# Differential equations with general highly oscillatory forcing terms

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**Abstract**

The concern of this paper is in expanding and computing initial-value problems of the form $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{h}_\omega(t)$ where the function $\boldsymbol{h}_\omega$ oscillates rapidly for $\omega \gg 1$. Asymptotic expansions for such equations are well understood in the case of modulated Fourier oscillators $\boldsymbol{h}_\omega(t) = \sum_m \boldsymbol{a}_m(t)\mathrm{e}^{\mathrm{i}m\omega t}$ and they can be used as an organising principle for very accurate and affordable numerical solvers. However, there is no similar theory for more general oscillators and there are sound reasons to believe that approximations of this kind are unsuitable in that setting. We follow in this paper an alternative route, demonstrating that, for a much more general family of oscillators, e.g. linear combinations of functions of the form $\mathrm{e}^{\mathrm{i}\omega g_k(t)}$, it is possible to expand $\boldsymbol{y}(t)$ in a different manner. Each $r$th term in the expansion is $\mathcal{O}\!\left(\omega^{-1/\varsigma}\right)$ for some $\varsigma > 0$ and it can be represented as an $r$-dimensional highly oscillatory integral. Since computation of multivariate highly oscillatory integrals is fairly well understood, this provides a powerful method for an effective discretisation of a numerical solution for a large family of highly oscillatory ordinary differential equations.

## 1 Introduction

The subject matter of this paper are highly oscillatory ODE systems of the form

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{h}_\omega(t), \quad t \ge 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0 \in \mathbb{C}^d, \tag{1.1}$$

where the function $\boldsymbol{h}_\omega$ oscillates rapidly for $\omega \gg 1$ and all (except possibly for a countable set) values of $t$. We assume that the functions $\boldsymbol{f}$ and $\boldsymbol{h}_\omega$ are analytic but our work extends straightforwardly to functions of lower smoothness.

In a previous paper we have considered the case of the *oscillator*

$$\boldsymbol{h}_\omega(t) = \sum_{m=-\infty}^{\infty} \boldsymbol{a}_m(t)\mathrm{e}^{\mathrm{i}m\omega t}$$

(Condon, Deaño & Iserles 2010$a$, Condon, Deaño & Iserles 2010$b$). In that case it is possible to represent the solution as an asymptotic expansion in inverse powers of $\omega$ of the form

$$\boldsymbol{y}(t) \sim \boldsymbol{p}_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} \boldsymbol{p}_{r,m}(t)\mathrm{e}^{\mathrm{i}m\omega t}, \qquad t \ge 0, \tag{1.2}$$

where the functions $\boldsymbol{p}_{r,m}$ are non-oscillatory and can be derived recursively: $\boldsymbol{p}_{r,0}$ by solving a non-oscillatory ODE and $\boldsymbol{p}_{r,m}$, $m \neq 0$, by recursion. Once (1.2) is truncated for $r \geq R+1$, we obtain a numerical approximation whose error, $\mathcal{O}(\omega^{-R-1})$, actually *improves* for growing frequency $\omega$.

Unfortunately, (1.2) is no longer true for more general oscillators and it cannot be easily generalised in an obvious manner. A good toy example that illustrates problems inherent in such generalisation is the linear system

$$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{h}_\omega(t), \quad t \geq 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{1.3}$$

whose exact solution is

$$\boldsymbol{y}(t) = \mathrm{e}^{tA}\boldsymbol{y}_0 + \mathrm{e}^{tA}\int_0^t \mathrm{e}^{-\tau A}\boldsymbol{h}_\omega(\tau)\,\mathrm{d}\tau, \qquad t \geq 0. \tag{1.4}$$

Let first $\boldsymbol{h}_\omega(t,\xi) = \boldsymbol{a}(t)\mathrm{e}^{\mathrm{i}\omega g(t)}$, where $g'(t) \neq 0$, $t \geq 0$. In that case an asymptotic expansion of the integral in (1.4) is readily available,

$$\int_0^t \mathrm{e}^{-\tau A}\boldsymbol{a}(\tau)\mathrm{e}^{\mathrm{i}\omega g(\tau)}\,\mathrm{d}\tau \sim -\sum_{r=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{r+1}}\left[\boldsymbol{x}_r(t)\frac{\mathrm{e}^{\mathrm{i}\omega g(t)}}{g'(t)} - \boldsymbol{x}_r(0)\frac{\mathrm{e}^{\mathrm{i}\omega g(0)}}{g'(0)}\right],$$

where

$$\boldsymbol{x}_0(t) = \mathrm{e}^{-tA}\boldsymbol{a}(t), \qquad \boldsymbol{x}_{r+1}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\boldsymbol{x}_r(t)}{g'(t)}, \quad r \in \mathbb{Z}_+$$

(Iserles & Nørsett 2005). Note that the functions $\boldsymbol{x}_r$ are non-oscillatory. This readily provides an expansion of the solution in the form (1.2). Alternatively, we can pursue the recursive approach of (Condon et al. 2010*b*). However, the paradigm (1.2) fails for other oscillators. For example, suppose that $g'(0) = 0$, $g''(0) \neq 0$, while $g'(t) \neq 0$, $t > 0$. Without loss of generality we may assume that $g(0) = 0$. In that case the asymptotic expansion is

$$\begin{aligned}\int_0^t \mathrm{e}^{-\tau A}\boldsymbol{a}(\tau)\mathrm{e}^{\mathrm{i}\omega g(\tau)}\,\mathrm{d}\tau &\sim \int_0^t \mathrm{e}^{\mathrm{i}\omega g(\tau)}\,\mathrm{d}\tau \sum_{r=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^r}\boldsymbol{x}_r(0) \\ &\quad - \sum_{r=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{r+1}}\left[\frac{\boldsymbol{x}_r(t)-\boldsymbol{x}_r(0)}{g'(t)}\mathrm{e}^{\mathrm{i}\omega g(t)} - \frac{\boldsymbol{x}_r'(0)}{g''(0)}\right],\end{aligned} \tag{1.5}$$

where

$$\boldsymbol{x}_0(t) = \mathrm{e}^{-tA}\boldsymbol{a}(t), \qquad \boldsymbol{x}_{r+1}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\boldsymbol{x}_r(t)-\boldsymbol{x}_r(0)}{g'(t)}, \quad r \in \mathbb{Z}_+.$$

The first thing to notice is that, by the *van der Corput Lemma* (Stein 1993),

$$\int_0^t \mathrm{e}^{\mathrm{i}\omega g(\tau)}\,\mathrm{d}\tau = \mathcal{O}\left(\omega^{-1/2}\right), \qquad \omega \gg 1.$$

This immediately precludes the expansion (1.2) yet we might hope that perhaps it can be easily generalised by expanding in $\omega^{-r/2}$ in place of $\omega^{-r}$. Unfortunately, once we do this, we run up against another problem. The simplest example of this kind is $g(t)=t^2$, in which case

$$\int_0^t \mathrm{e}^{\mathrm{i}\omega g(\tau)}\,\mathrm{d}\tau = \frac{\pi^{1/2}}{2(-\mathrm{i}\omega)^{1/2}}\mathrm{erf}\,((-\mathrm{i}\omega)^{1/2}t) \sim \frac{\pi^{1/2}}{2(-\mathrm{i}\omega)^{1/2}} - \frac{\mathrm{e}^{\mathrm{i}\omega t^2}}{2}\sum_{r=0}^{\infty}\frac{(-1)^r(\frac12)_r}{t^{2r+1}(-\mathrm{i}\omega)^{r+1}}$$

for all $\omega \gg 1$ and $t>0$ (Olver, Lozier, Boisvert & Clark 2010), therefore

$$\begin{aligned}\int_0^t \mathrm{e}^{-\tau A}\boldsymbol{a}(\tau)\mathrm{e}^{\mathrm{i}\omega g(\tau)}\,\mathrm{d}\tau \sim{}& \frac{\pi^{1/2}}{2}\sum_{r=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{r+\frac12}}\boldsymbol{x}_r(0)\\ &- \sum_{r=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{r+1}}\left\{\left[\frac{\boldsymbol{x}_r(t)-\boldsymbol{x}_r(0)}{2t}+\frac{(-1)^r(\frac12)_r}{2t^{2r+1}}\boldsymbol{x}_r(0)\right]\mathrm{e}^{\mathrm{i}\omega t^2}-\frac{\boldsymbol{x}_r'(0)}{2}\right\}.\end{aligned}$$

On the face of it all is well and, upon substitution in (1.4), we have an asymptotic expansion. Unfortunately, this expansion fails at the origin and makes little sense for $0<t\ll 1$. The goal surely must be an expansion which is uniformly valid for all $t\geq 0$!

An obvious solution to our predicament is to forego the asymptotic expansion of $\int_0^t \mathrm{e}^{\mathrm{i}\omega g(\tau)}\,\mathrm{d}\tau$ in (1.5), even when it is known. Indeed, we can go a step further and forego an explicit asymptotic expansion of $\boldsymbol{y}(t)$ at the first place. Instead, we go back to the variation-of-constants representation (1.4) and compute the integral therein by any of the many modern quadrature methods for highly oscillatory integrals (Asheim & Huybrechs 2010, Huybrechs & Olver 2009).

All this is fairly straightforward for linear equations: the goal of this paper is to accomplish this task for nonlinear functions $\boldsymbol{f}$. Thus, we intend to prove that the solution $\boldsymbol{y}(t)$ of the equation (1.1) can be expanded in the form

$$\boldsymbol{y}(t) \sim \sum_{r=0}^{\infty}\boldsymbol{p}_r(t,\omega), \qquad t\geq 0, \tag{1.6}$$

where $\boldsymbol{p}_r(t,\omega)=\mathcal{O}(\omega^{-\kappa_r})$ for some $0<\kappa_1<\kappa_2<\cdots$ which depend on the oscillator $\boldsymbol{h}_\omega$. The above expansion is not unique, because, for example, we can add an $\mathcal{O}(\omega^{-\kappa_{r+1}})$ term to $\boldsymbol{p}_r$ and subtract it from $\boldsymbol{p}_{r+1}$. Our assertion, though, is that it is possible to define (1.6) in a particularly 'clean' form, whereby each $\boldsymbol{p}_r$, $r\in\mathbb{N}$, can be represented as a multivariate highly oscillatory integral.

We assume for simplicity that

$$\boldsymbol{h}_\omega(t) = \sum_m \theta_m(t,\omega)\boldsymbol{a}_m(t), \tag{1.7}$$

where the oscillators $\theta_m$s are scalar: the sum might be finite or infinite and in the latter case we assume that it is convergent.

The most ubiquitous oscillators are $\theta_m(t,\omega)=\mathrm{e}^{\mathrm{i}\omega g_m(t)}$. Let $\varsigma\geq 1$ be the least integer such that $g_m^{(\ell)}(t_*)=0$, $\ell=1,\ldots,\varsigma-1$ and $g_m^{(\varsigma)}(t_*)\neq 0$ for some $t_*\in[0,\infty)$. Then

$$\int_0^t f(\tau)\mathrm{e}^{\mathrm{i}\omega g_m(\tau)}\,\mathrm{d}\tau = \mathcal{O}\Big(\omega^{-1/\varsigma}\Big), \qquad \omega\gg 1,$$

uniformly for $t \in (0, \infty)$ and sufficiently smooth function $f$ (Stein 1993). In that case $\kappa_r = r/\varsigma$, $r \in \mathbb{N}$. Other examples of oscillators include *Bessel functions* $\theta_m(t, \omega) = \mathrm{J}_{\nu_m}(t\omega)$, $\nu_m \in \mathbb{R}$. Since

$$\int_0^t f(\tau)\mathrm{J}_\nu(\tau\omega)\,\mathrm{d}\tau = \mathcal{O}\big(\omega^{-1}\big), \qquad \omega \gg 1,$$

it will follow that $\kappa_r = r$, $r \in \mathbb{N}$.

The functions $\boldsymbol{p}_r$ can be constructed recursively:

$$\boldsymbol{p}_0' = \boldsymbol{f}(\boldsymbol{p}_0), \quad t \geq 0, \qquad \boldsymbol{p}_0(0) = \boldsymbol{y}_0,$$

and

$$\boldsymbol{p}_r(t) = \sum_{j_1} \cdots \sum_{j_r} \Phi(t) \int_0^t \int_0^t \cdots \int_0^t \boldsymbol{s}_r(\xi_1, \ldots, \xi_r) \prod_{i=1}^r \theta_{j_i}(\xi_i, \omega)\,\mathrm{d}\xi_1 \ldots \mathrm{d}\xi_r, \qquad r \in \mathbb{N},$$

where $\Phi$ is the solution of the linear equation

$$\Phi' = \frac{\partial \boldsymbol{f}(\boldsymbol{p}_0)}{\partial \boldsymbol{y}}\Phi, \quad t \geq 0, \qquad \Phi(0) = I.$$

The *non-oscillatory* functions $\boldsymbol{s}_r$ can be constructed recursively from $\boldsymbol{f}$ and its derivatives and they are independent of $\omega$.

Once the functions $\boldsymbol{s}_r$ are known for $r = 1, 2, \ldots, R$ for some $R \in \mathbb{N}$, we construct an $\mathcal{O}(\omega^{-\kappa_{R+1}})$ approximation to $\boldsymbol{y}$ with highly oscillatory quadrature over cubes of an increasing dimension. The actual situation is somewhat more difficult: the functions $\boldsymbol{s}_r$ are piecewise-smooth and, for a high-order approximation, we will partition cubes into simplexes. This can be accomplished by any of the familiar quadrature methods for highly oscillatory integrals or by bespoke methods which will be discussed in a follow-up paper. Note that the number of necessary quadratures, as well as their dimension, increases rapidly with $r \geq 1$. This means that realistic computations are likely to be restricted to modest values of $R$. Yet, even small $R$s lead to a very good approximation to $\boldsymbol{y}$ which, perhaps counterintuitively, becomes more accurate as the frequency $\omega$ grows.

In Section 2 we develop the recursive mechanism leading to the expansion (1.6), while Section 3 is devoted to few simple examples which illustrate the potential of our approach. Finally, in Section 4 we discuss briefly the computation of the highly oscillatory multivariate integrals $\boldsymbol{p}_r$.

## 2 The recursive procedure

We assume for simplicity that $\boldsymbol{h}_\omega$ includes just one oscillator, hence that we are solving the equation

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}) + \theta(t, \omega)\boldsymbol{a}(t), \quad t \geq 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0 \in \mathbb{C}^d. \tag{2.1}$$

Once we can expand (2.1) into the series (1.6), a generalisation to

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}) + \sum_m \theta_m(t,\omega)\boldsymbol{a}_m(t), \quad t \geq 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0 \in \mathbb{C}^d \tag{2.2}$$

is straightforward, although the growth in the number of terms might be substantial.

Let us assume that

$$\int_0^t f(\tau)\theta(\tau,\omega)\,\mathrm{d}\tau = \mathcal{O}\big(\omega^{-\sigma}\big)\,, \qquad \omega \gg 1 \tag{2.3}$$

generically for all smooth functions $f$, $t > 0$ and some $\sigma > 0$. Note that $\mathcal{O}(\omega^{-\sigma})$ might be exceeded for some functions $f$ and values of $t$ – for example, while in general

$$\int_0^t f(\tau)\mathrm{e}^{\mathrm{i}\omega(1-\tau)^2}\,\mathrm{d}\tau = \mathcal{O}\Big(\omega^{-1/2}\Big)\,, \qquad \omega \gg 1,$$

it is true that

$$\int_0^t f(\tau)\mathrm{e}^{\mathrm{i}\omega(1-\tau)^2}\,\mathrm{d}\tau = \mathcal{O}\big(\omega^{-1}\big)\,, \qquad \omega \gg 1$$

for $t \in (0,1)$, as well as for all $t > 0$ in the case $f(1) = 0$.

## 2.1 The first few $r$s

We wish to find functions $\boldsymbol{p}_r(t,\omega)$ such that in (1.6) the function $\boldsymbol{p}_0$ is non-oscillatory (and independent of $\omega$) and

$$\boldsymbol{p}_r(t,\omega) = \mathcal{O}\big(\omega^{-r\sigma}\big)\,, \qquad \omega \gg 1, \quad r \in \mathbb{Z}_+. \tag{2.4}$$

Moreover, we stipulate that

$$\boldsymbol{p}_0(0) = \boldsymbol{y}_0, \qquad \boldsymbol{p}_r(0,\omega) \equiv \boldsymbol{0}, \quad r \in \mathbb{N},$$

hence the expansion is consistent with the initial conditions of (2.1). Setting

$$\boldsymbol{q}_r(t,\omega) = \sum_{j=r}^{\infty} \boldsymbol{p}_j(t,\omega), \qquad r \in \mathbb{Z}_+,$$

we note that, subject to (2.4), it is true that

$$\boldsymbol{q}_r(t,\omega) = \mathcal{O}\big(\omega^{-r\sigma}\big)\,, \quad \boldsymbol{q}_r(t,\omega) - \boldsymbol{p}_r(t,\omega) = \mathcal{O}\Big(\omega^{-(r+1)\sigma}\Big)\,, \qquad \omega \gg 1. \tag{2.5}$$

For $r = 0$ we let

$$\boldsymbol{p}_0' = \boldsymbol{f}(\boldsymbol{p}_0), \quad t \geq 0, \qquad \boldsymbol{p}_0(0) = \boldsymbol{y}_0. \tag{2.6}$$

Consider next $r = 1$. Since $\boldsymbol{q}_1 = \boldsymbol{y} - \boldsymbol{p}_0$, we have

$$\boldsymbol{q}_1' = \boldsymbol{f}(\boldsymbol{p}_0 + \boldsymbol{q}_1) + \theta(t,\omega)\boldsymbol{a}(t) - \boldsymbol{f}(\boldsymbol{p}_0) = \frac{\partial \boldsymbol{f}(\boldsymbol{p}_0(t))}{\partial \boldsymbol{y}}\boldsymbol{q}_1 + \theta(t,\omega)\boldsymbol{a}(t) + \mathcal{O}\big(\omega^{-2\sigma}\big)\,.$$

Using (2.5) we thus deduce that $\boldsymbol{p}_1$ obeys the ODE

$$\boldsymbol{p}_1' = \frac{\partial \boldsymbol{f}(\boldsymbol{p}_0(t))}{\partial \boldsymbol{y}} \boldsymbol{p}_1 + \theta(t,\omega)\boldsymbol{a}(t), \quad t \geq 0, \qquad \boldsymbol{p}_1(0) = \boldsymbol{0}.$$

The solution is highly oscillatory but it can be easily written down explicitly,

$$\boldsymbol{p}_1(t) = \Phi(t) \int_0^t \Phi^{-1}(\tau)\boldsymbol{a}(\tau)\theta(\tau,\omega)\,\mathrm{d}\tau, \qquad t \geq 0, \tag{2.7}$$

where $\Phi$ is the solution of the non-oscillatory linear equation

$$\Phi' = \frac{\partial \boldsymbol{f}(\boldsymbol{p}_0(t))}{\partial \boldsymbol{y}} \Phi, \quad t \geq 0, \qquad \Phi(0) = I.$$

Note that, according to (2.3), $\boldsymbol{p}_1(t) = \mathcal{O}(\omega^{-\sigma})$, consistently with (2.4).

Over to $r = 2$. We denote the derivative tensors of the analytic function $\boldsymbol{f}$ by

$$\boldsymbol{f}_m : \overbrace{\mathbb{C}^d \times \cdots \times \mathbb{C}^d}^{m+1 \text{ times}} \to \mathbb{C}, \quad m \in \mathbb{Z}_+, \quad \text{such that} \quad \boldsymbol{f}(\boldsymbol{y}+\boldsymbol{\delta}) = \sum_{m=0}^{\infty} \frac{1}{m!} \boldsymbol{f}_m(\boldsymbol{y})[\boldsymbol{\delta},\ldots,\boldsymbol{\delta}]$$

and note that each $\boldsymbol{f}_m$ is linear in all its arguments except for the first. Of course, $\boldsymbol{f}_0(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{y})$ and $\boldsymbol{f}_1(\boldsymbol{y})[\boldsymbol{\delta}] = [\partial \boldsymbol{f}(\boldsymbol{y})/\partial \boldsymbol{y}]\boldsymbol{\delta}$.

Since $\boldsymbol{y} = \boldsymbol{p}_0 + \boldsymbol{p}_1 + \boldsymbol{q}_2$, we have

$$\begin{aligned}
\boldsymbol{q}_2' &= (\boldsymbol{y} - \boldsymbol{p}_0 - \boldsymbol{p}_1)' = \{\boldsymbol{f}(\boldsymbol{p}_0 + \boldsymbol{p}_1 + \boldsymbol{q}_2) + \theta(t,\omega)\boldsymbol{a}(t)\} - \boldsymbol{f}(\boldsymbol{p}_0) \\
&\quad - \left\{ \frac{\partial \boldsymbol{f}(\boldsymbol{p}_0)}{\partial \boldsymbol{y}} \boldsymbol{p}_1 + \theta(t,\omega)\boldsymbol{a}(t) \right\} \\
&= \left\{ \boldsymbol{f}_0(\boldsymbol{p}_0) + \boldsymbol{f}_1(\boldsymbol{p}_0)[\boldsymbol{p}_1 + \boldsymbol{q}_2] + \frac{1}{2}\boldsymbol{f}_2(\boldsymbol{p}_0)[\boldsymbol{p}_1 + \boldsymbol{q}_2, \boldsymbol{p}_1 + \boldsymbol{q}_2] + \mathcal{O}\big(\omega^{-3\sigma}\big) \right\} \\
&\quad - \boldsymbol{f}_0(\boldsymbol{p}_0) - \boldsymbol{f}_1(\boldsymbol{p}_0)[\boldsymbol{p}_1] - \mathcal{O}\big(\omega^{-3\sigma}\big) \\
&= \boldsymbol{f}_1(\boldsymbol{p}_0)[\boldsymbol{q}_2] + \frac{1}{2}\boldsymbol{f}_2(\boldsymbol{p}_0)[\boldsymbol{p}_1, \boldsymbol{p}_1] + \mathcal{O}\big(\omega^{-3\sigma}\big).
\end{aligned}$$

Since, by (2.5), $\boldsymbol{p}_2 = \boldsymbol{q}_2 + \mathcal{O}\big(\omega^{-3\sigma}\big)$, we deduce that $\boldsymbol{p}_2$ obeys the ODE

$$\boldsymbol{p}_2' = \frac{\partial \boldsymbol{f}(\boldsymbol{p}_0)}{\partial \boldsymbol{y}} \boldsymbol{p}_2 + \frac{1}{2}\boldsymbol{f}_2(\boldsymbol{p}_0)[\boldsymbol{p}_1, \boldsymbol{p}_1], \quad t \geq 0, \qquad \boldsymbol{p}_2(0) = \boldsymbol{0}.$$

Although the equation is highly oscillatory, its linearity means that it can be solved readily by quadrature,

$$\boldsymbol{p}_2(t) = \frac{1}{2}\Phi(t) \int_0^t \Phi^{-1}(\tau)\boldsymbol{f}_2(\tau)(\boldsymbol{p}_0(\tau))[\boldsymbol{p}_1(\tau), \boldsymbol{p}_1(\tau)]\,\mathrm{d}\tau, \qquad t \geq 0.$$

Substituting the value of $\boldsymbol{p}_1$ from (2.7) and using the linearity of $\boldsymbol{f}_2$ in its second and third argument, we thus obtain, exchanging the order of integration,

$$\boldsymbol{p}_2(t) = \frac{1}{2}\Phi(t) \int_0^t \Phi^{-1}(\tau)\boldsymbol{f}_2(\boldsymbol{p}_0(\tau)) \Big[ \Phi(\tau) \int_0^\tau \Phi^{-1}(\xi_1)\boldsymbol{a}(\xi_1)\theta(\xi_1,\omega)\,\mathrm{d}\xi_1,$$

$$\Phi(\tau)\int_0^\tau \Phi^{-1}(\xi_2)\boldsymbol{a}(\xi_2)\theta(\xi_2,\omega)\,\mathrm{d}\xi_2,\Big]\,\mathrm{d}\tau$$

$$= \Phi(t)\int_0^t\int_0^t \boldsymbol{s}_2(\xi_1,\xi_2)\theta(\xi_1,\omega)\theta(\xi_2,\omega)\,\mathrm{d}\xi_1,\,\mathrm{d}\xi_2, \tag{2.8}$$

where

$$\boldsymbol{s}_2(\xi_1,\xi_2) = \frac{1}{2}\int_{\max\{\xi_1,\xi_2\}}^t \Phi^{-1}(\tau)\boldsymbol{f}_2(\boldsymbol{p}_0(\tau))[\Phi(\tau)\Phi^{-1}(\xi_1)\boldsymbol{a}(\xi_1),\Phi(\tau)\Phi^{-1}(\xi_2)\boldsymbol{a}(\xi_2)\,\mathrm{d}\tau.$$

Note that the function $\boldsymbol{s}_2$ is non-oscillatory and independent of $\omega$. Moreover, $\boldsymbol{s}_2$ is piecewise smooth in $[0,t]^2$: specifically, it is smooth in the simplexes with the vertices $\{(0,0),(0,t),(t,t)\}$ and $\{(0,0),(t,0),(t,t)\}$, but just continuous along their joint boundary $\xi_1=\xi_2$.

Because of (2.3), it follows that the double integral $\boldsymbol{p}_2$ in (2.8) is $\mathcal{O}(\omega^{-2\sigma})$, as required (Wong 1989).

## 2.2 $\boldsymbol{p}_r$ for $r \geq 1$

We are now in position to present and prove the main result of this paper. Let

$$\mathbb{I}_{m,r} = \left\{(k_1,k_2,\ldots,k_m)\,:\,1\leq k_1\leq k_2\leq\cdots\leq k_m,\,\sum_{i=1}^m k_i = r\right\},\qquad m,r\in\mathbb{N},$$

denote by $\nu_{m,r}$ the number of distinct $m$-tuples $\boldsymbol{k}\in\mathbb{N}^m$ such that $\sum_{i=1}^m k_i = r$ and set

$$\begin{aligned}
\boldsymbol{s}_1(\xi_1) &= \Phi^{-1}(\xi_1)\boldsymbol{a}(\xi_1),\\
\boldsymbol{s}_2(\xi_1,\xi_2) &= \frac{1}{2}\int_{\max\{\xi_1,\xi_2\}}^t \Phi^{-1}(\tau)\boldsymbol{f}_2(\boldsymbol{p}_0(\tau))[\Phi(\tau)\boldsymbol{s}_1(\xi_1),\Phi(\tau)\boldsymbol{s}_1(\xi_2)\,\mathrm{d}\tau,\\
\boldsymbol{s}_3(\xi_1,\xi_2,\xi_3) &= \int_{\max\{\xi_1,\xi_2,\xi_3\}}^t \Phi^{-1}(\tau)\boldsymbol{f}_2(\boldsymbol{p}_0(\tau))[\Phi(\tau)\boldsymbol{s}_1(\xi_1),\Phi(\tau)\boldsymbol{s}_2(\xi_1,\xi_2)]\,\mathrm{d}\tau\\
&\quad+\frac{1}{6}\int_{\max\{\xi_1,\xi_2,\xi_3\}}^t \Phi^{-1}(\tau)\boldsymbol{f}_3(\boldsymbol{p}_0(\tau))[\Phi(\tau)\boldsymbol{s}_1(\xi_1),\Phi(\tau)\boldsymbol{s}_1(\xi_2),\Phi(\tau)\boldsymbol{s}_1(\xi_3)]\,\mathrm{d}\tau
\end{aligned}$$

and, in general, for every $r\geq 2$,

$$\boldsymbol{s}_r(\boldsymbol{\xi}) = \sum_{m=2}^r \frac{1}{m!}\sum_{\boldsymbol{k}\in\mathbb{I}_{m,r}}\nu_{m,r}\int_{\max\{\xi_1,\ldots,\xi_r\}}^t \Phi^{-1}(\tau)\boldsymbol{f}_m(\boldsymbol{p}_0(\tau))[\Phi(\tau)\boldsymbol{s}_{k_1},\ldots,\Phi(\tau)\boldsymbol{s}_{k_m}]\,\mathrm{d}\tau, \tag{2.9}$$

where

$$\begin{aligned}
\boldsymbol{s}_{k_1} &= \boldsymbol{s}_{k_1}(\xi_1,\ldots,\xi_{k_1}),\\
\boldsymbol{s}_{k_j} &= \boldsymbol{s}_{k_j}(\xi_{k_1+\cdots+k_{j-1}+1},\xi_{k_1+\cdots+k_{j-1}+2},\cdots,\xi_{k_1+\cdots+k_j}),\qquad j=2,\ldots,m.
\end{aligned}$$

The following theorem establishes the form of $\boldsymbol{p}_r$ as a multivariate highly oscillatory integral, consistently with (2.7) and (2.8).

**Theorem 1** *The functions $\boldsymbol{p}_r$, $r \in \mathbb{N}$, in (1.6) can be expressed as $r$-dimensional highly oscillatory integrals,*

$$\boldsymbol{p}_r(t,\omega) = \Phi(t) \int_0^t \int_0^t \cdots \int_0^t \boldsymbol{s}_r(\xi_1,\ldots,\xi_r) \prod_{j=1}^{r} \theta(\xi_j,\omega)\,\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_r. \tag{2.10}$$

*This implies that*

$$\boldsymbol{p}_r(t,\omega) = \mathcal{O}\big(\omega^{-r\sigma}\big), \qquad r \in \mathbb{Z}_+. \tag{2.11}$$

*Proof* Let $r \geq 1$. We commence by proving by induction on $r$ that $\boldsymbol{p}_r$ obeys the linear ODE

$$\boldsymbol{p}_r' = \sum_{m=1}^{r} \frac{\nu_{m,r}}{m!} \sum_{\boldsymbol{k}\in\mathbb{I}_{m,r}} \boldsymbol{f}_m(\boldsymbol{p}_0)[\boldsymbol{p}_{k_1},\boldsymbol{p}_{k_2},\ldots,\boldsymbol{p}_{k_m}]. \tag{2.12}$$

This is certainly true for $r=1$. Otherwise, since

$$\boldsymbol{q}_r = \boldsymbol{y} - \sum_{i=1}^{r-1} \boldsymbol{p}_i,$$

exploring the fact that $\boldsymbol{p}_j = \mathcal{O}\big(\omega^{-j\sigma}\big)$ for $j = 0,\ldots,r-1$ and that $\boldsymbol{p}_j - \boldsymbol{q}_j = \mathcal{O}\big(\omega^{-(r+1)\sigma}\big)$,

$$\begin{aligned}
\boldsymbol{p}_r' &= \boldsymbol{q}_r' + \mathcal{O}\Big(\omega^{-(r+1)\sigma}\Big) = \boldsymbol{y}' - \sum_{i=0}^{r-1} \boldsymbol{p}_i' + \mathcal{O}\Big(\omega^{-(r+1)\sigma}\Big)\\
&= \boldsymbol{f}\left(\sum_{j=0}^{r-1} \boldsymbol{p}_j + \boldsymbol{q}_r\right) - \boldsymbol{f}(\boldsymbol{p}_0) - \sum_{i=1}^{r-1}\sum_{m=1}^{i} \frac{\nu_{m,i}}{m!} \sum_{\boldsymbol{k}\in\mathbb{I}_{m,i}} \boldsymbol{f}_m(\boldsymbol{p}_0)[\boldsymbol{p}_{k_1},\ldots,\boldsymbol{p}_{k,m}]\\
&\quad + \mathcal{O}\Big(\omega^{-(r+1)\sigma}\Big)\\
&= \sum_{m=1}^{\infty} \frac{1}{m!} \boldsymbol{f}_m(\boldsymbol{p}_0) \left[\sum_{j_1=1}^{r-1} \boldsymbol{p}_{j_1} + \boldsymbol{q}_r, \cdots, \sum_{j_m=1}^{r-1} \boldsymbol{p}_{j_m} + \boldsymbol{q}_r\right]\\
&\quad - \sum_{i=1}^{r-1}\sum_{m=1}^{i} \frac{\nu_{m,i}}{m!} \sum_{\boldsymbol{k}\in\mathbb{I}_{m,i}} \boldsymbol{f}_m(\boldsymbol{p}_0)[\boldsymbol{p}_{k_1},\ldots,\boldsymbol{p}_{k,m}] + \mathcal{O}\Big(\omega^{-(r+1)\sigma}\Big)\\
&= \sum_{m=1}^{r} \frac{1}{m!} \sum_{j_1=1}^{r} \cdots \sum_{j_m=1}^{r} \boldsymbol{f}_m(\boldsymbol{p}_0)[\boldsymbol{p}_{j_1},\ldots,\boldsymbol{p}_{j_m}]\\
&\quad - \sum_{i=1}^{r-1}\sum_{m=1}^{i} \frac{\nu_{m,i}}{m!} \sum_{\boldsymbol{k}\in\mathbb{I}_{m,i}} \boldsymbol{f}_m(\boldsymbol{p}_0)[\boldsymbol{p}_{k_1},\ldots,\boldsymbol{p}_{k,m}] + \mathcal{O}\Big(\omega^{-(r+1)\sigma}\Big)\\
&= \sum_{i=1}^{r}\sum_{m=1}^{i} \frac{\nu_{m,i}}{m!} \sum_{\boldsymbol{k}\in\mathbb{I}_{m,i}} \boldsymbol{f}_m(\boldsymbol{p}_0)[\boldsymbol{p}_{k_1},\ldots,\boldsymbol{p}_{k_m}]\\
&\quad - \sum_{i=1}^{r-1}\sum_{m=1}^{i} \frac{\nu_{m,i}}{m!} \sum_{\boldsymbol{k}\in\mathbb{I}_{m,i}} \boldsymbol{f}_m(\boldsymbol{p}_0)[\boldsymbol{p}_{k_1},\ldots,\boldsymbol{p}_{k,m}] + \mathcal{O}\Big(\omega^{-(r+1)\sigma}\Big)
\end{aligned}$$

$$= \sum_{m=1}^{r} \frac{\nu_{m,r}}{m!} \sum_{\boldsymbol{k}\in\mathbb{I}_{m,r}} \boldsymbol{f}_m(\boldsymbol{p}_0)[\boldsymbol{p}_{k_1},\ldots,\boldsymbol{p}_{k_m}],$$

where we have used $\mathbb{I}_{m,i}=\emptyset$ for $m \geq i+1$. Since we can discard $\mathcal{O}\big(\omega^{-(r+1)\sigma}\big)$ terms, $\boldsymbol{p}_r$ obeys the linear ODE (2.12).

We now solve (2.12) using variation of constants and use induction to substitute $\boldsymbol{p}_j$ for $j \leq r-1$ by relevant modification of (2.10),

$$\begin{aligned}
\boldsymbol{p}_r(t) &= \Phi(t)\int_0^t \Phi^{-1}(\tau) \sum_{m=2}^{r} \frac{\nu_{m,r}}{m!} \sum_{\boldsymbol{k}\in\mathbb{I}_{m,r}} \boldsymbol{f}_m(\boldsymbol{p}_0(\tau))[\boldsymbol{p}_{k_1}(\tau),\boldsymbol{p}_{k_2}(\tau),\ldots,\boldsymbol{p}_{k_m}(\tau)]\,\mathrm{d}\tau \\
&= \Phi(t)\int_0^t \Phi^{-1}(\tau) \sum_{m=2}^{r} \frac{\nu_{m,r}}{m!} \sum_{\boldsymbol{k}\in\mathbb{I}_{m,r}} \boldsymbol{f}_m(\boldsymbol{p}_0(\tau))[\boldsymbol{\psi}_1(\tau),\ldots,\boldsymbol{\psi}_m(\tau)]\,\mathrm{d}\tau,
\end{aligned}$$

where $k_0=0$ and

$$\begin{aligned}
\boldsymbol{\psi}_j(\tau) = \Phi(\tau)\int_0^\tau \cdots \int_0^\tau & \boldsymbol{s}_{k_j}(\xi_{k_0+\cdots+k_{j-1}+1},\ldots,\xi_{k_1+\cdots+k_j}) \\
&\times \prod_{i=k_0+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} \theta_i(\xi_i,\omega)\,\mathrm{d}\xi_{k_0+\cdots+k_{j-1}+1}\cdots\mathrm{d}\xi_{k_1+\cdots+k_j}
\end{aligned}$$

Therefore

$$\begin{aligned}
\boldsymbol{p}_r(t) = \Phi(t)\int_0^t\int_0^\tau \cdots \int_0^\tau \Phi^{-1}(\tau) \sum_{m=2}^{r} \frac{\nu_{m,r}}{m!} \sum_{\boldsymbol{k}\in\mathbb{I}_{m,r}} & \boldsymbol{f}_m(\boldsymbol{p}_0(\tau))[\Phi(\tau)\boldsymbol{s}_{k_1}(\boldsymbol{\eta}_1),\ldots, \\
& \Phi(\tau)\boldsymbol{s}_{k_m}(\boldsymbol{\eta}_m)] \prod_{j=1}^{r}\theta(\xi_j,\omega)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_r\,\mathrm{d}\tau,
\end{aligned}$$

where

$$\boldsymbol{\eta}_j = (\xi_{k_0+\cdots+k_{j-1}+1},\ldots,\xi_{k_0+\cdots+k_j}), \qquad j=1,\ldots,m.$$

We next note, consecutively exchanging the order of integration, that for any $\mathrm{C}^1$ function $F:[0,t]^{r+1}\to\mathbb{C}$

$$\begin{aligned}
&\int_0^t\int_0^\tau\cdots\int_0^\tau F(\tau,\xi_1,\ldots,\xi_r)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_r\,\mathrm{d}\tau \\
&= \int_0^t\int_{\xi_r}^t\int_0^\tau\cdots\int_0^\tau F(\tau,\xi_1,\cdots,\xi_r)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{r-1}\,\mathrm{d}\tau\,\mathrm{d}\xi_r \\
&= \int_0^t\int_0^t\int_{\max\{\xi_{r-1},\xi_r\}}^t\int_0^\tau\cdots\int_0^\tau F(\tau,\xi_1,\ldots,\xi_r)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{r-2}\,\mathrm{d}\tau\,\mathrm{d}\xi_{r-1}\,\mathrm{d}\xi_r \\
&= \cdots = \int_0^t\cdots\int_0^t\int_{\max\{\xi_1,\ldots,\xi_r\}}^t F(\tau,\xi_1,\ldots,\xi_r)\,\mathrm{d}\tau\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_r.
\end{aligned}$$

Therefore

$$\begin{aligned}\boldsymbol{p}_r(t) = \Phi(t)\int_0^t \cdots \int_0^t \sum_{m=2}^{r} \frac{\nu_{m,r}}{m!} \sum_{\boldsymbol{k}\in\mathbb{I}_{m,r}} \int_{\max\{\xi_1,\ldots,\xi_r\}}^t \Phi^{-1}(\tau)\boldsymbol{f}_m(\boldsymbol{p}_0(\tau))[\Phi(\tau)\boldsymbol{s}_{k_1}(\boldsymbol{\eta}_1),\\ \ldots,\Phi(\tau)\boldsymbol{s}_{k_m}(\boldsymbol{\eta}_m)]\,\mathrm{d}\tau \prod_{j=1}^{r}\theta(\xi_j,\omega)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_r\\ = \Phi(t)\int_0^t\cdots\int_0^t \boldsymbol{s}_k(\xi_1,\ldots,\xi_r)\prod_{j=1}^{r}\theta(\xi_j,\omega)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_r,\end{aligned}$$

proving (2.10). To conclude the proof of the theorem, we evoke the standard theory of highly oscillatory integrals (Wong 1989) to argue (2.11), being an $r$-fold integral of a function whose univariate integral is $\mathcal{O}(\omega^{-\sigma})$ and which has no resonance points on the boundary. □

The extension of the theorem to equation (2.2) is straightforward: for simplicity we assume that (2.3) is true (with the same $\sigma$) for all $\theta_m$. In that case we need to replace (2.10) by

$$\boldsymbol{p}_r(t) = \Phi(t)\sum_{\ell_1}\cdots\sum_{\ell_r}\int_0^t\cdots\int_0^t \boldsymbol{s}_r(\xi_1,\ldots,\xi_r)\prod_{j=1}^{r}\theta_{\ell_j}(\xi_j,\omega)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_r \qquad (2.13)$$

– the number of multivariate highly oscillatory integrals in need of computation increases rapidly. Clearly, unless there is a fairly small number of oscillators $\theta_j$, computing (2.13) for anything but modest values of $r$ becomes impractical.

# 3 Examples and numerics

The purpose of this section is to demonstrate our approach, rather than to conduct an exhaustive analytic and numerical investigation. Thus, our focus is on a single scalar equation with three different kinds of oscillators. Our first example is

$$y' = -y^2 + \frac{1}{2}\mathrm{e}^{\mathrm{i}\omega t}, \quad t \ge 0, \qquad y(0) = 1. \qquad (3.1)$$

The equation, to the best of our knowledge, cannot be integrated in terms of familiar functions. Its numerical solution for different values of $\omega$ is displayed in Fig. 3.1 and we note that it behaves according to the theory: the larger $\omega$, the greater the frequency of oscillation (which we can observe by looking at the imaginary part), but also the smaller its departure from $p_0(t) = (1+t)^{-1}$, the solution of (2.6). This is demonstrated in Fig. 3.2.

Since $f(y) = -y^2$, we have $f_1(y) = -2y$, $f_2 \equiv -2$ and $f_m \equiv 0$ for $m \ge 3$. Therefore $\Phi' = -2(1+t)^{-1}\Phi$, $\Phi(0) = 0$, with the solution $\Phi(t) = (1+t)^{-2}$. Therefore,

$$\begin{aligned}p_1(t) &= \frac{1}{2(1+t)^2}\int_0^t (1+\tau)^2\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau\\ &= -\frac{\mathrm{i}}{2\omega}\left[\mathrm{e}^{\mathrm{i}\omega t} - \frac{1}{(1+t)^2}\right] + \frac{1}{\omega^2}\frac{(1+t)\mathrm{e}^{\mathrm{i}\omega t}-1}{(1+t)^2} + \frac{\mathrm{i}}{\omega^3}\frac{\mathrm{e}^{\mathrm{i}\omega t}-1}{(1+t)^2}.\end{aligned}$$

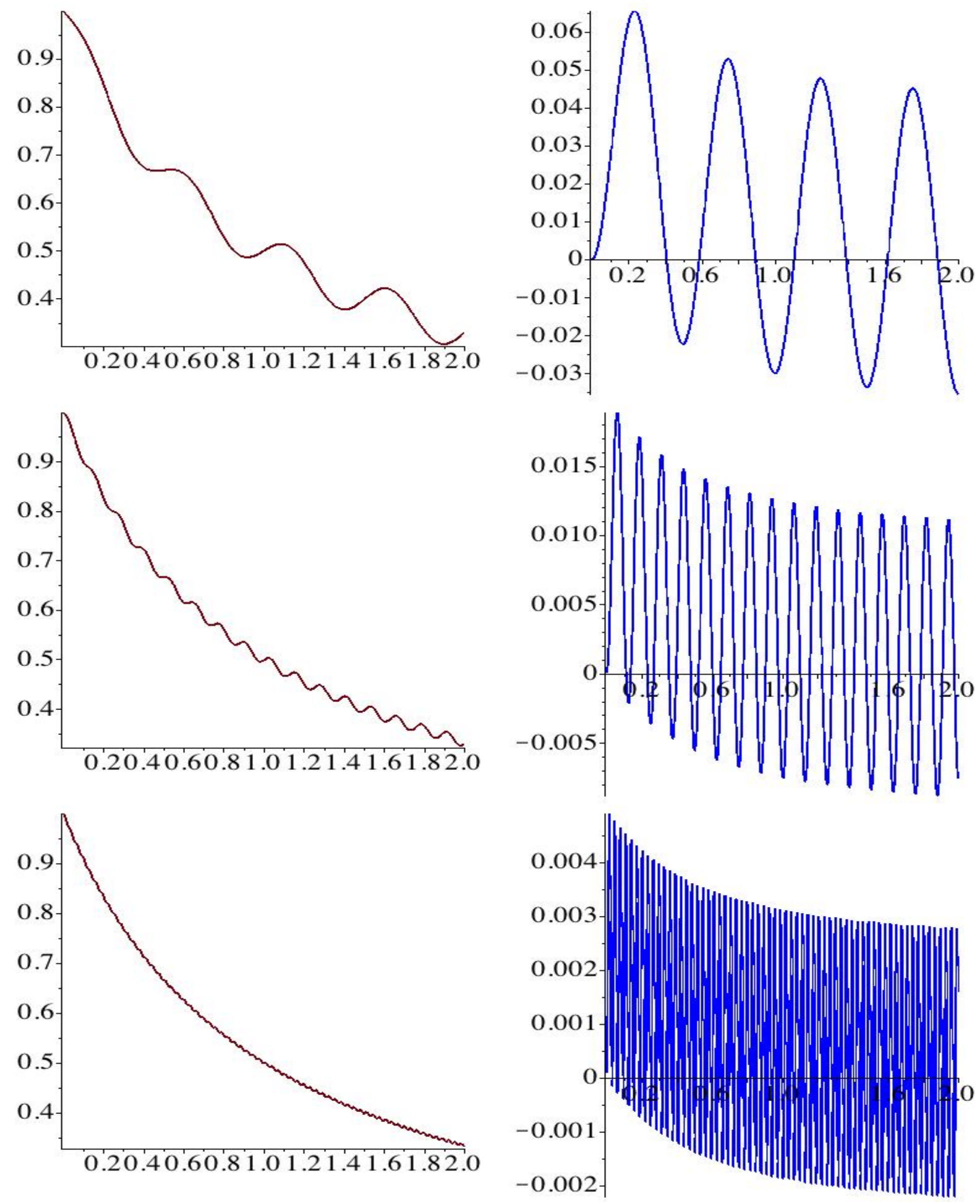


Figure 3.1: Real (on the left) and imaginary parts of the solution of (3.1) for $\omega = 12.5, 50$ and 200.

Fig. 3.3 displays the error $|p_0(t) + p_1(t,\omega) - y(t)|$ for different values of $\omega$. A comparison with Fig. 3.2 demonstrates that, consistently with the theory, the error decays considerably faster as $\omega$ grows, specifically as $\mathcal{O}(\omega^{-2})$ rather than $\mathcal{O}(\omega^{-1})$.

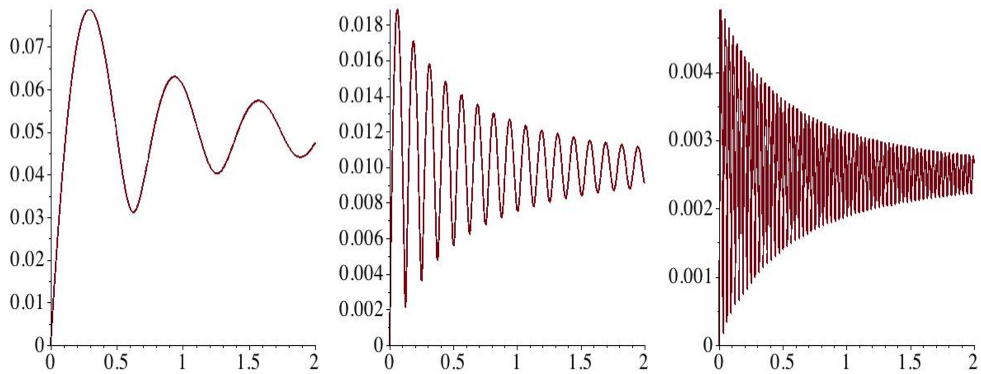

Figure 3.2: $|p_0(t) - y(t)|$ for $\omega = 12.5, 50, 200$.

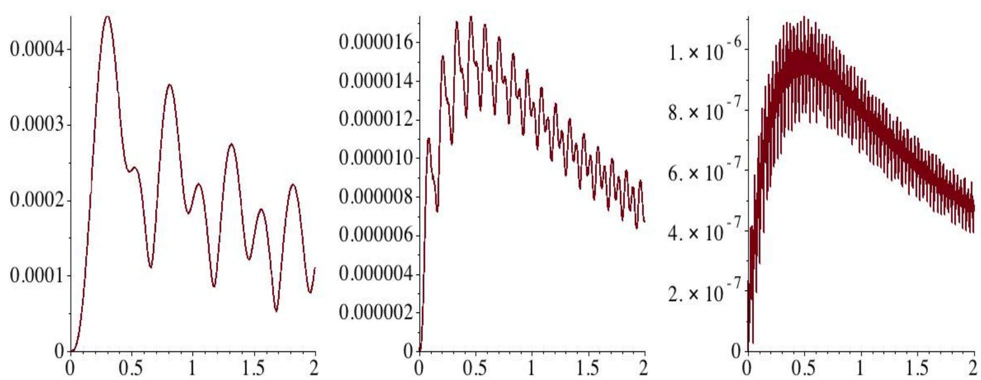

Figure 3.3: $|p_0(t) + p_1(t,\omega) - y(t)|$ for $\omega = 12.5, 50, 200$.

Next we compute

$$\begin{aligned}
p_2(t) &= -\frac{1}{4(1+t)^2}\int_0^t \frac{1}{(1+\tau)^2}\left[\int_0^\tau (1+\eta)^2 \mathrm{e}^{\mathrm{i}\omega\eta}\,\mathrm{d}\eta\right]^2 \mathrm{d}\tau \\
&= \frac{t}{4\omega^2(1+t)^3} + \frac{\mathrm{i}}{8\omega^3}\frac{-(3-5t)+4(1+t)\mathrm{e}^{\mathrm{i}\omega t}-(1+t)^3\mathrm{e}^{2\mathrm{i}\omega t}}{(1+t)^3} \\
&\quad + \frac{1}{8\omega^4}\frac{(11-5t)-8(2+t)\mathrm{e}^{\mathrm{i}\omega t}+5(1+t)^2\mathrm{e}^{2\mathrm{i}\omega t}}{(1+t)^3} \\
&\quad + \frac{\mathrm{i}}{16\omega^5}\frac{(27-5t)-16(3+t)\mathrm{e}^{\mathrm{i}\omega t}+21(1+t)\mathrm{e}^{2\mathrm{i}\omega t}}{(1+t)^3} + \frac{1}{\omega^6}\frac{-1+2\mathrm{e}^{\mathrm{i}\omega t}-\mathrm{e}^{2\mathrm{i}\omega t}}{(1+t)^3}.
\end{aligned}$$

Fig. 3.4 depicts the error $|p_0(t) + p_1(t,\omega) + p_2(t,\omega) - y(t)|$ for 'our' values of $\omega$. We have no easy explanation why the error does not decrease by as much as predicted

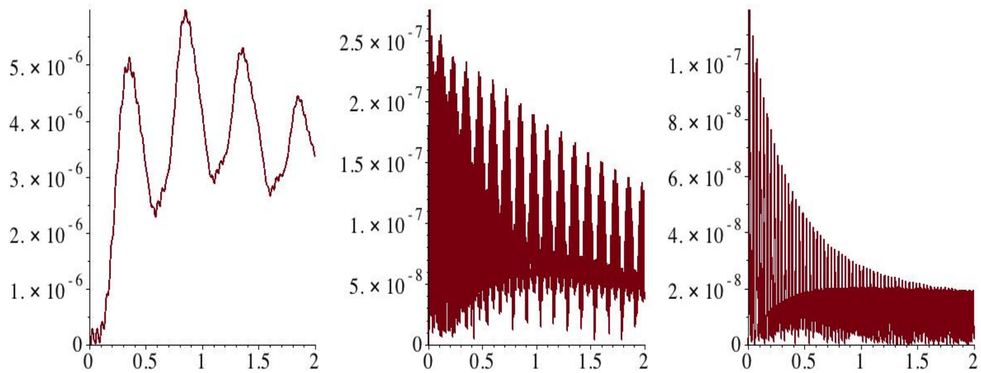


Figure 3.4: $|p_0(t) + p_1(t,\omega) + p_2(t,\omega) - y(t)|$ for $\omega = 12.5, 50, 200$.

by our theory for $\omega = 200$, a likely reason might be that the reference solution (using standard adaptive numerical method) is simply not accurate enough for the minute error tolerance called for in this computation or that, in the presence of a large number of minute steps, its error-control mechanism is distorted by significant roundoff error.

Few observations are in order. Firstly, it is hardly a surprise that $p_r = \mathcal{O}(\omega^{-r})$, $r = 0, 1, 2$, because this is predicted by general theory. Secondly, we can easily pluck from our expansion the leading terms of (1.2):

$$\begin{aligned}
p_{0,0}(t) &= p_0(t) = \frac{1}{1+t};\\
p_{1,0}(t) &= \frac{\mathrm{i}}{2(1+t)^2}, \qquad p_{1,1} \equiv \frac{\mathrm{i}}{2}, \qquad p_{1,m} \equiv 0, \quad m \neq 0, 1;\\
p_{2,0}(t) &= -\frac{4+3t}{(1+t)^3}, \qquad p_{2,1}(t) = \frac{1}{1+t}, \qquad p_{2,m} \equiv 0, \quad m \neq 0, 1.
\end{aligned}$$

This is a feature exclusive to the oscillator $\theta(t,\omega) = \mathrm{e}^{\mathrm{i}\omega t}$. Finally, we note that there is a transfer of wavelengths: while $p_1$, like the original equation, exhibits just the frequency $\mathrm{e}^{\mathrm{i}\omega t}$, we have in $p_2$ both $\mathrm{e}^{\mathrm{i}\omega t}$ and $\mathrm{e}^{2\mathrm{i}\omega t}$. This feature, which we called 'blossoming' in (Condon et al. 2010*a*), is characteristic of nonlinear systems and absent in (1.5), an expansion of a forced linear system.

We consider next

$$y' = -y^2 + \frac{1}{2}\mathrm{e}^{\mathrm{i}\omega(1-t)^2}, \quad t \geq 0, \qquad y(0) = 1. \tag{3.2}$$

Unlike (3.1), the oscillator has a stationary point at $t = 1$. The numerical solution is displayed in Fig. 3.5 and it is instructive to compare it with Fig. 3.1. As before, the oscillation becomes faster as $\omega$ grows, yet its amplitude decays – however, in the present case it decays at a different rate. In the interval $[0, 1)$ the decay, similarly to (3.1), is like $\mathcal{O}(\omega^{-1})$ but, once it crosses the stationary point, the amplitude increases

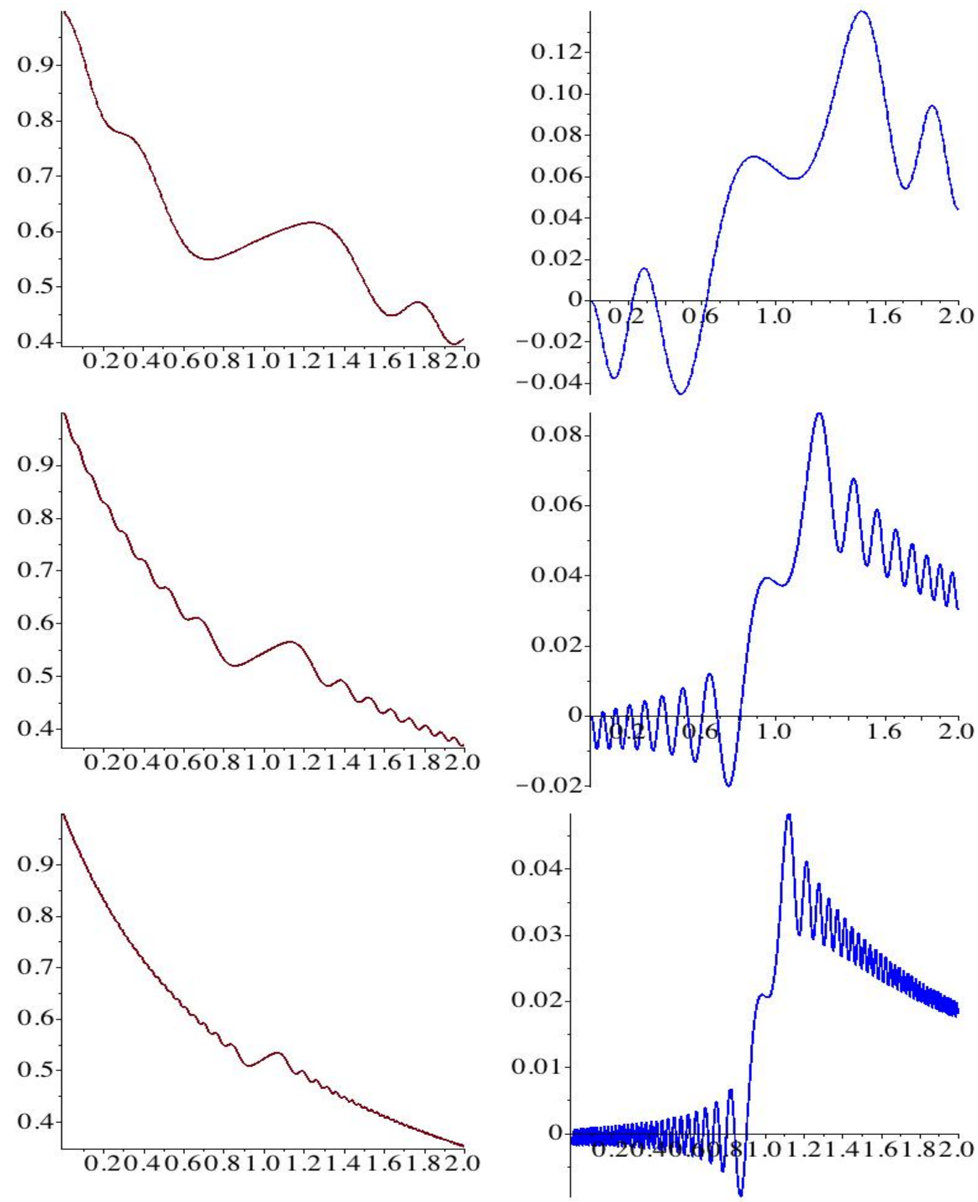


Figure 3.5: Real (on the left) and imaginary parts of the solution of (3.2) for $\omega = 12.5, 50$ and 200.

to $\mathcal{O}(\omega^{-1/2})$. In general, the entire character of the solution changes at the stationary point.

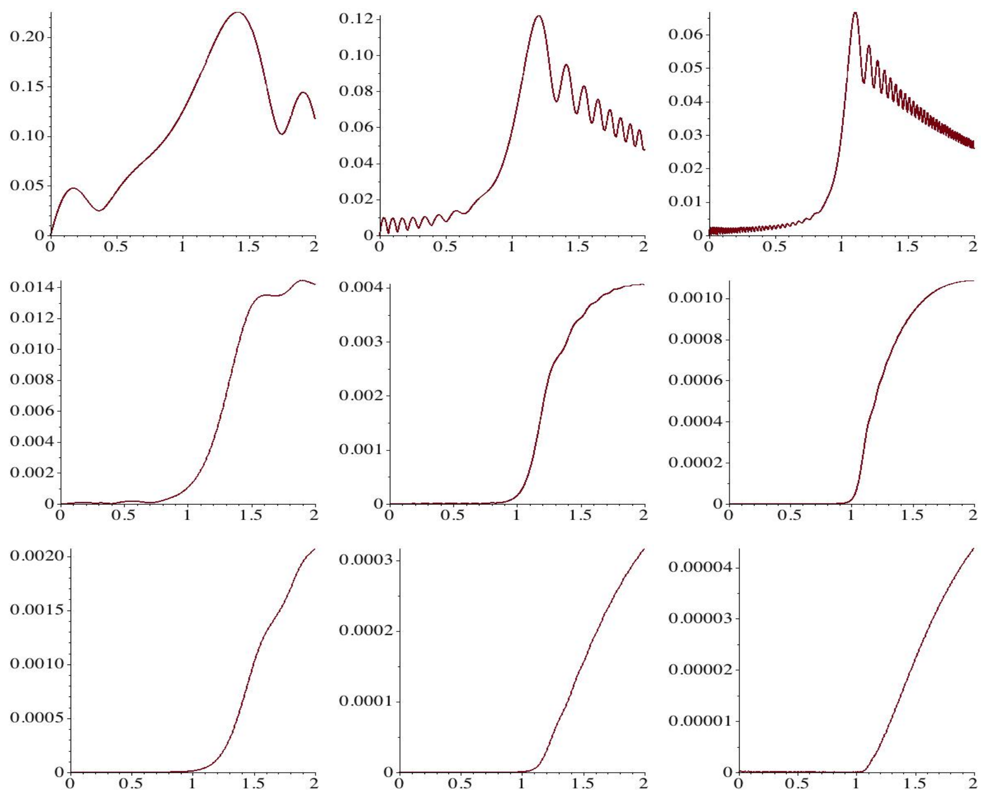

Figure 3.6: The error for $\omega = 12.5, 50, 200$ (in the three columns) for $p_0$ (top row), $p_0 + p_1$ (second row) and $p_0 + p_1 + p_2$ (bottom row) and equation (3.2).

The functions $p_0$ and $\Phi$ do not change, while $p_1$ can be computed explicitly,

$$\begin{aligned} p_1(t,\omega) &= \frac{1}{(1+t)^2}\int_0^t (1+\tau)^2 \mathrm{e}^{\mathrm{i}\omega(1-\tau)^2}\,\mathrm{d}\tau \\ &= \frac{\pi^{1/2}}{(-\mathrm{i}\omega)^{1/2}} \frac{\operatorname{erf}\left((-\mathrm{i}\omega)^{1/2}\right) - \operatorname{erf}\left((1-t)(-\mathrm{i}\omega)^{1/2}\right)}{(1+t)^2} \\ &\quad - \frac{1}{-\mathrm{i}\omega}\frac{-3\mathrm{e}^{\mathrm{i}\omega} + (3+t)\mathrm{e}^{\mathrm{i}\omega(1-t)^2}}{4(1+t)^2} \\ &\quad + \frac{\pi^{1/2}}{(-\mathrm{i}\omega)^{3/2}} \frac{\operatorname{erf}\left((-\mathrm{i}\omega)^{1/2}\right) - \operatorname{erf}\left((1-t)(-\mathrm{i}\omega)^{1/2}\right)}{8(1+t)^2}, \end{aligned}$$

where erf is the error function (Olver et al. 2010). Since $\operatorname{erf} z \sim 1 - \pi^{-1/2}\mathrm{e}^{-z^2}/z$ for $|\arg z| < \frac{3}{4}\pi$, this is the case for $\operatorname{erf}\left((-\mathrm{i}\omega)^{1/2}(1-t)\right)$ for $t \in (0,1)$, as well as for

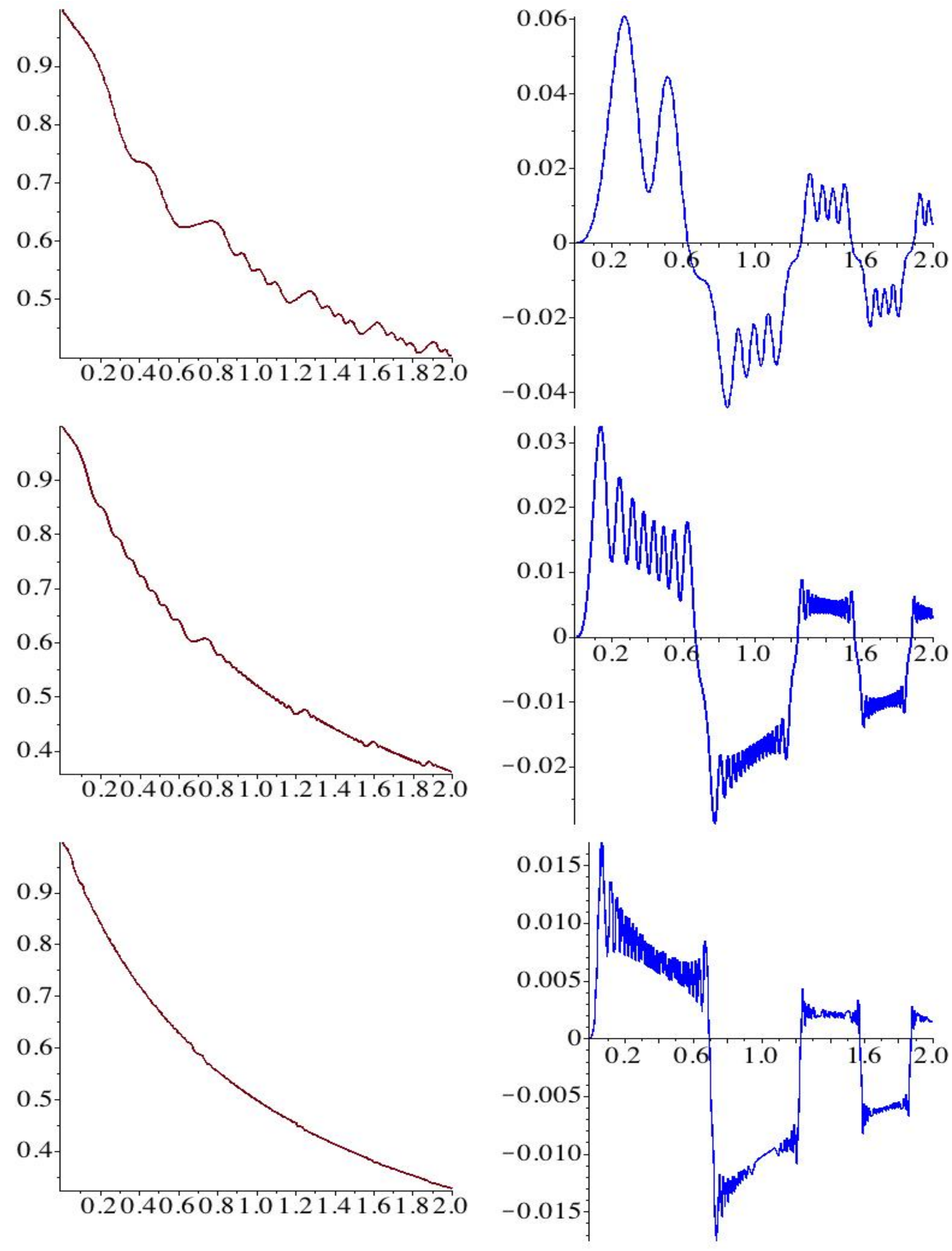


Figure 3.7: Real (on the left) and imaginary parts of the solution of (3.3) for $\omega = 12.5, 50$ and $200$.

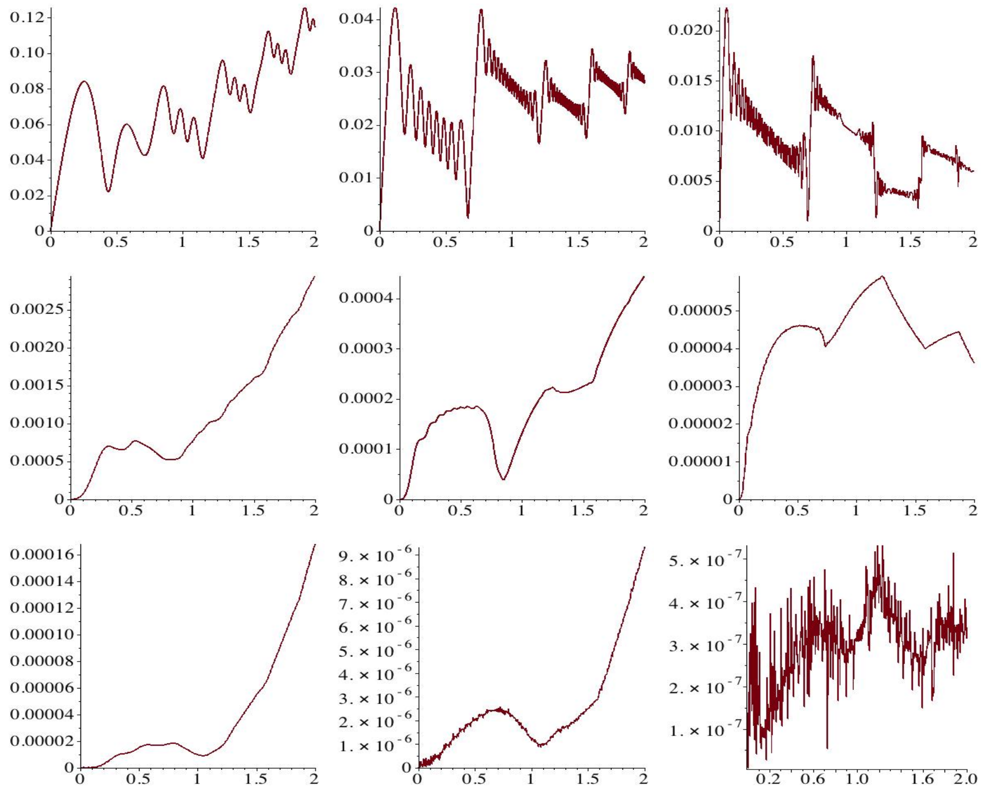


Figure 3.8: The error for $\omega = 12.5, 50, 200$ (in the three columns) for $p_0$ (top row), $p_0 + p_1$ (second row) and $p_0 + p_1 + p_2$ (bottom row) and equation (3.3).

$\operatorname{erf}((-\mathrm{i}\omega)^{1/2})$. Therefore

$$\operatorname{erf}((-\mathrm{i}\omega)^{1/2}) - \operatorname{erf}((1-t)(-\mathrm{i}\omega)^{1/2})) = \mathcal{O}\Big(\omega^{-1/2}\Big)$$

and $p_1(t,\omega) = \mathcal{O}(\omega^{-1})$. However, for $t > 1$ we need to use the parity of the error function, $\operatorname{erf}(-z) = -\operatorname{erf} z$, for $z = (-\mathrm{i}\omega)^{1/2}(1-t)$ to fit within $|\arg z| < \frac{3}{4}\pi$ and it readily follows that $p_1(t,\omega) = \mathcal{O}(\omega^{-1/2})$ for $t \geq 1$. The double integral $p_2$, however, cannot be computed explicitly and we have resorted to high precision numerical integration in our experiments.

Fig. 3.6 displays the errors of truncated expansions $\sum_{r=0}^{R} p_r$ for $R = 0, 1, 2$ and $\omega \in \{10, 50, 100\}$. All is in perfect accordance with the theory. Thus, the precision is much greater in $(0,1)$ but, once $t$ crosses a stationary point and the nature of the asymptotic expansion changes, the error is larger, while its decay as $\omega$ grows is significantly slower.

Our final example is

$$y' = -y^2 + \frac{1}{2}\mathrm{e}^{\mathrm{i}\omega \sin \pi t^2}, \quad t \ge 0, \qquad y(0) = 1. \tag{3.3}$$

The oscillator has stationary points at the origin and at $\sqrt{n-\frac{1}{2}}$, $n \in \mathbb{Z}_+$. Thus, $y(t) - p_0(t) = \mathcal{O}\big(\omega^{-1/2}\big)$ for all $t \ge 0$ and the solution undergoes an infinite sequence of changes to its behavior at the stationary points. This can be seen vividly in Fig. 3.7 by examining the imaginary part of the solution.

Fig. 3.8 displays the error committed by approximating $y$ by $\sum_{r=1}^{R} p_r$ for $R \in \{0,1,2\}$. The figure is fully consistent with our theory. Thus, the error drops down for increasing $R$ (in fact, it is $\mathcal{O}\big(\omega^{-(R+1)/2}\big)$), but also for increasing $\omega$.

# 4 The computation of $\boldsymbol{p}_r$: preliminary ideas

Let us assume that $\theta(t,\omega) = \mathrm{e}^{\mathrm{i}\omega g(t)}$, since in that case the behaviour of oscillatory integrals is well understood (Wong 1989).

To integrate (2.1) (or (2.2)) we advance by steps $t_0 = 0 < t_1 < t_2 < \cdots$. The steps need not be small: the accuracy of the method follows from the asymptotic expansion (1.6) rather than from conventional concepts of order. However, it is important that all stationary points of the oscillators are in the set $\{t_n\}$ of time steps and that if $t_n$ is a stationary point than neither $t_{n-1}$ nor $t_{n+1}$ can be one. Therefore we have two types of intervals of integration $(t_n, t_{n+1})$: *type I,* when neither $t_n$ nor $t_{n+1}$ is a stationary point, and *type II,* when either $t_n$ or $t_{n+1}$ (but not both!) is a stationary point. We may in that case assume that $t_n$ is a stationary point, since the other case follows by trivial change of variable.

The function $\Phi^{-1}(t)\boldsymbol{p}_r$ in (2.10) is formally an integral over the cube $(t_n, t_{n+1})^r$ but the function $\boldsymbol{s}_r$ is piecewise-smooth there and it is a good idea to partition the cube into $r!$ simplices where $\boldsymbol{s}_r$ is smooth (or, at any rate, shares the smoothness of $\boldsymbol{f}$ and $\boldsymbol{a}$). Each such simplex can be obtained by commencing with the vertex $(t_n, t_n, \ldots, t_n) \in \mathbb{R}^r$ and generating further $r$ vertices by replacing one $t_n$ at a time by $t_{n+1}$. After $r$ such steps we reach the vertex $(t_{n+1}, t_{n+1}, \ldots, t_{n+1})$.

For $r = 1$ we have just one simplex, the interval with vertices $t_n$ and $t_{n+1}$. For $r = 2$ our procedure yields two simplexes, with the vertices

$$(t_n,t_n),(t_n,t_{n+1}),(t_{n+1},t_{n+1}) \qquad \text{and} \qquad (t_n,t_n),(t_{n+1},t_n),(t_{n+1},t_{n+1}).$$

$r = 3$ results in six simplexes, whose vertices are

$$\begin{gathered}
(t_n,t_n,t_n),(t_n,t_n,t_{n+1}),(t_n,t_{n+1},t_{n+1}),(t_{n+1},t_{n+1},t_{n+1}),\\
(t_n,t_n,t_n),(t_n,t_n,t_{n+1}),(t_{n+1},t_n,t_{n+1}),(t_{n+1},t_{n+1},t_{n+1}),\\
(t_n,t_n,t_n),(t_n,t_{n+1},t_n),(t_n,t_{n+1},t_{n+1}),(t_{n+1},t_{n+1},t_{n+1}),\\
(t_n,t_n,t_n),(t_n,t_{n+1},t_n),(t_{n+1},t_{n+1},t_n),(t_{n+1},t_{n+1},t_{n+1}),\\
(t_n,t_n,t_n),(t_{n+1},t_n,t_n),(t_{n+1},t_n,t_{n+1}),(t_{n+1},t_{n+1},t_{n+1}),\\
(t_n,t_n,t_n),(t_{n+1},t_n,t_n),(t_{n+1},t_{n+1},t_n),(t_{n+1},t_{n+1},t_{n+1})
\end{gathered}$$

and so on.

The asymptotic behaviour for large $\omega$ of an integral of the form

$$\int_{\mathcal{S}} u(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\omega g(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x},$$

where $\mathcal{S} \in \mathbb{R}^r$ is a polytope, depends on three kinds of *critical points:*

1. *Stationary points* $\boldsymbol{x}_* \in \mathrm{cl}\,\mathcal{S}$ where $\boldsymbol{\nabla} g(\boldsymbol{x}_*) = \boldsymbol{0}$;

2. *Resonance points* $\boldsymbol{x}_* \in \partial\mathcal{S}$ where $\boldsymbol{\nabla} g(\boldsymbol{x}_*) \neq \boldsymbol{0}$ is orthogonal to the boundary; and

3. *Vertices* of the polytope $\mathcal{S}$

(Huybrechs & Olver 2009, Wong 1989). In our case

$$\boldsymbol{\nabla} g(\boldsymbol{x}) = \begin{bmatrix} g'(x_1) \\ g'(x_2) \\ \vdots \\ g'(x_r) \end{bmatrix}.$$

Therefore for Type I integrals there is no stationary point in a simplex, while for Type II integrals there is a single stationary point at $(t_{n+1}, t_{n+1}, \ldots, t_{n+1})$. Moreover, in neither case are there any resonance points. Therefore, the multivariate integral inherits its behaviour in a fairly straightforward manner from the univariate one. Specifically, in the case of Type I integrals $\boldsymbol{p}_r(t_{n+1}, \omega) = \mathcal{O}(\omega^{-r})$, while for Type II it is true that $\boldsymbol{p}_r(t_{n+1}, \omega) = \mathcal{O}\big(\omega^{-r/(\varsigma+1)}\big)$, where $g'(t_n) = g''(t_n) = \cdots = g^{(\varsigma)}(t_n) = 0$ and $g^{(\varsigma+1)}(t_n) \neq 0$.

A word of caution is in order: although formally $\boldsymbol{p}_r(t_{n+1}, \omega) = \mathcal{O}(\omega^{-r})$ for Type I, the amplitude of the $\mathcal{O}(\omega^{-r})$ term can be very large when either $t_n$ or $t_{n+1}$ is close to a stationary point. Therefore, a safe strategy is to employ Type I integrals only when the endpoints are well separated from stationary points. Once $t_{n+2} - t_{n+1}$, say, is small, it is a much better idea to go from $t_n$ all the way to $t_{n+2}$, subsequently backtracking from $t_{n+2}$ to $t_{n+1}$ – in other words, replacing one Type I and one Type II integral by two Type II integrals.

## 4.1 Asymptotic expansions

The computation of explicit asymptotic expansions of multivariate highly oscillatory integrals is possible, at least in principle, but, once dimensions increase, rapidly become very complicated, arguably not very helpful (Iserles & Nørsett 2006).

The situation is not unduly complicated for type I integrals, once we follow upon the ideas of (Iserles & Nørsett 2005). Let $f$ be a smooth function in the closure of the simplex in question. Recalling that $g' \neq 0$ in $(t_n, t_{n+1})$, a univariate expansion follows at once from (Iserles & Nørsett 2005),

$$\int_{t_n}^{t_{n+1}} f(\xi)\mathrm{e}^{\mathrm{i}\omega g(\xi)}\,\mathrm{d}\xi \sim -\sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} \left[ f_m(t_{n+1}) \frac{\mathrm{e}^{\mathrm{i}\omega g(t_{n+1})}}{g'(t_{n+1})} - f_m(t_n) \frac{\mathrm{e}^{\mathrm{i}\omega g(t_n)}}{g'(t_n)} \right],$$

where

$$f_0(\xi) = f(\xi), \qquad f_{m+1}(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{f_m(\xi)}{g'(\xi)}, \quad m \in \mathbb{Z}_+.$$

It takes more effort to derive a bivariate expansion yet, after fairly standard algebra using the methodology of (Iserles & Nørsett 2006), we obtain, for example,

$$\begin{aligned}
&\int_{t_n}^{t_{n+1}}\int_{\xi_1}^{t_{n+1}} f(\xi_1,\xi_2)\mathrm{e}^{\mathrm{i}\omega g(\xi_2)}\,\mathrm{d}\xi_2\mathrm{e}^{\mathrm{i}\omega g(\xi_1)}\,\mathrm{d}\xi_1\\
&\sim -\frac{\mathrm{e}^{\mathrm{i}\omega g(t_{n+1})}}{g'(t_{n+1})}\sum_{\ell=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{\ell+1}}\int_{t_n}^{t_{n+1}} f_\ell(\xi,t_{n+1})\mathrm{e}^{\mathrm{i}\omega g(\xi)}\,\mathrm{d}\xi\\
&\quad+\sum_{\ell=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{\ell+1}}\int_{t_n}^{t_{n+1}}\frac{f_\ell(\xi,\xi)}{g'(\xi)}\mathrm{e}^{2\mathrm{i}\omega g(\xi)}\,\mathrm{d}\xi,
\end{aligned}$$

where

$$f_0(\xi_1,\xi_2) = f(\xi_1,\xi_2), \qquad f_{\ell+1}(\xi_1,\xi_2) = \frac{\partial}{\partial\xi_2}\frac{f_{0,\ell}(\xi_1,\xi_2)}{g'(\xi_2)}, \quad \ell \in \mathbb{Z}_+.$$

Let

$$\begin{aligned}
f_{0,\ell}(\xi) &= f_\ell(\xi,t_{n+1}), & f_{m+1,\ell}(\xi) &= \frac{\partial}{\partial\xi}\frac{f_{m,\ell}(\xi)}{g'(\xi)},\\
\tilde{f}_{0,\ell}(\xi) &= \frac{f_\ell(\xi,\xi)}{g'(\xi)}, & \tilde{f}_{m+1,\ell}(\xi) &= \frac{\partial}{\partial\xi}\frac{\tilde{f}_{m,\ell}(\xi)}{g'(\xi)}, \quad m \in \mathbb{Z}_+.
\end{aligned}$$

Then, expanding again,

$$\begin{aligned}
&\int_{t_n}^{t_{n+1}}\int_{\xi_1}^{t_{n+1}} f(\xi_1,\xi_2)\mathrm{e}^{\mathrm{i}\omega g(\xi_2)}\,\mathrm{d}\xi_2\mathrm{e}^{\mathrm{i}\omega g(\xi_1)}\,\mathrm{d}\xi_1\\
&\sim -\frac{\mathrm{e}^{2\mathrm{i}\omega g(t_{n+1})}}{g'(t_{n+1})}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+2}}\sum_{\ell=0}^{m}\frac{\tilde{f}_{m,\ell}(t_{n+1})}{2^{m-\ell}}\\
&\quad+\frac{\mathrm{e}^{2\mathrm{i}\omega g(t_{n+1})}}{g'^2(t_{n+1})}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+2}}\sum_{\ell=0}^{m} f_{m,\ell}(t_{n+1})\\
&\quad-\frac{\mathrm{e}^{\mathrm{i}\omega[g(t_n)+g(t_{n+1})]}}{g'(t_n)g'(t_{n+1})}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+2}}\sum_{\ell=0}^{m} f_{m,\ell}(t_n)\\
&\quad+\frac{\mathrm{e}^{2\mathrm{i}\omega g(t_n)}}{g'(t_n)}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+2}}\sum_{\ell=0}^{m}\frac{\tilde{f}_{m,\ell}(t_n)}{2^{m-\ell}}.
\end{aligned}$$

This procedure can be carried out also in higher dimensions although, needless to say, its complexity increases rapidly.

Matters are becoming considerably more complicated for type II integrals. Assume thus that $g'(t_{n+1}) = 0$, $g''(t_{n+1})$ and that otherwise $g$ is strictly monotone. A univariate expansion is standard

$$\int_{t_n}^{t_{n+1}} f(\xi)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x \sim \int_{t_n}^{t_{n+1}} \mathrm{e}^{\mathrm{i}\omega g(\xi)}\,\mathrm{d}\xi \sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^m} f_m(t_{n+1})$$
$$- \sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} \left\{ f_m'(t_{n+1}) \frac{\mathrm{e}^{\mathrm{i}\omega g(t_{n+1})}}{g''(t_{n+1})} - [f_m(t_n) - f_m(t_{n+1})] \frac{\mathrm{e}^{\mathrm{i}\omega g(t_n)}}{g'(t_n)} \right\},$$

where
$$f_0(\xi) = f(\xi), \qquad f_{m+1}(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi} \frac{f_m(\xi) - f_m(t_{n+1})}{g'(\xi)}, \quad m \in \mathbb{Z}_+$$

(Iserles & Nørsett 2005). Unfortunately, a generalisation to a bivariate simplex, although possible in principle, rapidly leads to fairly unpleasant expressions. It is reasonable to rule out this approach as a practical computational tool for the different dimensions and diverse oscillators occuring within the framework of this paper.

## 4.2 Filon-type expansions

As pointed out in (Iserles & Nørsett 2006), while asymptotic expansions are next to useless as a direct computational tool in a multivariate setting, they are immensely useful in identifying the data necessary for the implementation of *Filon-type methods.*

Let $f : \mathcal{S} \to \mathbb{C}$ and $g : \mathcal{S} \to \mathbb{C}$ be smooth functions in a closed polytope $\mathcal{S} \subset \mathbb{R}^R$ with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_M \in \mathbb{R}^R$. Assume further that $\boldsymbol{\nabla} g$ is nonzero in $\mathcal{S}$, except possibly at the vertices and is nowhere orthogonal to the faces of the polytope. Then there exists an asymptotic expansion of the form

$$\mathcal{I}[f] := \int_{\mathcal{S}} f(\boldsymbol{\xi})\mathrm{e}^{\mathrm{i}\omega[g(\xi_1)+\cdots+g(\xi_R)]}\,\mathrm{d}\boldsymbol{\xi} \sim \sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{(m+R)/(\varsigma+1)}} \sum_{\ell=1}^{M} \mathcal{L}_{m,\ell}[f](\boldsymbol{v}_\ell), \quad (4.1)$$

where $\varsigma$ is the highest degree of a stationary point (that is, the largest number such that $\partial^{\boldsymbol{i}} g(\boldsymbol{v}_j) = 0$ for a vertex, $j \in \{1, 2, \ldots, M\}$ and all $|\boldsymbol{i}| = 1, 2, \ldots, \varsigma$ and $\varsigma = 0$ if $\boldsymbol{\nabla} g \neq \mathbf{0}$ at all the vertices). The functionals $\mathcal{L}_{m,\ell}$ are all linear and depend on $f$ and its derivatives. Let $\varphi$ be any function that interpolates $f$ and its derivatives at the vertices, specifically so that $\mathcal{L}_{m,\ell}[\varphi - f] = 0$ for all $m = 0, 1, \ldots, p$ and $\ell = 1, \ldots, M$. Then the Filon-type quadrature

$$\mathcal{F}[f] := \int_{\mathcal{S}} \varphi(\boldsymbol{\xi})\mathrm{e}^{\mathrm{i}\omega[g(\xi_1)+\cdots+g(\xi_R)]}\,\mathrm{d}\boldsymbol{\xi} \quad (4.2)$$

satisfies $\mathcal{F}[f] = \mathcal{I}[f] + \mathcal{O}\big(\omega^{-(p+R)/(\varsigma+1)}\big)$, as can be verified at once through a substitution of $\varphi - f$ in place of $f$ in (4.1) (Iserles & Nørsett 2006). Once the integral (4.2) can be evaluated explicitly, we can use it to obtain a high-quality approximation to (4.1).

An efficient computation of highly oscillatory integrals over simplexes using Filon-type and other methods is beyond the scope of the current paper and a matter for future investigation. In this paper we wish just to provide two examples, both of bivariate integrals, that demonstrate the feasibility of this approach, at least for some

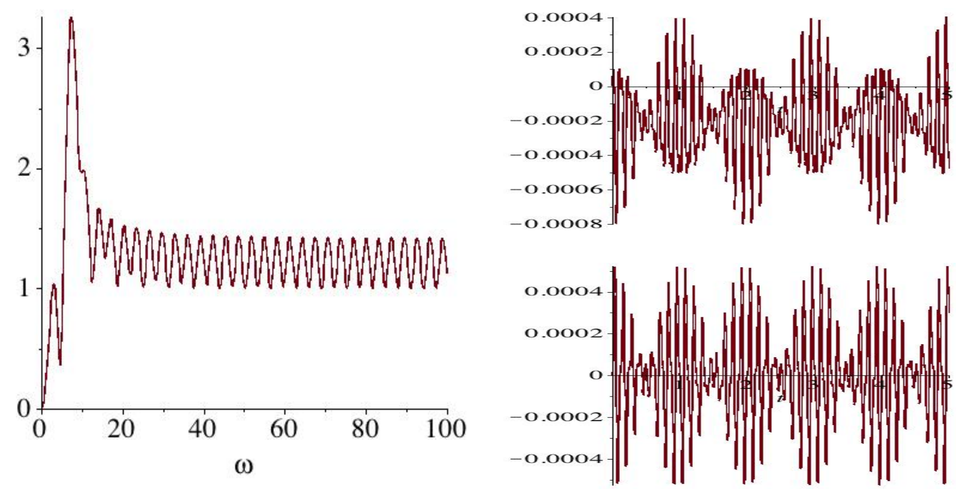


Figure 4.1: The integral $\boldsymbol{I}_\omega(t)$ for the type I example. On the left we display $\omega^2|\boldsymbol{I}_\omega(1)|$, to highlight the $\mathcal{O}(\omega^{-2})$ decay in amplitude. On the right we display the absolute value and the real (at the top) and imaginary parts of $\boldsymbol{I}_{50}(t)$, noting the very rapid oscillation of $\boldsymbol{I}_\omega(t)$ as a function of $t$.

oscillators. In both cases we consider a simplex with the vertices $(0,0)$, $(t,0)$ and $(t,t)$, that is

$$\mathcal{S} = \{(\xi_1,\xi_2) : 0 \le \xi_2 \le \xi_1, \quad 0 \le \xi_1 \le t, \}.$$

Firstly, consider $f(x_1,x_2) = \cos\pi(x_1 - 2x_2)$ and $g(x) = x$, a type I integral and let $\boldsymbol{I}_\omega(t) = \mathcal{I}[f]$, to highlight the dependence upon $t$ and $\omega$. The behaviour of the integral for varying $t$ and $\omega$ is highlighted in Fig. 4.1.

We consider two Filon-type schemes, both with polynomial bases. In the first case, henceforth denoted by $\boldsymbol{F}_\omega^{[1]}(t)$, we use the linear bivariate basis $\{1, x_1, x_2\}$ to interpolate $f$ at the vertices, while in the second, $\boldsymbol{F}_\omega^{[2]}(t)$, the basis of bivariate cubics is used to interpolate to both $f$ and its two directional derivatives at the vertices, as well as to the value of $f$ at $(\frac{2}{3}t, \frac{1}{3}t)$. In both cases the integral (4.2) can be easily computed explicitly.

We display in Fig. 4.2 the error for both methods for fixed $t$ and increasing $\omega$ in logarithmic scale – essentially, exhibiting the number of significant digits recovered by the numerical approximation. It is clear that, using just a small number of function values, we can recover the integral to a surprisingly high accuracy which increases with $\omega$ – everything is consistent with the theoretical estimates, $|\boldsymbol{F}_\omega^{[1]}(t) - \boldsymbol{I}_\omega(t)| = \mathcal{O}(\omega^{-3})$ and $|\boldsymbol{F}_\omega^{[2]}(t) - \boldsymbol{I}_\omega(t)| = \mathcal{O}(\omega^{-4})$.

As an example of a type II integral we consider (in the same simplex) the function $f(x_1,x_2) = \mathrm{e}^{x_1 - 2x_2}$ and the oscillator $g(x) = x^2$. We can use a polynomial basis because the *moments* $\mathfrak{m}_{m,n} = \int_{\mathcal{S}} x_1^m x_2^n \mathrm{e}^{\mathrm{i}\omega(x_1^2+x_2^2)}\,\mathrm{d}\boldsymbol{x}$ can be computed explicitly for

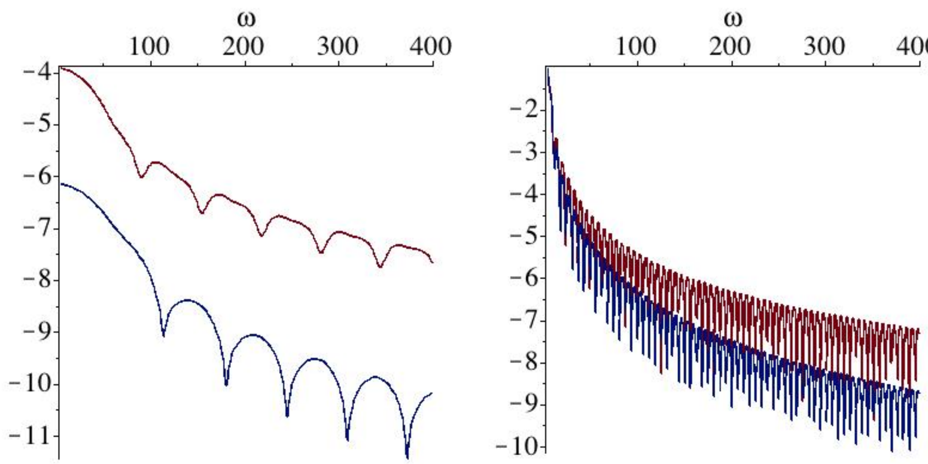


Figure 4.2: The values of $\log_{10}|\boldsymbol{F}_\omega^{[j]}(t)-\boldsymbol{I}_\omega(t)|$, $j=1,2$, for $t=\frac{1}{10}$ (on the left) and $t=1$ (on the right) and the type I integral. The top curve corresponds to $\boldsymbol{F}_\omega^{[1]}$.

all $m,n\in\mathbb{Z}_+$, e.g.

$$\begin{aligned}
\mathfrak{m}_{0,0}(t,\omega) &= \frac{\pi}{8}\frac{\operatorname{erf}^2((-\mathrm{i}\omega)^{1/2}t)}{-\mathrm{i}\omega},\\
\mathfrak{m}_{1,0}(t,\omega) &= \frac{\pi^{1/2}}{8}\frac{2^{1/2}\operatorname{erf}((-2\mathrm{i}\omega)^{1/2}t)-2\mathrm{e}^{\mathrm{i}\omega t^2}\operatorname{erf}((-\mathrm{i}\omega)^{1/2}t)}{(-\mathrm{i}\omega)^{3/2}},\\
\mathfrak{m}_{0,1}(t,\omega) &= \frac{\pi^{1/2}}{8}\frac{2\operatorname{erf}((-\mathrm{i}\omega)^{1/2}t)-2^{1/2}\operatorname{erf}((-2\mathrm{i}\omega)^{1/2}t)}{(-\mathrm{i}\omega)^{3/2}}.
\end{aligned}$$

Our first example uses linear functions to interpolate $f$ at the vertices and its first derivatives at $(0,0)$ – we have 5 interpolation conditions and 6 elements in our basis $\{1,x_1,x_2,x_1^2,x_1x_2,x_2^2\}$ and, to match conditions to degrees of freedom, we also interpolate at $(\frac{2}{3}t,\frac{1}{3}t)$. Using similar notation to the previous example, we denote the exact integral and the outcome of this Filon-type method by $\boldsymbol{I}_\omega$ and $\boldsymbol{F}_\omega^{[1]}$ respectively.

In our second example we interpolate to $f$ and its first two derivatives (6 conditions altogether) at $(0,0)$, $f$ and its first derivatives at $(t,0)$ and just $f$ at $(t,t)$. Thus we have 10 conditions, matching the number of bivariate cubics. Alas, it is impossible to use a base of cubics to interpolate to the above Hermite data. However, once we complement the cubic base with $x_1^4$, while adding an interpolation condition at $(\frac{2}{3}t,\frac{1}{3}t)$, all is well and this yields $\boldsymbol{F}_\omega^{[2]}$.

Fig. 4.3 depicts the number of significant digits in $\boldsymbol{F}_\omega^{[j]}(1)$ for $j=1,2$. (Unlike Fig. 4.2, there is little point in displaying the error for $t=\frac{1}{10}$ since then the integral is non-oscillatory.) As predicted by theory, $\boldsymbol{F}_\omega^{[2]}$ is better. However, the slower decay for $\omega\gg 1$ (compared to type I integrals and Filon-type methods – in the current setting we have $\boldsymbol{F}_\omega^{[1]}[f]-\boldsymbol{I}_\omega[f]=\mathcal{O}(\omega^{-2})$ and $\boldsymbol{F}_\omega^{[2]}[f]-\boldsymbol{I}_\omega[f]=\mathcal{O}(\omega^{-5/2})$) means that the

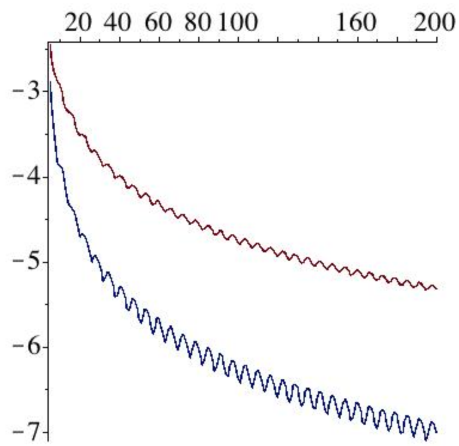


Figure 4.3: $\log_{10} |\boldsymbol{F}_\omega^{[j]}(1) - \boldsymbol{I}_\omega(1)|$, $j = 1, 2$, for the type II integral. The top curve corresponds to $\boldsymbol{F}_\omega^{[1]}$.

accuracy is lower (although not by much), in comparison to Fig. 4.2. This emphasises a broader point about our asymptotic expansions: once stationary points are present, we need more terms in the expansion (1.6) to attain the same accuracy – but also the approximation of each term requires more data and effort.

The purpose of this section is to emphasise the important point that the highly oscillatory integrals (2.10) (or indeed (2.13)) *can* be computed effectively by numerical approximation. We definitely do not seek here to discuss this issue at any great depth. Several important methodologies have emerged from modern theory of computational highly oscillatory quadrature – not just Filon-type (Huybrechs & Olver 2012, Iserles & Nørsett 2006) but also Levin-type (Olver 2006) and stationary phase (Huybrechs & Vandewalle 2007) methods, as well as complex-valued Gaussian quadrature (Deaño & Huybrechs 2009). An attractive option is to generalise the univariate approach of (Olver 2007), namely a Filon-type method without any need to calculate moments, to a multivariate setting. All this is a matter for further exploration.

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