N \beta_2(z) F(\alpha^{-1}z, |c|^2q^N, q), \qquad n \in \mathbb{Z}_+,$$

where β_1 and β_2 are determined from the initial conditions $\phi_0 \equiv 1$ and $\phi_1(z) = c\alpha + z$,

$$\beta_1(z) = \frac{zF(\alpha^{-1}z, |c|^2q, q) - (c\alpha + z)F(\alpha^{-1}z, |c|^2, q)}{zF(\alpha z^{-1}, |c|^2, q)F(\alpha^{-1}z, |c|^2q, q) - \alpha F(\alpha z^{-1}, |c|^2q, q)F(\alpha^{-1}z, |c|^2, q)}$$

$$\beta_2(z) = \frac{(c\alpha + z)F(\alpha z^{-1}, |c|^2, q) - \alpha F(\alpha z^{-1}, |c|^2q, q)}{zF(\alpha z^{-1}, |c|^2, q)F(\alpha^{-1}z, |c|^2q, q) - \alpha F(\alpha z^{-1}, |c|^2q, q)F(\alpha^{-1}z, |c|^2, q)}$$

To verify that the underlying Schur parameters are correct, we note that for all $N \in \mathbb{N}$

$$\phi_N(0) = c\alpha^N \frac{F(\infty; |c|^2 q^N, q}{F(\infty; |c|^2 q, q)} = c\alpha^N,$$

because $F(\infty; |c|^2 q^N, q) \equiv 1$ according to our computation.

3.3 A representation of the OPUC as q-Bessel functions

In this section we obtain an explicit representation of the OPUC sequence $\{\phi_n\}$ as a linear combination of the q-Bessel functions $\mathrm{J}_\nu^{(2)}$ (cf. (Gasper & Rahman 2004, p. 4) for the definition of q-Bessel functions). This is very much in line with the numerous explicit representations of orthogonal polynomials on the real line in terms of hypergeometric and q-hypergeometric functions (Chihara 1978, Ismail 2005). Bearing in mind the definition of F and the representation (3.8) of the OPUC $\{\phi_n\}$, a simple calculation leads to

$$\begin{aligned} F(\zeta,\tau,q) - F(\zeta,q\tau,q) &= \sum_{m=1}^{\infty} \frac{(1-q^m)\tau^m}{(q,q)_m(\zeta,q)_m} = \sum_{m=1}^{\infty} \frac{\tau^m}{(q,q)_{m-1}(\zeta,q)_m} \\ &= \sum_{m=0}^{\infty} \frac{\tau^{m+1}}{(q,q)_m(\zeta,q)_{m+1}} = \frac{\tau}{1-\zeta} F(q\zeta,\tau,q). \end{aligned}$$

We thus deduce the functional equation

$$F(\zeta,\tau,q) = F(\zeta,q\tau,q) + \frac{\tau}{1-\zeta} F(q\zeta,\tau,q), \tag{3.20}$$

given in tandem with the initial condition $F(\zeta,0,q) \equiv 1$.

Note, incidentally, that the function

$$\tilde{F}(\zeta,\tau,q) = \frac{1}{(\zeta,q)_\infty(\tau,q)_\infty}$$

is a solution of (3.20), as can be verify easly by direct substitution. Needless to say, $\tilde{F} \neq F$ (cf. (3.15)), but then there is absolutely no reason to claim that (3.20) has a unique solution.

We now start similarly to Subsection 3.2.2, yet progress differently,

$$\begin{aligned} F(\zeta,\tau,q) - F(q\zeta,\tau,q) &= \sum_{m=1} \frac{\tau^m}{(q,q)_m(\zeta,q)_{m+1}} [(1-\zeta q^m) - (1-\zeta)] \\ &= \zeta \sum_{m=1} \frac{\tau^m}{(q,q)_{m-1}(\zeta,q)_{m+1}} = \frac{\zeta\tau}{(1-\zeta)(1-q\zeta)} F(q^2\zeta,\tau,q). \end{aligned}$$

Let

$$\chi_r = F(q^r\zeta,\tau,q), \qquad r \in \mathbb{Z}_+.$$

(Needless to say, $\chi_r = \chi_r(\zeta,\tau,q)$, but it is convenient to suppress parameters). We have just proved that

$$\chi_0 = \chi_1 + \frac{\zeta\tau}{(\zeta,q)_2}\chi_2$$

and, replacing ζ with $q^r\zeta$ for $r \in \mathbb{Z}_+$, we deduce the recurrence

$$\chi_r = \chi_{r+1} + \frac{\zeta q^r \tau}{(\zeta q^r, q)_2}\chi_{r+2}, \qquad r \in \mathbb{Z}_+. \tag{3.21}$$

Proposition 3.6 *For every $s \in \mathbb{Z}_+$ it is true that*

$$\chi_0 = \sum_{\ell=0}^{s} \begin{bmatrix} s \\ \ell \end{bmatrix}_q \frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta, q)_\ell (q^s\zeta, q)_\ell}\chi_{s+\ell}. \tag{3.22}$$

Proof By induction on s. The statement is trivial for $s = 0$ and reduces to (3.21) for $s = 1$. In general, we assume (3.22) for s and use (3.21),

$$\begin{aligned}
\chi_0 &= \sum_{\ell=0}^{s} \begin{bmatrix} s \\ \ell \end{bmatrix}_q \frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta, q)_\ell (q^s\zeta, q)_\ell} \left[\chi_{s+1+\ell} + \frac{q^{s+\ell}\zeta\tau}{(q^{s+\ell}\zeta, q)_2}\chi_{s+\ell+2}\right] \\
&= \sum_{\ell=0}^{s} \begin{bmatrix} s \\ \ell \end{bmatrix}_q \frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta, q)_\ell (q^s\zeta, q)_\ell}\chi_{s+1+\ell} \\
&\quad + \sum_{\ell=1}^{s+1} \frac{q^{s+\ell-1}}{(q^{s+\ell-1}\zeta, q)_2} \begin{bmatrix} s \\ \ell-1 \end{bmatrix}_q \frac{q^{(\ell-2)(\ell-1)}(\zeta\tau)^\ell}{(\zeta, q)_{\ell-1}(q^s\zeta, q)_{\ell-1}}\chi_{s+1+\ell}.
\end{aligned}$$

Let us examine the ℓth term (we restrict our attention to $1 \le \ell \le s$, cases $\ell = 0$ and $\ell = s + 1$ being trivial):

$$\begin{aligned}
&\begin{bmatrix} s \\ \ell \end{bmatrix}_q \frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta, q)_\ell (q^s\zeta, q)_\ell} + \begin{bmatrix} s \\ \ell-1 \end{bmatrix}_q \frac{q^{s+\ell-1+(\ell-2)(\ell-1)}(\zeta\tau)^\ell}{(\zeta, q)_{\ell-1}(q^s\zeta, q)_{\ell+1}} \\
&= \frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta, q)_\ell (q^s\zeta, q)_{\ell+1}} \left\{ \begin{bmatrix} s \\ \ell \end{bmatrix}_q (1 - q^{s+\ell}\zeta) + \begin{bmatrix} s \\ \ell-1 \end{bmatrix}_q q^{s-\ell+1}(1 - q^{\ell-1}\zeta) \right\}.
\end{aligned}$$

But

$$\begin{aligned}
&\begin{bmatrix} s \\ \ell \end{bmatrix}_q (1 - q^{s+\ell}\zeta) + \begin{bmatrix} s \\ \ell-1 \end{bmatrix}_q q^{s-\ell+1}(1 - q^{\ell-1}\zeta) \\
&= \frac{(q, q)_s}{(q, q)_\ell (q, q)_{s+1-\ell}} [(1 - q^{s+1-\ell})(1 - q^{s+\ell}\zeta) + (1 - q^\ell)(q^{s-\ell+1} - q^s\zeta)] \\
&= \frac{(q, q)_s}{(q, q)_\ell (q, q)_{s+1-\ell}} (1 - q^{s+1})(1 - q^s\zeta) = \begin{bmatrix} s+1 \\ \ell \end{bmatrix}_q (1 - q^s\zeta),
\end{aligned}$$

therefore the ℓth term is

$$\frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta, q)_\ell (q^s\zeta, q)_{\ell+1}} \begin{bmatrix} s+1 \\ \ell \end{bmatrix}_q (1 - q^s\zeta) = \begin{bmatrix} s+1 \\ \ell \end{bmatrix}_q \frac{q^{(\ell-1)\ell}(\zeta\tau)^\ell}{(\zeta, q)_\ell (q^{s+1}\zeta, q)_\ell}.$$

This is precisely (3.22) for $s + 1$ and an inductive proof is complete. □

We now let $s \to \infty$ in (3.22),

$$\lim_{s\to\infty} \chi_{s+\ell} = \lim_{s\to\infty} F(q^{s+\ell}\zeta,\tau,q) = F(0,\tau,q) = \mathrm{e}_q(\tau) = \frac{1}{(\tau,q)_\infty},$$
$$\lim_{s\to\infty} (q^s\zeta,q)_\ell = (0,q)_\ell = 1,$$
$$\lim_{s\to\infty} \begin{bmatrix} s \\ \ell \end{bmatrix}_q = \lim_{s\to\infty} \frac{(q,q)_s}{(q,q)_\ell (q,q)_{s-\ell}} = \frac{(q,q)_\infty}{(q,q)_\ell (q,q)_\infty} = \frac{1}{(q,q)_\ell}.$$

Therefore

$$F(\zeta,\tau,q) = \mathrm{e}_q(x) \sum_{\ell=0}^{\infty} q^{(\ell-1)\ell} \frac{(\zeta\tau)^\ell}{(q,q)_\ell(\zeta,q)_\ell} = \frac{1}{(\tau,q)_\infty} {}_0\phi_1\left[\begin{matrix} -; \\ \zeta; \end{matrix}\; q, \zeta\tau\right].$$

There exist several generalisations of Bessel functions into the realm of q-functions. In particular, the second q-Bessel function is

$$\mathrm{J}_\nu^{(2)}(x,q) = \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty} \left(\frac{x}{2}\right)^\nu {}_0\phi_1\left[\begin{matrix} -; \\ q^{\nu+1}; \end{matrix}\; q, -\frac{1}{4}x^2 q^{\nu+1}\right]$$

(Gasper & Rahman 2004, p. 25). Letting

$$\mu = \frac{\log\zeta}{\log q}$$

(in other words, $q^\mu = \zeta$) we thus have

$$\mathrm{J}_{\mu-1}^{(2)}(2\sqrt{\tau},q) = \frac{(\zeta,q)_\infty}{(q,q)_\infty} \tau^{(\mu-1)/2} {}_0\phi_1\left[\begin{matrix} -; \\ \zeta; \end{matrix}\; q, -\zeta\tau\right]$$

and we conclude that

$$F(\zeta,\tau,q) = \frac{(q,q)_\infty}{(\tau,q)_\infty(\zeta,q)_\infty} (-\tau)^{-(\mu-1)/2} \mathrm{J}_{\mu-1}^{(2)}(2\mathrm{i}\sqrt{\tau},q). \tag{3.23}$$

We now use (3.8) to obtain an explicit representation of the ϕ_ms in terms of q-Bessel functions. To this end we note that we need to reckon for both $F(\alpha z^{-1}, q^m\tau, q)$ and $F(\alpha^{-1}z, q^m\tau, q)$. However, if $q^{\mu(z)} = \alpha^{-1}z$ then $q^{-\mu(z)} = \alpha z^{-1}$. Therefore,

$$\phi_m(z) = \frac{(q,q)_\infty}{(q^m\tau,q)_\infty} \left\{ \frac{\alpha^m\beta_1(z)}{(\alpha z^{-1},q)_\infty} (-q^m\tau)^{[\mu(z)+1]/2} \mathrm{J}_{-\mu(z)-1}^{(2)}(2\mathrm{i}(q^m\tau)^{1/2},q) \right.$$
$$\left. + \frac{z^m\beta_2(z)}{(\alpha^{-1}z,q)_\infty} (-q^m\tau)^{[\mu(z)+1]/2} \mathrm{J}_{\mu(z)-1}^{(2)}(2\mathrm{i}(q^m\tau)^{1/2},q) \right\}, \qquad m \in \mathbb{Z}_+.$$

3.4 Limiting behaviour

In this subsection we consider the three instances when the Schur parameters $a_n = c\alpha^n$ are allowed to approach their limiting values, which correspond to known OPUC:

$\alpha = 0$ (Lebesgue), $c = 1$ (Rogers–Szegő) and $\alpha = 1$ (Geroniums). The first is trivial, the second relatively straightforward while the third confronts us with the greatest difficulty.

In the Lebesgue case the pantograph equation (3.1) becomes the (trivial) ODE $\Phi'' = z\Phi'$ which, in tandem with the initial conditions $\Phi(0) = 1$, $\Phi'(0) = z$, results in the explicit solution $\Phi(t, z) = \mathrm{e}^{tz}$. Hence $\phi_m(z) = z^m$ – not great surprise here! To deduce this from directly from the representation (3.11), we note first that in the current case $F(\zeta, \tau, q) = (1 - \tau)^{-1}$ is independent of ζ and this implies that also $H_m(\zeta, \tau, q) = (1 - \tau)^{-1}$. Therefore, by (3.12), $\eta_2(z) \equiv 1 - \tau$ and, α being zero, we recover $\phi_m(z) = z^m$ from (3.11).

3.4.1 Rogers–Szegő polynomials

We recall that the Schur parameters is given by $a_n = \alpha^n = (-1)^n q^{n/2}$, where $q \in (0, 1)$, give raise to the *Rogers–Szegő* polynomials (Simon 2005), whose explicit form is

$$\phi_m(z) = \sum_{j=0}^{m} (-1)^{m-j} \begin{bmatrix} m \\ j \end{bmatrix}_q q^{\frac{1}{2}(m-j)} z^j, \qquad m \in \mathbb{Z}_+. \tag{3.24}$$

Setting $\alpha = -q^{1/2}$ presents absolutely no problems in our analysis, since $q = |\alpha|^2$, is consistent with the current setting. Thus, we can readily deduce from (3.15) that

$$F(\zeta, q^m, q) = \frac{1}{(\zeta, q)_\infty (q^m, q)_\infty} \sum_{\ell=0}^{\infty} (-1)^\ell \begin{bmatrix} m + \ell - 1 \\ \ell \end{bmatrix}_q q^{\frac{1}{2}(\ell-1)\ell} \zeta^\ell, \qquad m \in \mathbb{N}.$$

In particular,

$$\begin{aligned} F(\zeta, q, q) &= \frac{1}{(\zeta, q)_\infty (q, q)_\infty} r(\zeta), \\ F(\zeta, q^2, q) &= \frac{1}{(\zeta, q)_\infty (q^2, q)_\infty} \sum_{\ell=0}^{\infty} (-1)^\ell \frac{1 - q^{\ell+1}}{1 - q} q^{\frac{1}{2}(\ell-1)\ell} \zeta^\ell \\ &= \frac{1}{(\zeta, q)_\infty (q, q)_\infty} [r(\zeta) - q r(q\zeta)], \end{aligned}$$

where

$$r(\zeta) = G(\zeta, q, q) = \sum_{\ell=0}^{\infty} (-1)^\ell q^{\frac{1}{2}(\ell-1)\ell} \zeta^\ell$$

is an entire function of order zero: all of the analysis in Subsection 3.2.2 applies here. With greater generality, it follows from (Gasper & Rahman 2004, p. 235) that

$$F(\zeta, q^m, q) = \frac{1}{(\zeta, q)_\infty (q, q)_\infty} \sum_{j=0}^{m-1} \begin{bmatrix} m - 1 \\ j \end{bmatrix}_q q^{\frac{1}{2}j(j+1)} \frac{r(q^j \zeta)}{r(\zeta)}, \qquad m \in \mathbb{N}.$$

Therefore

$$H_m(\zeta, q, q) = \frac{F(\zeta, q^{m+1}, q)}{F(\zeta, q, q)} = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q q^{\frac{1}{2}j(j+1)} \frac{r(q^j \zeta)}{r(\zeta)}. \tag{3.25}$$

It can be verified at once by elementary algebra that

$$r(q\zeta) = \frac{1 - r(\zeta)}{\zeta}, \qquad \zeta \in \mathbb{C},$$

therefore, by induction,

$$r(q^m\zeta) = (-1)^m q^{-\frac{1}{2}(m-1)m} \zeta^{-m} \left[r(\zeta) - \sum_{\ell=0}^{m-1} (-1)^\ell q^{\frac{1}{2}(\ell-1)\ell} \zeta^\ell \right].$$

In principle (3.25) can be reformulated employing just $r(\zeta)$, but this adds little to our understanding.

Intriguingly, the function r resembles a Jacobi theta function. Specifically, $r(\zeta) + r(\zeta^{-1}) = -1 + \sum_{l=-\infty}^{\infty} (-1)^\ell q^{\frac{1}{2}(\ell-1)\ell} \zeta^\ell$. But, letting $p = q^{1/2}$, we have

$$\sum_{\ell=-\infty}^{\infty} (-1)^\ell q^{\frac{1}{2}(\ell-1)\ell} \zeta^\ell = p^{-1/4} \sum_{\ell=-\infty}^{\infty} (-1)^\ell p^{(\ell-\frac{1}{2})^2} \zeta^\ell = p^{-1/4} \sum_{\ell=-\infty}^{\infty} p^{(\ell+\frac{1}{2})^2} (-\zeta^{-1})^\ell.$$

Let $z = \log(-\zeta)/(2\mathrm{i})$. Then

$$\sum_{\ell=-\infty}^{\infty} (-1)^\ell q^{\frac{1}{2}(\ell-1)\ell} \zeta^\ell = p^{-1/4} \mathrm{e}^{-\mathrm{i}z} \theta_2(z, p),$$

where θ_2 is the second *Jacobi theta function* (Rainville 1960, p. 316). Unfortunately, this intriguing connection with theta functions does not provide, insofar as we can see, much insight into Rogers–Szegő polynomials.

Abandoning the theta connection, we substitute (3.25) into (3.11) to recover an alternative representation of Rogers–Szegő polynomials, substituting (3.25) into

$$\phi_m(z) = (-1)^m q^{m/2} \eta_1(z) H_m(-q^{1/2} z^{-1}, q^{m+1}, q) + z^m \eta_2(z) H_m(-q^{1/2} z, q^{m+1}, q),$$

where η_1 and η_2 can be also expressed using the form for H_1 from (3.25).

3.4.2 Geronimus polynomials

The limiting case $\alpha = 1$, therefore $q = 1$, corresponding to *Geronimus polynomials*, is substantially more complicated, because the q-factorials $(q, q)_m$ littering our denominators, become zero and naive progression to the limit does not work.

We recall that in this case the generating function Φ obeys an ODE with the explicit solution (3.3). Using the notation therein, we let

$$\varrho_+^*(z) = z\bar{\varrho}_+(z^{-1})$$

and observe that, conjugation flipping the sign of a square root, it is true that $\varrho_+^*(z) = \varrho_-(z)$. Consequently, for Geronimus polynomials,

$$\phi_m(z) = \beta_+(z)\varrho_+^m(z) + \beta_-(z)(\varrho_+^*)^m(z), \qquad m \in \mathbb{Z}_+, \tag{3.26}$$

where we recall that

$$\varrho_+(z) = \frac{1}{2}[1 + z + \sqrt{(1-z)^2 + 4|c|^2 z}].$$

Our intent is to demonstrate that, as $\alpha \to 1$, the expression (3.13) tends to (3.26). This is not a straightforward statement since, as $\alpha \to 1$, so does q and the function F becomes unbounded. Fortunately the functions $F(\zeta, q\tau, q)$ and $F(\zeta, \tau, q)$, at the numerator and denominator of H_1 respectively, blow up at a commensurable rate and their quotient $H_1(\zeta, \tau, q)$ remains bounded.

Lemma 3.7 *Bearing in mind that $q = |\alpha|^2$, it is true that*

$$\lim_{\alpha \to 1} H_1(\alpha^{-1} z, \tau, q) = \frac{\varrho_-(z)}{z} = \frac{\varrho_+^*(z)}{z}. \tag{3.27}$$

Proof Set

$$R(\zeta, \tau, q) = \frac{F(q\zeta, \tau, q)}{F(\zeta, \tau, q)}$$

and denote

$$H_1^o(z, c) = \lim_{\alpha \to 1} H(\alpha^{-1} z, \tau, q), \qquad R^o(z, c) = \lim_{\alpha \to 1} R(\alpha^{-1} z, \tau, q).$$

(Recall that $\tau = q|c|^2$). We have

$$\begin{aligned} F(\zeta, \tau, q) - F(\zeta, q\tau, q) &= \sum_{m=1}^{\infty} \frac{\tau^m}{(q, q)_{m-1} (\zeta, q)_m} = \frac{\tau}{1-\zeta} \sum_{m=0}^{\infty} \frac{\tau^m}{(q, q)_m (q\zeta, q)_m} \\ &= \frac{\tau}{1-\zeta} F(q\zeta, \tau, q), \end{aligned}$$

while we have already proved in Section 3.3 that

$$F(\zeta, \tau, q) - F(q\zeta, \tau, q) = \frac{\zeta\tau}{(1-\zeta)(1-q\tau)} F(q^2\zeta, \tau, q).$$

Dividing the first identity by $F(\zeta, \tau, q)$, we have

$$1 - H_1(\zeta, \tau, q) = \frac{\tau}{1-\zeta} \times \frac{F(q\zeta, \tau, q)}{F(\zeta, \tau, q)} \quad \Rightarrow \quad H_1(\zeta, \tau, q) = 1 - \frac{\tau}{1-\zeta} R(\zeta, \tau, q),$$

while similar division in the second identity yields

$$\begin{aligned} 1 - R(\zeta, \tau, q) &= \frac{\zeta\tau}{(1-\zeta)(1-q\tau)} \frac{F(q\zeta, \tau, q)}{F(\zeta, \tau, q)} \times \frac{F(q^2\zeta, \tau, q)}{F(q\zeta, \tau, q)} \\ &= \frac{\zeta\tau}{(1-\zeta)(1-q\tau)} R(\zeta, \tau, q) R(q\zeta, \tau, q). \end{aligned}$$

Letting $\alpha \to 1$, hence $q \to 1$, $\zeta \to z$ and $\tau \to |c|^2$, we obtain the quadratic equation

$$|c|^2 z R^{o2}(z, c) + (1-z)^2 R^o(z, c) - (1-z)^2 = 0,$$

therefore

$$R^o(z,c) = -\frac{1-z}{2|c|^2 z}[(1-z) \pm \sqrt{(1-z)^2+4|c|^2 z}]$$

and analyticity at the origin means that we need to take a minus sign inside the square brackets. Therefore

$$\begin{aligned} H_1^o(z,c) &= \lim_{\alpha\to 1}\left[1-\frac{\tau}{1-\zeta}R(\zeta,\tau,q)\right] = 1-\frac{|c|^2}{1-z}R^o(z,c) \\ &= 1+\frac{(1-z)-\sqrt{(1-z)^2+4|c|^2 z}}{2z} = \frac{1+z-\sqrt{(1-z)^2+4|c|^2 z}}{2z} = \frac{\varrho_-(z)}{z} \end{aligned}$$

and the proof follows. □

Formulæ(3.3) and (3.13) are both linear combinations of two components: for (3.3) these are powers of ϱ_+ and ϱ_-. Let us restrict the attention to the curve of orthogonality, $|z|=1$. The functions $\varrho_\pm$ have two brach points, $1-2|c|^2 \pm \mathrm{i}|c|\sqrt{1-|c|^2}$, both of unit modulus. Since conjugations flips the sign of a square root, it follows from (3.27) that

$$H_1(\alpha \mathrm{e}^{\mathrm{i}\theta},\tau,q) = \overline{H_1(\alpha^{-1}\mathrm{e}^{\mathrm{i}\theta},\tau,q)} \xrightarrow{\alpha\to 1} \overline{\frac{\mathrm{e}^{-\mathrm{i}\theta}}{2}\left[1+\mathrm{e}^{\mathrm{i}\theta}-\sqrt{(1-\mathrm{e}^{-\mathrm{i}\theta})^2+4|c|^2}\right]} = \varrho_+(\mathrm{e}^{\mathrm{i}\theta})$$

for every $\theta \in [-\pi,\pi]$. We deduce that

$$\lim_{\alpha\to 1}\phi_m(z) = [\lim_{\alpha\to 1}\eta_1(z)]\lambda_+^m(z) + [\lim_{\alpha\to 1}\eta_2(z)]\lambda_-^m(z), \qquad m \in \mathbb{Z}_+.$$

Although it is possible to prove directly (and messily) that $\lim_{\alpha\to 1}\eta_1 = \beta_+$ and $\lim_{\alpha\to 1}\eta_2 = \beta_-$, this is not necessary, because $\eta_{1,2}$ and $\beta_\pm$ are determined by the equations

$$\beta_+(z)+\beta_-(z) = 1, \qquad \beta_+(z)\varrho_+(z)+\beta_-(z)\varrho_-(z) = z+c$$

and

$$\eta_1(z)+\eta_2(z) = 1, \qquad \alpha H_1\left(\frac{\alpha}{z},\tau,q\right)\eta_1(z) + zH_1\left(\frac{z}{\alpha},\tau,q\right)\eta_2(z) = z+c\alpha.$$

Thus, once $\alpha \to 1$, the second set of equations tends to the first, we obtain the right limits to η_1 and η_2 and our polynomials indeed converge to Geronimus polynomials.

4 Generating functions and orthogonal polynomials of the second kind

Given a sequence of OPUC, $\{\phi_n\}$, the polynomials defined by the recurrence relation

$$\Omega_n(z) = z\Omega_{n-1}(z) - \phi_n(0)\Omega_{n-1}^*(z), \qquad n \in \mathbb{N},$$

are the so-called *orthogonal polynomials of the second kind* corresponding to $\{\phi_n\}$. Note that the underlying sequence of Schur parameters satisfies $\Omega_n(0) = -\phi_n(0)$,

$n \in \mathbb{N}$. These polynomials also obey the difference equation (1.1), except that the initial conditions are $\Omega_0(z) \equiv 1$, $\Omega_1(z) = z - a_1$.

Throughout the paper we seek to analyse orthogonal polynomials whose Schur parameters are given by (1.2). The corresponding sequence of OPUCs of the second kind, also satisfies the difference equation (1.3), but with the initial conditions

$$\Omega_0(z) \equiv 1, \qquad \Omega_1(z) = z - c\alpha.$$

For the generating function given by

$$\tilde{\Xi}(z,t) = \sum_{n=0}^{\infty} \Omega_n(z) t^n.$$

the entire discussion of Section 2 remains valid for the second kind polynomials. However, due to the different initial conditions, equation (2.2) becomes

$$\tilde{\Xi}(z,t) = \sum_{m=0}^{\infty} \frac{q^{m^2}(\alpha z t^2 |c|^2)^m}{(\alpha t, q)_{m+1}(zt,q)_{m+1}} - \alpha(c-1)t \sum_{m=0}^{\infty} \frac{q^{m(m+1)}(\alpha z t^2 |c|^2)^m}{(\alpha t, q)_{m+1}(zt,q)_{m+1}},$$

while (2.4) for the OPUC of the second kind is given by

$$\Omega_m(z) = f_m(\alpha z |c|^2)) - \alpha(c-1) f_{m-1}(\alpha z q |c|^2), \qquad m \in \mathbb{N},$$

equivalently by

$$\begin{aligned}\Omega_m(z) = &\sum_{k=0}^{m} \sum_{\ell=0}^{\min\{k,m-k\}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} m-k \\ \ell \end{bmatrix}_q q^{\ell^2} |c|^{2\ell} \alpha^{m-k} z^k \\ &- (c-1) \sum_{k=0}^{m-1} \sum_{\ell=0}^{\min\{k,m-k-1\}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} m-k-1 \\ \ell \end{bmatrix}_q q^{\ell(\ell+1)} |c|^{2\ell} \alpha^{m-k} z^k.\end{aligned} \tag{4.1}$$

To recover Geronimus polynomials of the second kind, it is enough to let $\alpha \to 1$ in (4.1), whereby

$$\begin{aligned}\Omega_m(z) = &\sum_{k=0}^{m} \sum_{\ell=0}^{\min\{k,m-k\}} \binom{k}{\ell} \binom{m-k}{\ell} |c|^{2\ell} z^k \\ &- (c-1) \sum_{k=0}^{m-1} \sum_{\ell=0}^{\min\{k,m-k-1\}} \binom{k}{\ell} \binom{m-k-1}{\ell} |c|^{2\ell} z^k.\end{aligned}$$

Note that $\Omega_m(0) = -c$ as it is required. For the Rogers–Szegő case, the expression (2.9) remains valid.

For the generating function given by

$$\tilde{\Phi}(t,z) = \sum_{n=0}^{\infty} \frac{\Omega_n(z)}{n!} t^n,$$

the formulas for the OPUC in terms of the corresponding hypergeometric functions, developed in Section 3, remain equally valid,

$$\phi_m(z) = \alpha^m \tilde{\eta}_1(z) \prod_{l=1}^{m} H_1(\alpha z^{-1}, q^l\tau, q) + z^m \tilde{\eta}_2(z) \prod_{l=1}^{m} H_1(\alpha^{-1} z, q^l\tau, q),$$
$$\Omega_m(z) = \alpha^m \hat{\eta}_1(z) \prod_{l=1}^{m} H_1(\alpha z^{-1}, q^l\tau, q) + z^m \hat{\eta}_2(z) \prod_{l=1}^{m} H_1(\alpha^{-1} z, q^l\tau, q),$$

where $H_1((\zeta, \tau, q)$ and $F(\zeta, \tau, q)$ are given by (3.10) and (3.9) respectively,

$$H_1(\zeta, \tau, q) = \frac{F(\zeta, q\tau, q)}{F(\zeta, \tau, q)}, \qquad F(\zeta, \tau, q) = \sum_{m=0}^{\infty} \frac{\tau^m}{(q,q)_m(\zeta,q)_m},$$

and the coefficients $\tilde{\eta}_1(z)$, $\tilde{\eta}_2(z)$, are

$$\tilde{\eta}_1(z) = \frac{z + c\alpha - zH_1(\alpha^{-1}z, \tau, q)}{\alpha H_1(\alpha z^{-1}, \tau, q) - zH_1(\alpha^{-1}z, \tau, q)},$$
$$\tilde{\eta}_2(z) = \frac{\alpha H_1(\alpha z^{-1}, \tau, q) - z - c\alpha}{\alpha H_1(\alpha z^{-1}, \tau, q) - zH_1(\alpha^{-1}z, \tau, q)},$$
$$\hat{\eta}_1(z) = \frac{z - c\alpha - zH_1(\alpha^{-1}z, \tau, q)}{\alpha H_1(\alpha z^{-1}, \tau, q) - zH_1(\alpha^{-1}z, \tau, q)},$$
$$\hat{\eta}_2(z) = \frac{\alpha H_1(\alpha z^{-1}, \tau, q) - z + c\alpha}{\alpha H_1(\alpha z^{-1}, \tau, q) - zH_1(\alpha^{-1}z, \tau, q)},$$

taking into account the initial conditions $\Omega_0(z) \equiv 1$, $\Omega_1(z) = z - c\alpha$.

We next extend our analysis to reciprocal polynomials. The analogue of (1.1) for reciprocal polynomials $\{u_n\}$ is

$$(z\bar{a}_{n+1} + \bar{a}_n)u_n(z) = \bar{a}_n u_{n+1}(z) + (1 - |a_n|^2)\bar{a}_{n+1} z u_{n-1}(z), \qquad n \in \mathbb{N},$$

thus for $a_n = c\alpha^n$, $n \in \mathbb{N}$ we obtain the recurrence

$$(\bar{\alpha}z + 1)u_n(z) = u_{n+1}(z) + (1 - |c|^2|\alpha|^{2n})\bar{\alpha} z u_{n-1}(z), \qquad n \in \mathbb{Z}.$$

Following the ideas developed in Section 3, we consider the generating function

$$U(t, z) = \sum_{n=0}^{\infty} \frac{u_n(z)}{n!} t^n.$$

Multiplying (1.3) by $t^n/n!$ and summing up for $n \in \mathbb{N}$ results in

$$(\bar{\alpha}z+1) \sum_{n=1}^{\infty} \frac{u_n(z)}{n!} t^n = \sum_{n=1}^{\infty} \frac{u_{n+1}(z)}{n!} t^n + \bar{\alpha}z \sum_{n=1}^{\infty} \frac{u_{n-1}(z)}{n!} t^n - \bar{\alpha}z|c|^2 \sum_{n=1}^{\infty} \frac{u_{n-1}(z)}{n!} |\alpha|^{2n} t^n$$

or, equivalently,

$$(\bar{\alpha}z + 1)[U(t) - U(0)] = U'(t) - U'(0) + \bar{\alpha}z \int_0^t U(x)\,\mathrm{d}x - \bar{\alpha}z|c|^2 \int_0^{|\alpha|^2 t} U(x)\,\mathrm{d}x.$$

Differentiating this expression with respect to t, we obtain a functional differential equation *á la* (3.1),

$$U''(t) = (\bar{\alpha}z + 1)U'(t) - \bar{\alpha}zU(t) + \bar{\alpha}\tau zU(|\alpha|^2 t) \tag{4.2}$$

where $q = |\alpha|^2$, $\tau = q|c|^2$ and $q, \tau \in (0, 1)$, with the initial conditions $U(0) = \phi_0^*(z) = 1$, $U'(0) = \phi_1^*(z) = \bar{c}\bar{\alpha}z + 1$ for the reciprocal polynomials and $U(0) = \Omega_0^*(z) = 1$, $U'(0) = \Omega_1^*(z) = -\bar{c}\bar{\alpha}z + 1$ for the reciprocal polynomials of the second kind.

A similar approach to the one developed in Section 3 allows us to state

Theorem 4.1 *The generating function can be expressed explicitly in the form*

$$U(t, z) = \beta_1^i(z) \sum_{m=0}^{\infty} \frac{\tau^m}{(q, q)_m((\bar{\alpha}z)^{-1}, q)_m} \mathrm{e}^{q^m t} + \beta_2^i(z) \sum_{m=0}^{\infty} \frac{\tau^m}{(q, q)_m(\bar{\alpha}z, q)_m} \mathrm{e}^{\bar{\alpha}zq^m t},$$

with $i = 1, 2$. The coefficients $\beta_1^1(z), \beta_2^1(z)$ are determined by the initial conditions $U(0, z) \equiv 1$, $\partial U(0, z)/\partial t = \bar{c}\bar{\alpha}z + 1$, whereas $\beta_1^2(z), \beta_2^2(z)$ are obtained from the initial conditions $U(0, z) \equiv 1$, $\partial U(0, z)/\partial t = -\bar{c}\bar{\alpha}z + 1$.

As a consequence of Theorem 4.1, we can obtain explicit expressions for the reciprocal polynomials and for the reciprocal polynomials of the second kind.

Corollary 4.2 *The sequences of reciprocal polynomials (ϕ_m^*), (Ω_m^*) can be expressed in the form*

$$\phi_m^*(z) = \beta_1^1(z)F((\bar{\alpha}z)^{-1}, q^m\tau, q) + \bar{\alpha}^m z^m \beta_2^1(z)F(\bar{\alpha}z, q^m\tau, q)$$

with $\beta_1^1(z), \beta_2^1(z)$ determined by the conditions $\phi_0^*(z) = 1$, $\phi_1^*(z) = \bar{c}\bar{\alpha}z + 1$ and

$$\Omega_m^*(z) = \beta_1^2(z)F((\bar{\alpha}z)^{-1}, q^m\tau, q) + \bar{\alpha}^m z^m \beta_2^2(z)F(\bar{\alpha}z, q^m\tau, q)$$

where $\beta_1^2(z), \beta_2^2(z)$ follow from the conditions $\Omega_0^*(z) = 1$, $\Omega_1^*(z) = -\bar{c}\bar{\alpha}z + 1$.

Letting

$$H_m(\zeta, \tau, q) = \frac{F(\zeta, q^m\tau, q)}{F(\zeta, \tau, q)},$$
$$\eta_j^i(z) = \beta_j^i(z)F(\zeta, \tau, q), \qquad i, j = 1, 2,$$

and reformulating the above expressions, we can enunciate the following theorem.

Theorem 4.3 *The reciprocal polynomials admit the representation*

$$\phi_m^*(z) = \eta_1^1(z) \prod_{l=1}^{m} H_1((\bar{\alpha}z)^{-1}, q^l\tau, q) + \eta_2^1(z)\bar{\alpha}^m z^m \prod_{l=1}^{m} H_1(\bar{\alpha}z, q^l\tau, q),$$
$$\Omega_m^*(z) = \eta_1^2(z) \prod_{l=1}^{m} H_1((\bar{\alpha}z)^{-1}, q^l\tau, q) + \eta_2^2(z)\bar{\alpha}^m z^m \prod_{l=1}^{m} H_1(\bar{\alpha}z, q^l\tau, q),$$

with the same initial conditions as in Corollary 4.2

To calculate the coefficients, we impose the initial conditions in the expressions of Corollary 4.2 to obtain the system

$$\beta_1^1 F((\bar{\alpha}z)^{-1},\tau,q)+\beta_2^1 F(\bar{\alpha}z,\tau,q)=1$$
$$\beta_1^1 F\left((\bar{\alpha}z)^{-1},q\tau,q\right)+\bar{\alpha}z\beta_2^1 F\left(\bar{\alpha}z,q\tau,q\right)=\bar{c}\bar{\alpha}z+1,$$

whose solution is

$$\beta_1^1(z)=\frac{(1+\bar{c}\bar{\alpha})F(\bar{\alpha}z,\tau,q)-\bar{\alpha}zF(\bar{\alpha}z,q\tau,q)}{F(\bar{\alpha}z,\tau,q)F((\bar{\alpha}z)^{-1},q\tau,q)-\bar{\alpha}zF(\bar{\alpha}z,q\tau,q)F((\bar{\alpha}z)^{-1},\tau,q)},$$
$$\beta_2^1(z)=\frac{(1+\bar{c}\bar{\alpha}z)F((\bar{\alpha}z)^{-1},\tau,q)-F((\bar{\alpha}z)^{-1},q\tau,q)}{F(\bar{\alpha}z,\tau,q)F((\bar{\alpha}z)^{-1},q\tau,q)-\bar{\alpha}zF(\bar{\alpha}z,q\tau,q)F((\bar{\alpha}z)^{-1},\tau,q)}.$$

Analogously, the system

$$\beta_1^2 F((\bar{\alpha}z)^{-1},\tau,q)+\beta_2^2 F(\bar{\alpha}z,\tau,q)=1$$
$$\beta_1^2 F\left((\bar{\alpha}z)^{-1},q\tau,q\right)+\bar{\alpha}z\beta_2^2 F\left(\bar{\alpha}z,q\tau,q\right)=-\bar{c}\bar{\alpha}z+1$$

is solved by

$$\beta_1^2(z)=\frac{(1-\bar{c}\bar{\alpha})F(\bar{\alpha}z,\tau,q)-\bar{\alpha}zF(\bar{\alpha}z,q\tau,q)}{F(\bar{\alpha}z,\tau,q)F((\bar{\alpha}z)^{-1},q\tau,q)-\bar{\alpha}zF(\bar{\alpha}z,q\tau,q)F((\bar{\alpha}z)^{-1},\tau,q)},$$
$$\beta_2^2(z)=\frac{(1-\bar{c}\bar{\alpha}z)F((\bar{\alpha}z)^{-1},\tau,q)-F((\bar{\alpha}z)^{-1},q\tau,q)}{F(\bar{\alpha}z,\tau,q)F((\bar{\alpha}z)^{-1},q\tau,q)-\bar{\alpha}zF(\bar{\alpha}z,q\tau,q)F((\bar{\alpha}z)^{-1},\tau,q)}.$$

Concerning to the expressions of reciprocal polynomials given by Theorem 4.3, imposition of the initial conditions results in the following representation for the coefficients,

$$\eta_1^1(z)=\frac{\bar{\alpha}zH_1\left(\bar{\alpha}z,q\tau,q\right)-\bar{c}\bar{\alpha}z-1}{\bar{\alpha}zH_1\left(\bar{\alpha}z,q\tau,q\right)-H_1\left((\bar{\alpha}z)^{-1},q\tau,q\right)},$$
$$\eta_2^1(z)=\frac{\bar{c}\bar{\alpha}z+1-H_1\left((\bar{\alpha}z)^{-1},q\tau,q\right)}{\bar{\alpha}zH_1\left(\bar{\alpha}z,q\tau,q\right)-H_1\left((\bar{\alpha}z)^{-1},q\tau,q\right)}$$

and

$$\eta_1^2(z)=\frac{\bar{\alpha}zH_1\left(\bar{\alpha}z,q\tau,q\right)+\bar{c}\bar{\alpha}z-1}{\bar{\alpha}zH_1\left(\bar{\alpha}z,q\tau,q\right)-H_1\left((\bar{\alpha}z)^{-1},q\tau,q\right)},$$
$$\eta_2^2(z)=\frac{-\bar{c}\bar{\alpha}z+1-H_1\left((\bar{\alpha}z)^{-1},q\tau,q\right)}{\bar{\alpha}zH_1\left(\bar{\alpha}z,q\tau,q\right)-H_1\left((\bar{\alpha}z)^{-1},q\tau,q\right)}.$$

5 The Carathéodory function

An analytic function $\mathcal{F}$ in the open unit disk $\mathbb{D}$ is called a *Carathéodory function* if $F(0)=1$ and $\operatorname{Re}F(z)>0$ on $\mathbb{D}$. Such functions play a major role in the theory of

OPUC (Simon 2005). In fact, the coefficients of its Maclaurin series

$$\mathcal{F}(z) = 1 + 2\sum_{n=1}^{\infty} \mu_{-n} z^n, \qquad |z| < 1,$$

provide the moments $\{\mu_n\}$ of the orthogonality measure μ. At the same time, such measure has a decomposition

$$\mathrm{d}\mu(\mathrm{e}^{i\theta}) = \omega(\theta)\frac{\mathrm{d}\theta}{2\pi} + \mu_s(\mathrm{e}^{\mathrm{i}\theta}),$$

where

$$\omega(\theta) = \lim_{r\uparrow 1} \operatorname{Re} \mathcal{F}(r\mathrm{e}^{\mathrm{i}\theta}),$$

and the support of the singular part μ_s lies on the set

$$\{\mathrm{e}^{\mathrm{i}\theta} : \lim_{r\uparrow 1} \mathcal{F}(r\mathrm{e}^{i\theta}) = \infty\}.$$

In particular, $\mathrm{e}^{\mathrm{i}\theta_0}$ is a mass point of μ with the mass $\mu(\{\mathrm{e}^{\mathrm{i}\theta_0}\})$ if and only if

$$\mu\left(\{\mathrm{e}^{\mathrm{i}\theta_0}\}\right) = \lim_{r\uparrow 1} \frac{1-r}{2}\mathcal{F}\left(r\mathrm{e}^{\mathrm{i}\theta_0}\right) \neq 0. \tag{5.1}$$

In our case $|\alpha| < 1$ implies that the Schur parameters are ℓ_2-bounded, and this implies that there is no singular part of μ (Simon 2005, p. 4).

Furthermore, given a sequence of OPUC $\{\phi_n\}$ and its corresponding polynomials of the second kind $\{\Omega_n\}$, the link between the Carathéodory functions and the sequences of OPUCs rests upon the equality

$$\mathcal{F}(z) = \lim_{m\to\infty} \frac{\Omega_m^*(z)}{\phi_m^*(z)}, \tag{5.2}$$

(Peherstorfer & Steinbauer 1995, Simon 2005), where ϕ_m^* and Ω_m^* are the reciprocal polynomials of ϕ_m and Ω_m, respectively.

We seek to obtain the Carathéodory function corresponding to the sequence of OPUCs studied in the paper. To this end we use (5.2) with ϕ_m^*, Ω_m^* given by Corollary 4.2 , or equivalently, by Theorem 4.3 . Using Corollary 4.2 and taking into account the properties of the function $F(\zeta, q\tau, q)$, we have

$$\begin{aligned}\mathcal{F}(z) &= \lim_{m\to\infty} \frac{\Omega_m^*(z)}{\phi_m^*(z)} \qquad (5.3)\\ &= \lim_{m\to\infty}\left(\frac{\beta_1^2(z)F((\bar{\alpha}z)^{-1}, q^m\tau, q) + \bar{\alpha}^m z^m \beta_2^2(z)F(\bar{\alpha}z, q^m\tau, q)}{\beta_1^1(z)F((\bar{\alpha}z)^{-1}, q^m\tau, q) + \bar{\alpha}^m z^m \beta_2^1(z)F(\bar{\alpha}z, q^m\tau, q)}\right)\\ &= \frac{(1-\bar{c}\bar{\alpha})F(\bar{\alpha}z, \tau, q) - \bar{\alpha}zF(\bar{\alpha}z, q\tau, q)}{(1+\bar{c}\bar{\alpha})F(\bar{\alpha}z, \tau, q) - \bar{\alpha}zF(\bar{\alpha}z, q\tau, q)}.\end{aligned}$$

As a reality check, applying Theorem 4.3,

$$\mathcal{F}(z) = \lim_{m\to\infty} \frac{\Omega_m^*(z)}{\phi_m^*(z)} \tag{5.4}$$

$$= \lim_{m\to\infty}\left(\frac{\eta_1^2(z)H_1((\bar\alpha z)^{-1},q^m\tau,q)+\bar\alpha^m z^m\eta_2^2(z)H_1(\bar\alpha z,q^m\tau,q)}{\eta_1^1(z)H_1((\bar\alpha z)^{-1},q^m\tau,q)+\bar\alpha^m z^m\eta_2^1(z)H_1(\bar\alpha z,q^m\tau,q)}\right)$$
$$= \frac{\bar\alpha z H_1\left(\bar\alpha z,q\tau,q\right)+\bar c\bar\alpha z-1}{\bar\alpha z H_1\left(\bar\alpha z,q\tau,q\right)-\bar c\bar\alpha z-1}$$

which can be obtained directly from (5.3) dividing the numerator and denominator by $F(\bar\alpha z,\tau,q)$.

The two expressions for $\mathcal{F}$ have complementary advantages. While (5.3), expressing F by its definition (3.9), lends itself more easily to computation for all values of $|\alpha|,|c|<1$, the expression (5.4) is more conducive to expansion of $\mathcal{F}$ into series in z. The reason is that, substitituing (3.15) into (3.10), we can write H_1 in the form

$$H_1(\zeta,\tau,q)=\frac{\displaystyle\sum_{m=0}^{\infty}\frac{(\tau,q)_{m+1}}{(q,q)_m}q^{\frac12(m-1)m}(-\zeta)^m}{\displaystyle\sum_{m=0}^{\infty}\frac{(\tau,q)_m}{(q,q)_m}q^{\frac12(m-1)m}(-\zeta)^m}, \tag{5.5}$$

with the term $(\zeta,q)_\infty$ (problematic in both expansion into series and computation for $0\ll|q|<1$) gone. Substitution of (5.5) into (5.4), letting $q=|\alpha|^2$, $\tau=|\alpha c|^2$ and expansion into series with Maple yield the McLaurin expansion of $\mathcal{F}$,

$$\begin{aligned}\mathcal{F}(z)=1&-2\bar c(\bar\alpha z)-2\bar c[(1-\bar c)-|c|^2|\alpha|^4](\bar\alpha z)^2-2\bar c[(1-\bar c)^2+2|c|^2\bar c+2|\alpha|^4\\&-3|\alpha|^4|c|^2+2|\alpha|^8|c|^4](\bar\alpha z)^3-2\bar c[(1-\bar c)^3-6|\alpha|^4|c|^2-3|\alpha|^4|c|^2\bar c^2\\&+|\alpha|^4|c|^2(8-5|\alpha|^4|c|^2)\bar c+|\alpha|^8(9+|\alpha|^2)|c|^4-|\alpha|^{12}(4+|\alpha|^2)|c|^6](\bar\alpha z)^4+\cdots\end{aligned}$$

Therefore

$$\begin{aligned}\mu_{-1}&=-\bar c\bar\alpha,\\ \mu_{-2}&=-\bar c[(1-\bar c)-|c|^2|\alpha|^4]\bar\alpha^2,\\ \mu_{-3}&=-\bar c[(1-\bar c)^2+2|c|^2|\alpha|^4\bar c+2|\alpha|^4-3|\alpha|^4|c|^2+2|\alpha|^8|c|^4]\bar\alpha z^3,\\ \mu_{-4}&=-\bar c[(1-\bar c)^3-6|\alpha|^4|c|^2-3|\alpha|^4|c|^2\bar c^2+|\alpha|^4|c|^2(8-5|\alpha|^4|c|^2)\bar c\\&\qquad+|\alpha|^8(9+|\alpha|^2)|c|^4-|\alpha|^{12}(4+|\alpha|^2)|c|^6]\bar\alpha^4\end{aligned}$$

and so on. Long calculation demonstrates that for $|\alpha c|<1$, $|\alpha(1-c)|<1$

$$\begin{aligned}\mathcal{F}(z)=1&-\frac{2\bar c\bar\alpha z}{1-(1-\bar c)\bar\alpha z}+\frac{2\bar c|c|^2|\alpha|^4(\bar\alpha z)^2}{(1-\bar\alpha z)^3}-\frac{4\bar c^2|c|^2|\alpha|^4(\bar\alpha z)^3}{(1-\bar\alpha z)^4}\\&-\frac{2\bar c|c|^4|\alpha|^8[2-(\bar\alpha z)^2](\bar\alpha z)^3}{(1-\bar\alpha z)^5}+\mathcal{O}\big(|c|^5|\alpha|^{12}|z|^4\big)\,.\end{aligned}$$

The above expressions for the Carathéodory function provide the absolutely continuous part of the orthogonality measure

$$\omega_{\alpha,c}(\theta)=\lim_{r\uparrow1}\operatorname{Re}\left(\mathcal{F}(r\mathrm{e}^{\mathrm{i}\theta}\right)=\operatorname{Re}\mathcal{G}(\bar\alpha\mathrm{e}^{\mathrm{i}\theta},|\alpha|,c), \tag{5.6}$$

where $\mathcal{F}(z) = \mathcal{G}(\bar{\alpha}z, |\alpha|, c)$ and we let $\omega = \omega_{\alpha,c}$ to emphasise its dependence on the two parameters: note that in both (5.3) and (5.4) each z is always multiplied by $\bar{\alpha}$, any instance of α which does not appear in the product $\bar{\alpha}z$ features as an absolute value (cf. (5.5)) and, subject to $|\alpha c| < 1$, $|\alpha(1-c)| < 1$, $\mathcal{F}$ is bounded for $|z| = 1$. The latter fact implies that the singular part, of ω which is supported in the set $\{e^{i\theta} : \lim_{r\uparrow 1} \mathcal{F}(re^{i\theta}) = \infty\}$, is empty. Finally, replacing α by $\alpha e^{i\psi}$ merely shifts periodically $\omega(\theta)$ to $\omega(\theta - \psi)$, hence we can focus entirely on the case $\alpha \in (0, 1)$.

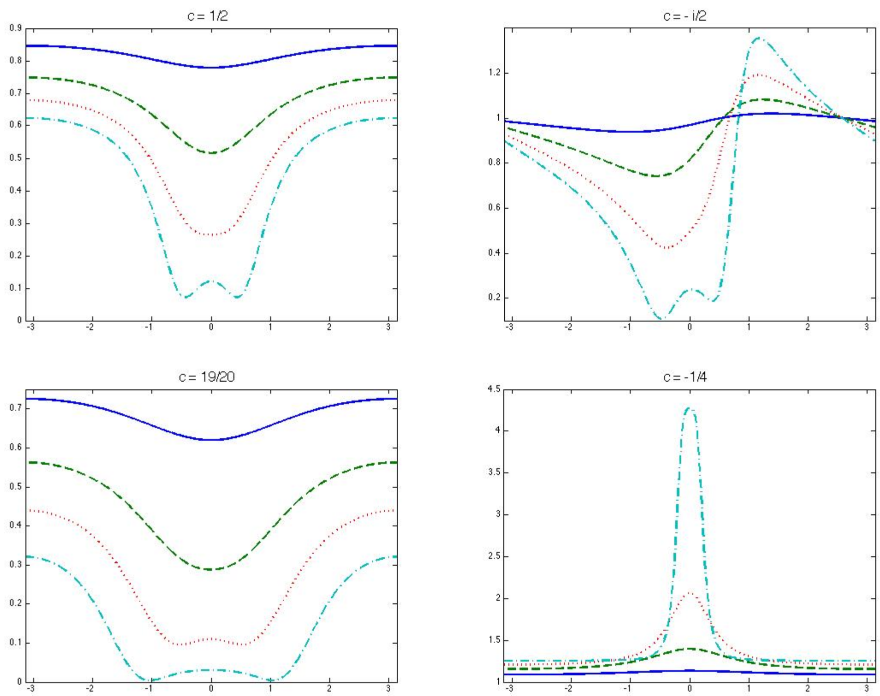

Figure 5.1: The measure ω for $\alpha = \frac{1}{5}$ (solid line), $\alpha = \frac{2}{5}$ (dash line), $\alpha = \frac{3}{5}$ (dotted line) and $\alpha = \frac{4}{5}$ (dash-dot line) for different values of c.

In Fig. 5.1 we demonstrate four instances of the measure ω from (5.6). Note that we do it always for $\alpha \in (0, 1)$ for reasons that we have elucidated in the previous paragraph. We present in each plot four different functions ω, for $\alpha \in \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$. (Recall that replacing $\alpha > 0$ by $\alpha e^{i\psi}$ merely shifts ω by $-\psi$.) In the left column we display the plots for positive values of c. For small α the measure is fairly flat (not a big surprise, since $\alpha = 0$ corresponds to $\omega \equiv 1$) but, as α grows, we obtain more interesting shapes. In particular, once α nears 1, the plot is increasingly near the origin in the middle of the range: this is consistent with the case $\alpha = 1$, the Geronimus polynomials, of which more later. The top right plot corresponds to a complex value $c = -\frac{1}{2}i$. The plot is no longer symmetric with respect to the origin. In the bottom left corner c is near 1, the case of Rogers–Szegő polynomials – except that for the latter $\alpha < 0$ which (as we have already discussed) corresponds to a shift of the argument of ω by π. We obtain curves which are fully consistent with the "wrapped Gaussian" weight function of Rogers–Szegő polynomials (Simon 2005, p. 77). Finally, in the right bottom corner we have $c = -\frac{1}{4}$, whose measure increasingly resembles a Gaussian curve sharply peaked at the origin. Since H_1 depends upon $|c|$, rather than c itself, it follows from (5.4) that

$$\mathcal{G}(z, \alpha, -c) = \frac{1}{\mathcal{G}(z, \alpha, c)}.$$

Therefore

$$\omega_{\alpha,-c}(\theta) = \omega_{\alpha,c}(-\theta)/|\mathcal{G}(e^{i\theta}, \alpha, c)|^2, \qquad |\theta| \leq \pi. \tag{5.7}$$

Fig. 5.2 demonstrates this behaviour.

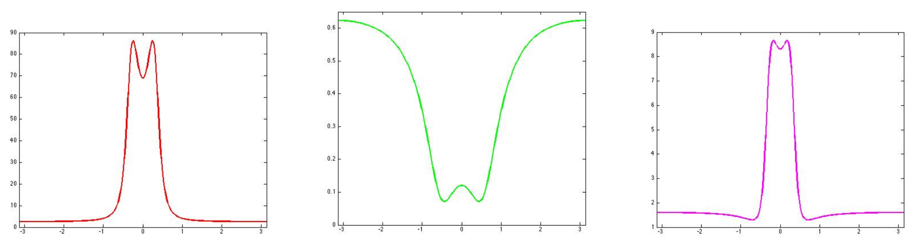

Figure 5.2: From the left: $1/|\mathcal{G}(e^{i\theta}, \frac{4}{5}, \frac{1}{2})|^2$, $\omega_{\frac{4}{5},\frac{1}{2}}(\theta)$ and $\omega_{\frac{4}{5},-\frac{1}{2}}(\theta)$. The rightmost curve is the product of the other two.

While in Fig. 5.1 the parameter c is fixed and α varies in each plot, the situation is reversed in Fig. 5.3. Here in each plot α is fixed and c is allowed to vary. We note that, once $\alpha < 1$ is large, the measure is very small in a large portion of the range. This is only to be expected since, as $\alpha \to 1$, we obtain Geronimus polynomials, whose measure vanishes for $|\theta| \leq \arccos(1 - 2|c|^2)$.[1]

For the degree one *Bernstein–Szegő polynomials* (Simon 2005, p. 72) with Schur parameters $a_1 = c$, $a_n = 0$ for all $n \geq 2$, we recover the expression of the Carathéodory function from (5.3) taking into account that the corresponding sequences of OPUC and their reciprocal counterparts are

$$\begin{aligned} \phi_n(z) &= z^n + cz^{n-1}, & \Omega_n(z) &= z^n - cz^{n-1}, \\ \phi_n^*(z) &= cz + 1, & \Omega_n^*(z) &= -\bar{c}z + 1. \end{aligned}$$

Although the direct computation of the Carathéodory function is fairly straight-

[1]Geronimus polynomials have a nonzero singular measure, but this is not the case for $|\alpha| < 1$, since, by our analysis, $\mathcal{F}$ is bounded for $|z| = 1$.

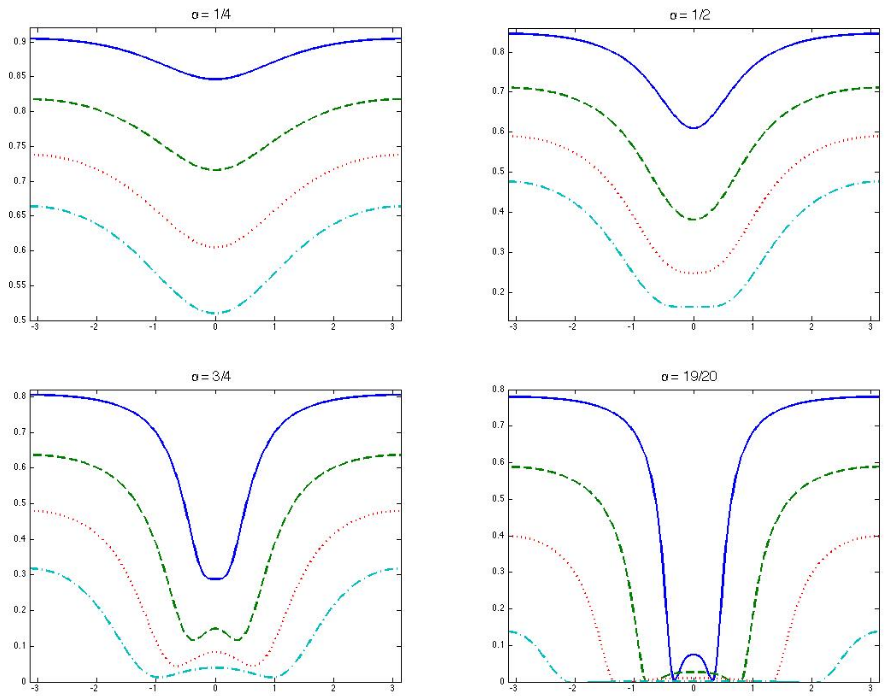

Figure 5.3: The measure ω for $c = \frac{1}{4}$ (solid line), $c = \frac{1}{2}$ (dash line), $c = \frac{3}{4}$ (dotted line) and $c = 1$ (dash-dot line) for different values of c.

forward for the *Bernstein–Szegő polynomials*, corresponding to the Schur parameters

$$a_n = \begin{cases} 1, & n = 0, \\ d, & n = 1, \\ 0, & n \geq 2, \end{cases}$$

as a reality check we verify that it forms a limiting case of our polynomials as $\alpha \to 0$ and $c\alpha \to d$, $|d| < 1$. Since in that case $H_1 \to 0$, we have

$$\mathcal{F}(z) = \frac{zH_1\left(z, q\tau, q\right) + \bar{d}z - 1}{zH_1\left(z, q\tau, q\right) - \bar{d}z - 1} = \frac{1 - \bar{d}z}{1 + \bar{d}z},$$

the correct expression (Simon 2005, p. 85).

Next we recover the Carathéodory function corresponding to the Geronimus polynomials. We recall that in this case $\alpha = 1$, hence $q = 1$, $\tau = |c|^2$ and the generating function obeys an ODE

$$\Phi''(t) - (1 + z)\Phi'(t) + (1 - |c|^2)z\Phi(t) = 0, \quad t \geq 0, \qquad \Phi(0) = 1, \quad \Phi'(0) = z + c,$$

whose solution gives the known representation of Geronimus polynomials,

$$\phi_n(z) = \beta_+(z)\rho_+(z)^n + \beta_-(z)\rho_-(z)^n,$$

where

$$\varrho_\pm(z) = \frac{1+z \pm \sqrt{(1-z)^2+4|c|^2 z}}{2}, \qquad \beta_\pm(z) = \frac{1}{2} \mp \frac{(1-z)-2c}{\sqrt{(1-z)^2+4|c|^2 z}}.$$

If, instead, we impose the initial conditions $\Phi(0)=1$, $\Phi'(0)=z-c$. the outcome is Geronimus polynomials of the second kind. while letting $\Phi(0)=1$, $\Phi'(0)=\bar{c}z+1$ and $\Phi(0)=1$, $\Phi'(0)=-\bar{c}z+1$ results in the reciprocal polynomials.

Using (5.4), and the properties of the function $H_1(\zeta,\cdot,\cdot)$ developed in Subsection 3.2, after straightforward computation the Carathéodory function can be written in the form

$$\begin{aligned}\mathcal{F}(z) &= \frac{\varrho_-(z)+\bar{c}z-1}{\varrho_-(z)-\bar{c}z-1} = \frac{-1+(2\bar{c}+1)z-\sqrt{(1-z)^2+4|c|^2 z}}{-1-(2\bar{c}-1)z-\sqrt{(1-z)^2+4|c|^2 z}} \\ &= \frac{\bar{c}z+c-\sqrt{(1-z)^2+4|c|^2 z}}{(1-\bar{c})z+c-1},\end{aligned}$$

which is the right expression (see (Geronimus 1961, p. 158)).

Finally, we recall that $\omega=\omega_{\alpha,c}$ obeys (5.6). Therefore, formalising our earlier argument,

$$\omega_{\alpha,c}(\theta) = \operatorname{Re}\mathcal{G}(|\alpha|e^{i(\theta-\arg\alpha)},|\alpha|,c) = \omega_{|\alpha|,c}(\theta-\arg\alpha).$$

Let $c=|c|e^{i\kappa}$. Recall that the second and third arguments of H_1 depend only on the nonnegative numbers $|\alpha|$ and $|c|$, because $q=|\alpha|^2$ and $\tau=|\alpha c|^2$. Therefore, using (5.4),

$$\begin{aligned}\mathcal{G}(z,\alpha,c) &= \frac{\bar{\alpha}zH_1+e^{-i\kappa}|c|\bar{\alpha}z-1}{\bar{\alpha}zH_1-e^{-i\kappa}|c|\bar{\alpha}z-1} \\ &= \frac{(\bar{\alpha}zH_1+e^{-i\kappa}|c|\bar{\alpha}z-1)(\alpha\bar{z}\bar{H}_1-e^{i\kappa}|c|\alpha\bar{z}-1)}{|\bar{\alpha}zH_1-e^{-i\kappa}|c|\bar{\alpha}z-1|^2} \\ &= \frac{|\alpha|^2|H_1|^2-2\mathrm{Re}\,(\bar{\alpha}zH_1)-|\alpha c|^2+1}{|\bar{\alpha}zH_1-e^{-i\kappa}|c|\bar{\alpha}z-1|^2} - 2|c|i\,\mathrm{Im}\,\frac{e^{i\kappa}(|\alpha|^2H_1-\alpha\bar{z})}{|\bar{\alpha}zH_1-e^{-i\kappa}|c|\bar{\alpha}z-1|^2}.\end{aligned}$$

Therefore

$$\begin{aligned}\omega_{\alpha,c}(\theta) &= \left|\frac{\bar{\alpha}e^{i\theta}H_1+|c|\bar{\alpha}e^{i\theta}-1}{\bar{\alpha}e^{i\theta}H_1+|c|\bar{\alpha}e^{i(\theta-\arg c)}-1}\right|^2 \omega_{\alpha,|c|}(\theta) \\ &= \left|\frac{|\alpha|e^{i(\theta-\arg\alpha)}(H_1+|c|)-1}{|\alpha|e^{i(\theta-\arg\alpha)}(H_1+\bar{c})-1}\right|^2 \omega_{|\alpha|,|c|}(\theta-\arg\alpha). \qquad (5.8)\end{aligned}$$

This connection between the values of the measure for real and complex values of α and c might be useful. In particular, it is easy, using (5.4), to prove that (5.7) is a special case of (5.8).

Acknowledgements

The work of the first author was partially supported by the research projects MTM2008-06689-C02-01 and MTM2011-28952-C02-01 from the Ministry of Science and Innovation of Spain and the European Regional Development Fund (ERDF), and by Project E-64 of Diputación General de Aragón (Spain).

References

Chihara, T. S. (1978), *An Introduction to Orthogonal Polynomials*, Gordon & Breach, New York.

Gasper, G. & Rahman, M. (2004), *Basic Hypergeometric Series*, 2nd edn, Cambridge University Press, Cambridge.

Geronimus, L. Y. (1961), *Orthogonal Polynomials*, Consultants Bureau, New York.

Hille, E. (1962), *Analytic Function Theory*, Vol. II, Blaisdell, Waltham, MA.

Iserles, A. (1993), 'On the generalized pantograph functional-differential equation', *Europ. J. Appl. Maths* **4**, 1–38.

Ismail, M. E. H. (2005), *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, Cambridge.

Peherstorfer, P. & Steinbauer, R. (1995), 'Characterization of general orthogonal polynomials with respect to a functional', *J. Comput. Appl. Maths* **65**, 339–355.

Rainville, E. D. (1960), *Special Functions*, Macmillan, New York.

Simon, B. (2005), *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, Amer. Math. Soc., Providence, RI.