Asymptotic Solvers for Highly Oscillatory Semi-explicit DAEs

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Abstract

The paper is concerned with the discretization and solution of DAEs of index 1 and subject to a highly oscillatory forcing term. Separate asymptotic expansions in inverse powers of the oscillatory parameter are constructed to approximate the differential and algebraic variables of the DAEs. The series are truncated to enable practical implementation. Numerical experiments are provided to illustrate the effectiveness of the method.

1 Introduction

Differential Algebraic Equations (DAEs) arise in numerous applications and in particular in circuit and device simulation [10, 11]. Accurate numerical solvers or discretization of such equations presents many challenges [7, 8]. These challenges are compounded in the presence of highly oscillatory forcing terms and it is the purpose of this paper to address these. We wish to analyse the behaviour of the system on a time scale which is much larger

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than the period of the forcing term and obtain an efficient method that is not restricted by the high frequency of the forcing term.

We are concerned with semi-explicit time-varying highly oscillatory DAEs of the form

$$\begin{aligned} \boldsymbol{x}'(t) &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) + \sum_{m=-\infty}^{\infty} \boldsymbol{a}_m(t)\mathrm{e}^{\mathrm{i}m\omega t}, \quad t \ge 0, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) + \frac{1}{\omega}\sum_{m=-\infty}^{\infty} \boldsymbol{b}_m(t)\mathrm{e}^{\mathrm{i}m\omega t}. \end{aligned} \tag{1.1}$$

where $\boldsymbol{x}(t) : \mathbb{R} \to \mathbb{C}^{d_1}$, $\boldsymbol{y}(t) : \mathbb{R} \to \mathbb{C}^{d_2}$, $\omega \gg 1$, while $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) : \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \to \mathbb{C}^{d_1}$ and $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) : \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \to \mathbb{C}^{d_2}$ are two analytic functions. We further assume that the Jacobian $\partial\boldsymbol{g}/\partial\boldsymbol{y}$ is nonsingular which means that the DAEs are of index 1.

In this paper, we introduce an asymptotic expansion of the solution of the DAE (1.1) in inverse powers of ω. Each term in this expansion can be obtained recursively using operations which, being independent of ω, are non-oscillatory. Besides their intrinsic value as an analytic tool, our expansions can be employed as an exceedingly affordable and precise numerical method.

In Section 2, we formulate the asymptotic expansion for linear DAEs with highly oscillatory terms. A numerical example is provided to illustrate the theoretical results. Section 3 is concerned with nonlinear DAEs and again a numerical example is provided.

2 Linear DAEs

Consider a set of linear DAEs

$$\begin{aligned} \boldsymbol{x}'(t) &= A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{y}(t) + \sum_{m=-\infty}^{\infty} \boldsymbol{a}_m(t)\mathrm{e}^{\mathrm{i}m\omega t}, \quad t \ge 0, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \\ \boldsymbol{0} &= C(t)\boldsymbol{x}(t) + D(t)\boldsymbol{y}(t) + \frac{1}{\omega}\sum_{m=-\infty}^{\infty} \boldsymbol{b}_m(t)\mathrm{e}^{\mathrm{i}m\omega t}. \end{aligned} \tag{2.1}$$

where $\boldsymbol{x}(t) : \mathbb{R} \to \mathbb{C}^{d_1}$, $\boldsymbol{y}(t) : \mathbb{R} \to \mathbb{C}^{d_2}$, $\omega \gg 1$, and the matrix functions are

$$A(t) : \mathbb{R} \to \mathbb{C}^{d_1 \times d_1}, \qquad B(t) : \mathbb{R} \to \mathbb{C}^{d_1 \times d_2},$$

$$C(t):\mathbb{R}\to\mathbb{C}^{d_2\times d_1},\qquad D(t):\mathbb{R}\to\mathbb{C}^{d_2\times d_2},$$

The condition that $D(t)$ is invertible is imposed so that the index of the DAE set is 1. The first equation of (2.1) is a differential equation. Had it been independent of $\boldsymbol{y}$, we could have expressed its solution $\boldsymbol{x}$ as an asymptotic series

$$\boldsymbol{x}(t)\sim\boldsymbol{p}_{0,0}(t)+\sum_{r=1}^{\infty}\frac{1}{\omega}\sum_{m=-\infty}^{\infty}\boldsymbol{p}_{r,m}(t)\mathrm{e}^{\mathrm{i}m\omega t}$$

[3, 4]. Our contention is that, although the underlying framework is considerably more complicated, this is also the case for $\boldsymbol{x}$ in (2.1).

When $D(t)$ is not singular, the solution of the algebraic equation in (2.1) is

$$\boldsymbol{y}(t)=-D^{-1}(t)\left[C(t)\boldsymbol{x}(t)+\frac{1}{\omega}\sum_{m=-\infty}^{\infty}\boldsymbol{b}_m(t)\mathrm{e}^{\mathrm{i}m\omega t}\right].\tag{2.2}$$

Thus, once we can write $\boldsymbol{x}$ as the above asymptotic expansion, we can do so also for $\boldsymbol{y}$. This motivates us to assume (and subsequently verify) the *ansatz* that both $\boldsymbol{x}$ and $\boldsymbol{y}$ possess an asymptotic expansion of this form.

Substituting the expression for $\boldsymbol{y}(t)$ in (2.2) into the differential equation in (2.1) yields,

$$\begin{aligned}\boldsymbol{x}'&=\left(A-BD^{-1}C\right)\boldsymbol{x}+\sum_{m=-\infty}^{\infty}\left[\boldsymbol{a}_m-\frac{1}{\omega}BD^{-1}\boldsymbol{b}_m\right]\mathrm{e}^{\mathrm{i}m\omega t}\\&=E\boldsymbol{x}-\frac{1}{\omega}\sum_{m=-\infty}^{\infty}\boldsymbol{c}_m\mathrm{e}^{\mathrm{i}m\omega t}+\sum_{m=-\infty}^{\infty}\boldsymbol{a}_m\mathrm{e}^{\mathrm{i}m\omega t},\end{aligned}$$

where

$$E(t)=A(t)-B(t)D^{-1}(t)C(t),\qquad\boldsymbol{c}_m(t)=B(t)D^{-1}(t)\boldsymbol{b}_m(t).$$

We now assume that the matrices A,B,C and D (hence also E) are constant. Prior to giving a procedure for the determination of the coefficients of the asymptotic expansions, we recall from [6] that, given a smooth continuous function $h(\tau):\mathbb{R}\to\mathbb{C}^d$, it is true that

$$\int_0^t h(\tau)\mathrm{e}^{\mathrm{i}m\omega\tau}\mathrm{d}\tau\sim-\sum_{k=1}^{\infty}\frac{1}{(-\mathrm{i}m\omega)^k}\left[h^{(k-1)}(t)\mathrm{e}^{\mathrm{i}m\omega t}-h^{(k-1)}(0)\right]$$

for $m \neq 0$ and $\omega \gg 1$.

Using variation of constants, the solution $\boldsymbol{x}$ has the form

$$\begin{aligned}
\boldsymbol{x}(t) = {} & \mathrm{e}^{tE}\boldsymbol{x}_0 - \frac{1}{\omega}\mathrm{e}^{tE}\int_0^t \mathrm{e}^{-\tau E}\boldsymbol{c}_0(\tau)\mathrm{d}\tau + \mathrm{e}^{tE}\int_0^t \mathrm{e}^{-\tau E}\boldsymbol{a}_0(\tau)\mathrm{d}\tau \\
& - \frac{\mathrm{e}^{tE}}{\omega}\sum_{m\neq 0}\int_0^t \mathrm{e}^{-\tau E}\boldsymbol{c}_m(\tau)\mathrm{e}^{\mathrm{i}m\omega\tau}\mathrm{d}\tau + \sum_{m\neq 0}\mathrm{e}^{tE}\int_0^t \mathrm{e}^{-\tau E}\boldsymbol{a}_m(\tau)\mathrm{e}^{\mathrm{i}m\omega\tau}\mathrm{d}\tau \\
\sim {} & \mathrm{e}^{tE}\boldsymbol{x}_0 - \frac{1}{\omega}\mathrm{e}^{tE}\int_0^t \mathrm{e}^{-\tau E}\boldsymbol{c}_0(\tau)\mathrm{d}\tau + \mathrm{e}^{tE}\int_0^t \mathrm{e}^{-\tau E}\boldsymbol{a}_0(\tau)\mathrm{d}\tau \\
& + \frac{\mathrm{e}^{tE}}{\omega}\sum_{m\neq 0}\sum_{r=1}^{\infty}\frac{1}{(-\mathrm{i}m\omega)^r}\left[\frac{\mathrm{d}^{r-1}}{\mathrm{d}\tau^{r-1}}[\mathrm{e}^{-\tau E}\boldsymbol{c}_m(\tau)]\,|_{\tau=t}\,\mathrm{e}^{\mathrm{i}m\omega t}\right. \\
& \qquad\qquad \left. - \frac{\mathrm{d}^{r-1}}{\mathrm{d}\tau^{r-1}}[\mathrm{e}^{-\tau E}\boldsymbol{c}_m(\tau)]\,|_{\tau=0}\right] \\
& - \mathrm{e}^{tE}\sum_{m\neq 0}\sum_{r=1}^{\infty}\frac{1}{(-\mathrm{i}m\omega)^r}\frac{\mathrm{d}^{r-1}}{\mathrm{d}\tau^{r-1}}[\mathrm{e}^{-\tau E}\boldsymbol{a}_m(\tau)]\,|_{\tau=t}\,\mathrm{e}^{\mathrm{i}m\omega t} \\
& \qquad\qquad - \frac{\mathrm{d}^{r-1}}{\mathrm{d}\tau^{r-1}}[\mathrm{e}^{-\tau E}\boldsymbol{a}_m(\tau)]\,|_{\tau=0}] \\
= {} & \mathrm{e}^{tE}\boldsymbol{x}_0 + \mathrm{e}^{tE}\int_0^t \mathrm{e}^{-\tau E}\boldsymbol{a}_0(\tau)\mathrm{d}\tau \\
& + \frac{1}{\omega}\left[\sum_{m\neq 0}\frac{1}{\mathrm{i}m}(\boldsymbol{a}_m(t)\mathrm{e}^{\mathrm{i}m\omega t} - \mathrm{e}^{tE}\boldsymbol{a}_m(0)) - \mathrm{e}^{tE}\int_0^t \mathrm{e}^{-\tau E}\boldsymbol{c}_0(\tau)\mathrm{d}\tau\right] \\
& + \sum_{r=2}^{\infty}\frac{1}{\omega^r}\sum_{m\neq 0}\left\{\left[\frac{1}{(-\mathrm{i}m)^{r-1}}\sum_{j=0}^{r-2}\binom{r-2}{j}(-1)^{r-2-j}E^{r-2-j}\boldsymbol{c}_m^{(j)}(t)\right.\right. \\
& \qquad \left. - \frac{1}{(-\mathrm{i}m)^r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}E^{r-1-j}\boldsymbol{a}_m^{(j)}(t)\right]\mathrm{e}^{\mathrm{i}m\omega t} \\
& \qquad - \left[\frac{\mathrm{e}^{tE}}{(-\mathrm{i}m)^{r-1}}\sum_{j=0}^{r-2}\binom{r-2}{j}(-1)^{r-2-j}E^{r-2-j}\boldsymbol{c}_m^{(j)}(0)\right. \\
& \qquad \left.\left. - \frac{\mathrm{e}^{tE}}{(-\mathrm{i}m)^r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}E^{r-1-j}\boldsymbol{a}_m^{(j)}(0)\right]\right\}
\end{aligned}$$

$$= \boldsymbol{p}_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} \boldsymbol{p}_{r,m}(t) \mathrm{e}^{\mathrm{i}m\omega t},$$

where

$$\boldsymbol{p}_{0,0}(t) = \mathrm{e}^{tE}\boldsymbol{x}_0 + \mathrm{e}^{tE} \int_0^t \mathrm{e}^{-\tau E} \boldsymbol{a}_0(\tau) \mathrm{d}\tau, \tag{2.3}$$

$$\boldsymbol{p}_{1,0}(t) = -\mathrm{e}^{tE} \sum_{m \neq 0} \frac{1}{\mathrm{i}m} \boldsymbol{a}_m(0) - \mathrm{e}^{tE} \int_0^t \mathrm{e}^{-\tau E} \boldsymbol{c}_0(\tau) \mathrm{d}\tau, \tag{2.4}$$

$$\boldsymbol{p}_{1,m}(t) = \frac{1}{\mathrm{i}m} \boldsymbol{a}_m(t), \qquad m \neq 0, \tag{2.5}$$

$$\begin{aligned}\boldsymbol{p}_{r,0}(t) = -\mathrm{e}^{tE} \sum_{m \neq 0} \Bigg[&\frac{1}{(-\mathrm{i}m)^{r-1}} \sum_{j=0}^{r-2} \binom{r-2}{j} (-1)^{r-2-j} E^{r-2-j} \boldsymbol{c}_m^{(j)}(0) \\ &- \frac{1}{(-\mathrm{i}m)^{r}} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} E^{r-1-j} \boldsymbol{a}_m^{(j)}(0) \Bigg],\end{aligned} \tag{2.6}$$

$$\begin{aligned}\boldsymbol{p}_{r,m}(t) = &\frac{1}{(-\mathrm{i}m)^{r-1}} \sum_{j=0}^{r-2} \binom{r-2}{j} (-1)^{r-2-j} E^{r-2-j} \boldsymbol{c}_m^{(j)}(t) \\ &- \frac{1}{(-\mathrm{i}m)^{r}} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} E^{r-1-j} \boldsymbol{a}_m^{(j)}(t), \quad m \neq 0,\end{aligned} \tag{2.7}$$

for $r \in \mathbb{N}$ and $r \geq 2$.

We substitute the terms (2.3–7) into (2.2) for $\boldsymbol{y}$,

$$\boldsymbol{y}(t) \sim \boldsymbol{q}_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} \boldsymbol{q}_{r,m}(t) \mathrm{e}^{\mathrm{i}m\omega t},$$

where

$$\boldsymbol{q}_{0,0}(t) = -D^{-1}C\boldsymbol{p}_{0,0}(t), \tag{2.8}$$

$$\boldsymbol{q}_{1,0}(t) = -D^{-1}C\boldsymbol{p}_{1,0}(t) - D^{-1}\boldsymbol{b}_0(t), \tag{2.9}$$

$$\boldsymbol{q}_{1,m}(t) = -D^{-1}C\boldsymbol{p}_{1,m}(t) - D^{-1}\boldsymbol{b}_m(t), \tag{2.10}$$

$$\boldsymbol{q}_{r,0}(t) = -D^{-1}C\boldsymbol{p}_{r,0}(t), \tag{2.11}$$

$$\boldsymbol{q}_{r,m}(t) = -D^{-1}C\boldsymbol{p}_{r,m}(t). \tag{2.12}$$

To illustrate the procedure just described, we consider a circuit as shown in Fig. 2.1 and which is governed by linear DAEs

$$\begin{aligned} C_{in}\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{v(t)}{R} + i(t) &= h(t), \\ L\frac{\mathrm{d}i(t)}{\mathrm{d}t} - v(t) &= 0, \qquad t \geq 0, \qquad \begin{bmatrix} v(0) \\ i(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \\ e(t) - Rh(t) - v(t) &= 0, \end{aligned} \tag{2.13}$$

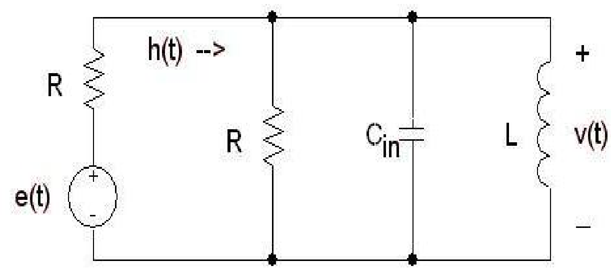

Figure 2.1: The linear circuit (2.13).

This can be rewritten in the standard form

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} &= A\boldsymbol{x}(t) + Bh(t) \\ 0 &= C\boldsymbol{x}(t) + Dh(t) + e(t), \end{aligned}$$

where

$$\begin{aligned} A &= \begin{bmatrix} -(C_{\mathrm{in}}R)^{-1} & -C_{\mathrm{in}}^{-1} \\ L^{-1} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} C_{\mathrm{in}}^{-1} \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}(t) = \begin{bmatrix} v(t) \\ i(t) \end{bmatrix}, \\ C &= [-1, 0], \qquad D = -R. \end{aligned}$$

The forcing term is $e(t) = A_{\mathrm{inj}} \sin(2\pi f_\omega t)/(2\pi f_\omega) = \omega^{-1} A_{\mathrm{inj}} \left(\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{e}^{-\mathrm{i}\omega t}\right)/(2\mathrm{i})$, $\omega = 2\pi f_\omega$ and f_ω is the oscillatory parameter. Thus, $\boldsymbol{a}_m(t) \equiv \boldsymbol{0}$, $b_1(t) = A_{\mathrm{inj}}/(2\mathrm{i})$ and $b_{-1}(t) = -A_{\mathrm{inj}}/(2\mathrm{i})$. The remaining values C_{in}, R, L are the circuit capacitance, resistance and inductance, while A_{inj} is a constant.

Our expansions are

$$\boldsymbol{x}(t) = \boldsymbol{p}_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} \boldsymbol{p}_{r,m}(t)\mathrm{e}^{\mathrm{i}m\omega t},$$

$$h(t) = q_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} q_{r,m}(t) \mathrm{e}^{\mathrm{i}m\omega t}.$$

It is instructive to compare the absolute error at different values of ω for the asymptotic method,

$$\boldsymbol{e}_s(t) = \boldsymbol{x}(t) - \boldsymbol{p}_{0,0}(t) - \sum_{r=1}^{s} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} \boldsymbol{p}_{r,m}(t) \mathrm{e}^{\mathrm{i}m\omega t},$$
$$\epsilon_s(t) = h(t) - q_{0,0}(t) - \sum_{r=1}^{s} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} q_{r,m}(t) \mathrm{e}^{\mathrm{i}m\omega t}.$$

We compute the first few terms to prove that the error tends to zero for $\omega \to \infty$. In each case, we compare the pointwise error incurred by a truncated expansion with either the exact solution or the Maple routine `rkf45-dae` with a high error tolerance `AbsErr` $= 10^{-10}$ and `RelErr` $= 10^{-10}$. Before calculating the coefficients $\boldsymbol{p}_{r,m}$ and $q_{r,m}$, we let

$$E = \begin{bmatrix} -2/(C_{\mathrm{in}}R) & -1/C_{\mathrm{in}} \\ L^{-1} & 0 \end{bmatrix}, \qquad \boldsymbol{c}_m = BD^{-1}b_m,$$
$$c_1 = \begin{bmatrix} -A_{\mathrm{inj}}/(2\mathrm{i}C_{\mathrm{in}}R) \\ 0 \end{bmatrix}, \qquad c_{-1} = \begin{bmatrix} A_{\mathrm{inj}}/(2\mathrm{i}C_{\mathrm{in}}R) \\ 0 \end{bmatrix}.$$

Employing (2.3–7) and (2.8–12), we obtain the coefficients

$$\boldsymbol{p}_{0,0} = \mathrm{e}^{tE}\boldsymbol{x}_0, \qquad \boldsymbol{p}_{1,m} = \boldsymbol{0},$$
$$\boldsymbol{p}_{r,m} = \frac{(-1)^r}{(-\mathrm{i}m)^{r-1}} E^{r-2}\boldsymbol{c}_m, \quad m = -1, 1,$$
$$\boldsymbol{p}_{r,0} = \frac{1}{\mathrm{i}^{r-1}} \mathrm{e}^{tE} E^{r-2} \left[\boldsymbol{c}_1(0) + (-1)^{r-1}\boldsymbol{c}_{-1}(0)\right];$$
$$q_{0,0} = \left[-R^{-1}\ 0\right] \mathrm{e}^{tE}\boldsymbol{x}_0, \qquad q_{1,0} = 0, \qquad q_{1,m} = R^{-1}b_m(t),$$
$$q_{r,0} = \left[-R^{-1}\ 0\right] \boldsymbol{p}_{r,0}, \qquad q_{r,m} = \left[-R^{-1}\ 0\right] \boldsymbol{p}_{r,m}, \quad m \neq 0.$$

Figures 2.2–4 display the real part of the error in computing $v(t)$, $i(t)$ and $h(t)$, respectively, for $s = 0, 1, 2, 3$ and $\omega = 200\pi, 2000\pi$. For this example, the parameters are chosen as $L = 0.1$, $R = 10$, $C_{\mathrm{in}} = 0.2533$ and $A_{\mathrm{inj}} = 10$. It is shown that the error decreases significantly with increasing s and ω, in accordance with our theoretical results. Note that e_0 and e_1 for $v(t)$ and $i(t)$ have the same behaviour since the value of $\boldsymbol{p}_{1,m}$ is $\boldsymbol{0}$ in this example.

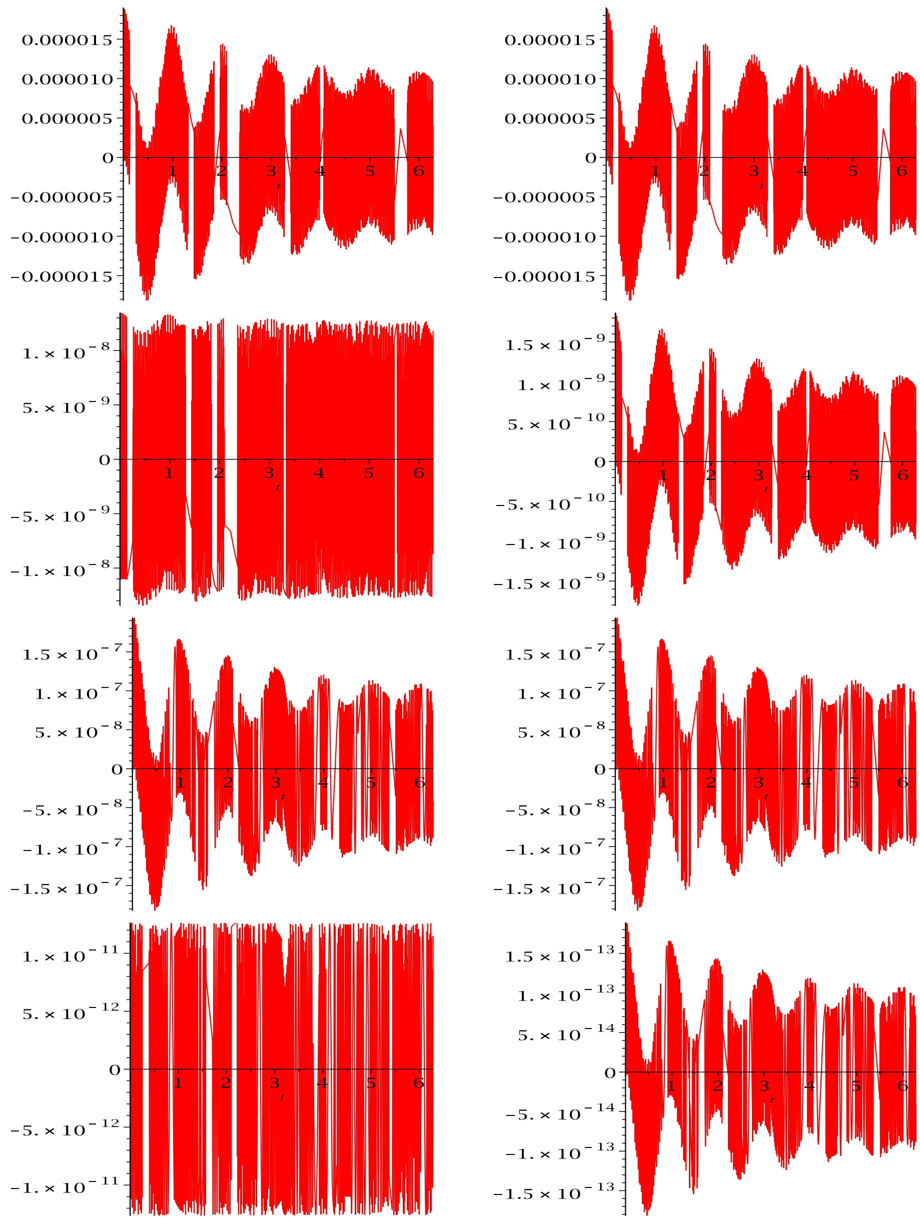

Figure 2.2: The errors for $v(t)$. The top row: e_0 (the left) and e_1 (the right) with $\omega = 200\pi$. The second row: e_2 (the left) and e_3 (the right) with $\omega = 200\pi$. The third row: e_0 (the left) and e_1 (the right) with $\omega = 2000\pi$. The bottom row: e_2 (the left) and e_3 (the right) with $\omega = 2000\pi$.

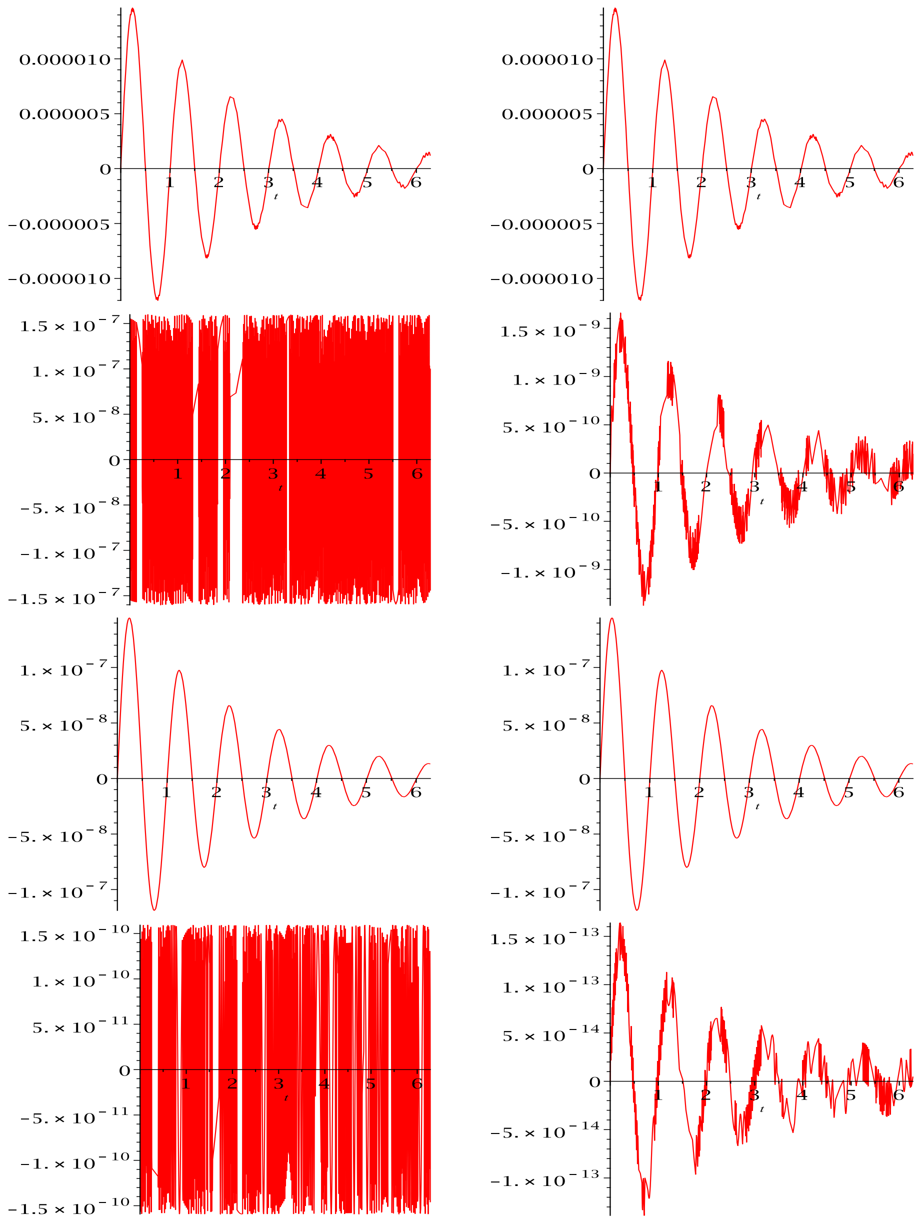

Figure 2.3: The errors for $i(t)$. The top row: e_0 (the left) and e_1 (the right) with $\omega = 200\pi$. The second row: e_2 (the left) and e_3 (the right) with $\omega = 200\pi$. The third row: e_0 (the left) and e_1 (the right) with $\omega = 2000\pi$. The bottom row: e_2 (the left) and e_3 (the right) with $\omega = 2000\pi$.

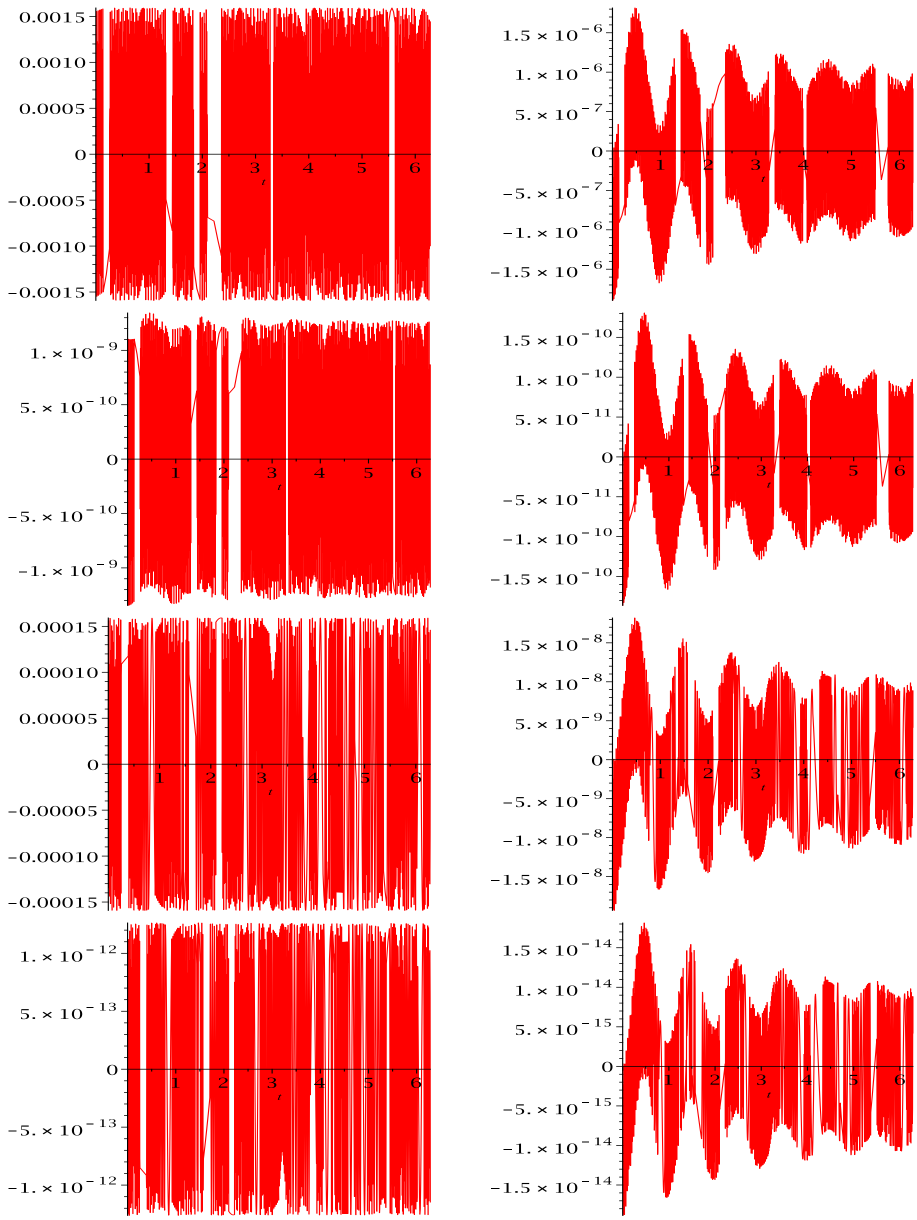

Figure 2.4: The errors for $h(t)$. The top row: ϵ_0 (the left) and ϵ_1 (the right) with $\omega = 200\pi$. The second row: ϵ_2 (the left) and ϵ_3 (the right) with $\omega = 200\pi$. The third row: ϵ_0 (the left) and ϵ_1 (the right) with $\omega = 2000\pi$. The bottom row: ϵ_2 (the left) and ϵ_3 (the right) with $\omega = 2000\pi$.

3 Nonlinear DAEs

3.1 The general theory

We now proceed to the considerably more complicated case of nonlinear highly oscillatory DAEs.

$$\begin{aligned} \boldsymbol{x}'(t) &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) + \sum_{m\in\mathbb{Z}} \boldsymbol{a}_m(t)\mathrm{e}^{\mathrm{i}m\omega t}, \quad t\ge 0, \qquad \boldsymbol{x}(0)=\boldsymbol{x}_0, \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) + \frac{1}{\omega}\sum_{m\in\mathbb{Z}} \boldsymbol{b}_m(t)\mathrm{e}^{\mathrm{i}m\omega t}, \end{aligned} \tag{3.1}$$

Insofar as the ordinary differential system

$$\boldsymbol{x}' = \boldsymbol{f}(t,\boldsymbol{x}) + \sum_{m\in\mathbb{Z}} \boldsymbol{a}_m(t)\mathrm{e}^{\mathrm{i}m\omega t}, \quad t\ge 0, \qquad \boldsymbol{x}(0)=\boldsymbol{x}_0,$$

is concerned, Condon et al. have presented in [3, 4] the asymptotic expansion,

$$\boldsymbol{x}(t) = \boldsymbol{p}_{0,0}(t) + \sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m=-\infty}^{\infty}\boldsymbol{p}_{r,m}(t)\mathrm{e}^{\mathrm{i}m\omega t}. \tag{3.2}$$

Matters are more complicated for the DAE (3.1) since we additionally need to consider the algebraic part. To this end, we differentiate the second algebraic equation formally,

$$\boldsymbol{0} = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\boldsymbol{x}' + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{y}}\boldsymbol{y}' + \frac{1}{\omega}\sum_{m\in\mathbb{Z}}\left[\boldsymbol{b}_m'(t) + \mathrm{i}m\omega\boldsymbol{b}_m(t)\right]\mathrm{e}^{\mathrm{i}m\omega t}. \tag{3.3}$$

Since the DAE index is one, $(\partial\boldsymbol{g}/\partial\boldsymbol{y})^{-1}$ exists. Therefore we can rewrite (3.3) in the form

$$\begin{aligned} \boldsymbol{y}' &= -\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{y}}\right)^{-1}\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\boldsymbol{x}' - \frac{1}{\omega}\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{y}}\right)^{-1}\sum_{m\in\mathbb{Z}}\left[\boldsymbol{b}_m'(t) + \mathrm{i}m\omega\boldsymbol{b}_m(t)\right]\mathrm{e}^{\mathrm{i}m\omega t} \\ &= -\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{y}}\right)^{-1}\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) - \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{y}}\right)^{-1}\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\sum_{m\in\mathbb{Z}}\boldsymbol{a}_m(t)\mathrm{e}^{\mathrm{i}m\omega t} \\ &\quad - \frac{1}{\omega}\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{y}}\right)^{-1}\sum_{m\in\mathbb{Z}}\left[\boldsymbol{b}_m'(t) + \mathrm{i}m\omega\boldsymbol{b}_m(t)\right]\mathrm{e}^{\mathrm{i}m\omega t} \end{aligned}$$

This is similar to the differential equation. Therefore, the *ansatz* is that the variable $\boldsymbol{y}(t)$ has an asymptotic expansion of the form

$$\boldsymbol{y}(t) = \boldsymbol{q}_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} \boldsymbol{q}_{r,m}(t) \mathrm{e}^{\mathrm{i}m\omega t}, \tag{3.4}$$

which we proceed to confirm. Substituting (3.2) and (3.4) into the DAEs (3.1) and comparing the terms by scale and frequency, the coefficients in (3.2) and (3.4) can be obtained in a recursive manner. Before giving these explicitly, we introduce and explain some notation that shall be employed in what follows.

The functions $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ and $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})$ are analytic and can be expanded in Taylor series about the function $(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})$. Thus, for any $\boldsymbol{p} \in \mathbb{C}^{d_1}$, $\boldsymbol{q} \in \mathbb{C}^{d_2}$, both sufficiently small in norm, we expand

$$\boldsymbol{f}(\boldsymbol{p}_{0,0}+\boldsymbol{p},\boldsymbol{q}_{0,0}+\boldsymbol{q}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \boldsymbol{f}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})[\boldsymbol{p},\boldsymbol{p},\ldots,\boldsymbol{p}][\boldsymbol{q},\boldsymbol{q},\ldots,\boldsymbol{q}],$$

where

$$\boldsymbol{f}_{n,k} : \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \overbrace{\mathbb{C}^{d_1} \times \cdots \mathbb{C}^{d_1}}^{k \text{ times}} \times \overbrace{\mathbb{C}^{d_2} \times \cdots \times \mathbb{C}^{d_2}}^{n-k \text{ times}} \to \mathbb{C}^{d_1}$$

is *linear* in all its arguments except for $\boldsymbol{p}_{0,0}$ and $\boldsymbol{q}_{0,0}$ and *symmetric* in each of the two groups of arguments enclosed by square brackets: of course, it is the *derivative tensor*

$$\boldsymbol{f}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0}) = \left.\frac{\partial^n \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{x}^k \partial \boldsymbol{y}^{n-k}}\right|_{(\boldsymbol{x},\boldsymbol{y})=(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})}.$$

We similarly define

$$\boldsymbol{g}_{n,k} : \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \overbrace{\mathbb{C}^{d_1} \times \cdots \mathbb{C}^{d_1}}^{k \text{ times}} \times \overbrace{\mathbb{C}^{d_2} \times \cdots \times \mathbb{C}^{d_2}}^{n-k \text{ times}} \to \mathbb{C}^{d_2}, \qquad 0 \le k \le n,$$

so that

$$\boldsymbol{g}(\boldsymbol{p}_{0,0}+\boldsymbol{p},\boldsymbol{q}_{0,0}+\boldsymbol{q}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \boldsymbol{g}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})[\boldsymbol{p},\boldsymbol{p},\ldots,\boldsymbol{p}][\boldsymbol{q},\boldsymbol{q},\ldots,\boldsymbol{q}].$$

Using (3.2) and (3.4), we expand

$$
\begin{aligned}
&\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0}) + \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\boldsymbol{f}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\\
&\left[\sum_{\ell_1=1}^{\infty}\frac{1}{\omega^{\ell_1}}\sum_{\sigma_1=-\infty}^{\infty}\boldsymbol{p}_{\ell_1,\sigma_1}\mathrm{e}^{\mathrm{i}\sigma_1\omega t},\cdots,\sum_{\ell_k=1}^{\infty}\frac{1}{\omega^{\ell_k}}\sum_{\sigma_k=-\infty}^{\infty}\boldsymbol{p}_{\ell_k,\sigma_k}\mathrm{e}^{\mathrm{i}\sigma_k\omega t}\right]\\
&\left[\sum_{\ell_{k+1}=1}^{\infty}\frac{1}{\omega^{\ell_{k+1}}}\sum_{\sigma_{k+1}=-\infty}^{\infty}\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}}\mathrm{e}^{\mathrm{i}\sigma_{k+1}\omega t},\cdots,\sum_{\ell_n=1}^{\infty}\frac{1}{\omega^{\ell_n}}\sum_{\sigma_n=-\infty}^{\infty}\boldsymbol{q}_{\ell_n,\sigma_n}\mathrm{e}^{\mathrm{i}\sigma_n\omega t}\right]\\
&= \boldsymbol{f}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0}) + \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\sum_{\ell_1=1}^{\infty}\cdots\sum_{\ell_n=1}^{\infty}\frac{1}{\omega^{\ell_1+\cdots+\ell_n}}\sum_{\sigma_1=-\infty}^{\infty}\cdots\sum_{\sigma_n=-\infty}^{\infty}\\
&\quad \boldsymbol{f}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\left[\boldsymbol{p}_{\ell_1,\sigma_1},\cdots,\boldsymbol{p}_{\ell_k,\sigma_k}\right]\left[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}},\cdots,\boldsymbol{q}_{\ell_n,\sigma_n}\right]\mathrm{e}^{\mathrm{i}(\sigma_1+\cdots+\sigma_n)\omega t}\\
&= \boldsymbol{f}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0}) + \sum_{n=1}^{\infty}\sum_{r=n}^{\infty}\frac{1}{\omega^r}\sum_{\boldsymbol{\ell}\in\mathbb{I}^o_{n,r}}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\sum_{m\in\mathbb{Z}}\sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,m}}\boldsymbol{f}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\\
&\quad \left[\boldsymbol{p}_{\ell_1,\sigma_1},\cdots,\boldsymbol{p}_{\ell_k,\sigma_k}\right]\left[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}},\cdots,\boldsymbol{q}_{\ell_n,\sigma_n}\right]\mathrm{e}^{\mathrm{i}m\omega t}\\
&= \boldsymbol{f}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0}) + \sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m\in\mathbb{Z}}\sum_{n=1}^{r}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_{\boldsymbol{\ell}\in\mathbb{I}^o_{n,r}}\sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,m}}\boldsymbol{f}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\\
&\quad \left[\boldsymbol{p}_{\ell_1,\sigma_1},\cdots,\boldsymbol{p}_{\ell_k,\sigma_k}\right]\left[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}},\cdots,\boldsymbol{q}_{\ell_n,\sigma_n}\right]\mathrm{e}^{\mathrm{i}m\omega t}
\end{aligned}
$$

where the two index sets above are

$$\mathbb{I}^o_{n,r} = \{\boldsymbol{\ell}\in\mathbb{N}^n \,:\, \mathbf{1}^\top\boldsymbol{\ell} = r\},\qquad \mathbb{J}_{n,m} = \{\boldsymbol{\sigma}\in\mathbb{Z}^n \,:\, \mathbf{1}^\top\boldsymbol{\sigma} = m\}.$$

There is a measure of redundancy in the set $\mathbb{I}^o_{n,r}$. For example,

$$\mathbb{I}^o_{3,5} = \{(1,1,3),(1,3,1),(3,1,1),(1,2,2),(2,1,2),(2,2,1)\},$$

consisting of six elements. Because of the built-in symmetry of $\boldsymbol{f}_{3,k}$, some of the elements above correspond to the same term and can be aggregated, but this depends on the value of k. For example, for $k=1$ the second and third entry of each triplet can be permuted, hence we can associate a weight $\theta_{n,k,r}(\boldsymbol{\ell})$ with each element and consider instead

$$\mathbb{I}_{3,1,5} = \{(1,1,3),(3,1,1),(1,2,2),(2,1,2)\},$$

say, with the multiplicities $\theta_{3,1,5}(3,1,1)=\theta_{3,1,5}(1,2,2)=1$, $\theta_{3,1,5}(1,1,3)=\theta_{3,1,5}(2,1,2)=2$.

More formally, let

$$\mathbb{I}_{n,k,r}=\{\boldsymbol{\ell}\in\mathbb{N}^n\,:\,\mathbf{1}^\top\boldsymbol{\ell}=r,\ \ell_1\le\ell_2\le\cdots\le\ell_k,\ \ell_{k+1}\le\ell_{k+2}\cdots\le\ell_n\}$$

and let $\theta_{n,k,r}(\boldsymbol{\ell})$ stand for the *multiplicity* of $\boldsymbol{\ell}\in\mathbb{I}_{n,k,r}$, i.e. the number of terms of $\mathbb{I}^o_{n,r}$ that can be brought into the form $\boldsymbol{\ell}$ by permutations of the first k entries and of the last $n-k$ entries. Note that $k=0$ and $k=n$ make perfect sense: in each of these cases there is in $\mathrm{I}_{n,k,r}$ just a single monotone sequence.

For example, bearing in mind that $n\le r$,

$$\begin{aligned}
n=1: &\quad \mathbb{I}_{1,1,r}=\{(r)\},\qquad \theta_{1,1,r}(r)=1;\\
n=2: &\quad \mathbb{I}_{2,0,r}=\mathbb{I}_{2,2,r}=\{(i,r-i)\,:\,i=1,\ldots,\lfloor r/2\rfloor\},\\
&\qquad\qquad \theta_{2,0,r}(i,r-i)=\theta_{2,2,r}(i,r-i)=\begin{cases}1, & 2i=r,\\ 2, & 2i<r;\end{cases}\\
&\quad \mathbb{I}_{2,1,r}=\{(i,r-i)\,:\,i=1,\ldots,r-1\},\qquad \theta_{2,1,r}(i,r-i)=1;\\
n=3: &\quad \mathbb{I}_{3,0,r}=\mathbb{I}_{3,3,r}=\{\boldsymbol{\ell}\,:\,\ell_1\le\ell_2\le\ell_3,\ 1^\top\boldsymbol{\ell}=r\},\\
&\qquad\qquad \theta_{3,0,r}(\boldsymbol{\ell})=\theta_{3,3,r}(\boldsymbol{\ell})=\begin{cases}1, & \ell_1=\ell_2=\ell_3,\\ 2, & \ell_1=\ell_2\ne\ell_3,\\ 2, & \ell_1=\ell_3\ne\ell_2,\\ 2, & \ell_2=\ell_3\ne\ell_1,\\ 3, & \text{otherwise};\end{cases}\\
&\quad \mathbb{I}_{3,1,r}=\{\boldsymbol{\ell}\,:\,\ell_2\le\ell_3,\ 1^\top\boldsymbol{\ell}=r\},\\
&\qquad\qquad \theta_{3,1,r}(\boldsymbol{\ell})=\begin{cases}1, & \ell_2=\ell_3,\\ 2, & \ell_2<\ell_3,\end{cases}\\
&\quad \mathbb{I}_{3,2,r}=\{\boldsymbol{\ell}\,:\,\ell_1\le\ell_2,\ 1^\top\boldsymbol{\ell}=r\},\\
&\qquad\qquad \theta_{3,2,r}(\boldsymbol{\ell})=\begin{cases}1, & \ell_1=\ell_2,\\ 2, & \ell_1<\ell_2\end{cases}
\end{aligned}$$

and so on.

We can now rewrite the Taylor expansion of $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ in the form

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{f}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})$$

$$+\sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m\in\mathbb{Z}}\sum_{n=1}^{r}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,k,r}}\theta_{n,k,r}(\boldsymbol{\ell})\sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,m}} \tag{3.5}$$

$$\boldsymbol{f}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\left[\boldsymbol{p}_{\ell_1,\sigma_1},\cdots,\boldsymbol{p}_{\ell_k,\sigma_k}\right]\left[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}},\cdots,\boldsymbol{q}_{\ell_n,\sigma_n}\right]\mathrm{e}^{\mathrm{i}m\omega t}.$$

Likewise,

$$\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{g}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})$$

$$+\sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m\in\mathbb{Z}}\sum_{n=1}^{r}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,k,r}}\theta_{n,k,r}(\boldsymbol{\ell})\sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,m}} \tag{3.6}$$

$$\boldsymbol{g}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\left[\boldsymbol{p}_{\ell_1,\sigma_1},\cdots,\boldsymbol{p}_{\ell_k,\sigma_k}\right]\left[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}},\cdots,\boldsymbol{q}_{\ell_n,\sigma_n}\right]\mathrm{e}^{\mathrm{i}m\omega t}.$$

where

$$\boldsymbol{g}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})=\left.\frac{\partial^n\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{x}^k\partial\boldsymbol{y}^{n-k}}\right|_{(\boldsymbol{x},\boldsymbol{y})=(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})}.$$

Substitute (3.5) and (3.6) on both sides of the DAE (3.1),

$$\boldsymbol{p}_{0,0}'+\sum_{m\in\mathbb{Z}}\mathrm{i}m\boldsymbol{p}_{1,m}\mathrm{e}^{\mathrm{i}m\omega t}+\sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{m=-\infty}^{\infty}\left(\boldsymbol{p}_{r,m}'+\mathrm{i}m\boldsymbol{p}_{r+1,m}\right)\mathrm{e}^{\mathrm{i}m\omega t}$$

$$=\boldsymbol{f}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})+\sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{n=1}^{r}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_{m\in\mathbb{Z}}\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,k,r}}\theta_{n,k,r}(\boldsymbol{\ell})$$

$$\sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,m}}\boldsymbol{f}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\left[\boldsymbol{p}_{\ell_1,\sigma_1},\cdots,\boldsymbol{p}_{\ell_k,\sigma_k}\right]\left[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}},\cdots,\boldsymbol{q}_{\ell_n,\sigma_n}\right]\mathrm{e}^{\mathrm{i}m\omega t}$$

$$+\sum_{m\in\mathbb{Z}}\boldsymbol{a}_m\mathrm{e}^{\mathrm{i}m\omega t},$$

$$\boldsymbol{0}=\boldsymbol{g}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})+\sum_{r=1}^{\infty}\frac{1}{\omega^r}\sum_{n=1}^{r}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_{m\in\mathbb{Z}}\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,k,r}}\theta_{n,k,r}(\boldsymbol{\ell})$$

$$\sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,m}}\boldsymbol{g}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\left[\boldsymbol{p}_{\ell_1,\sigma_1},\cdots,\boldsymbol{p}_{\ell_k,\sigma_k}\right]\left[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}},\cdots,\boldsymbol{q}_{\ell_n,\sigma_n}\right]\mathrm{e}^{\mathrm{i}m\omega t}$$

$$+\frac{1}{\omega}\sum_{m\in\mathbb{Z}}\boldsymbol{b}_m\mathrm{e}^{\mathrm{i}m\omega t}.$$

We next separate the equations by different powers of ω. Thus, for $r=0$,

$$\boldsymbol{p}_{0,0}'+\sum_{m\in\mathbb{Z}}\mathrm{i}m\boldsymbol{p}_{1,m}\mathrm{e}^{\mathrm{i}m\omega t}=\boldsymbol{f}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})+\sum_{m\in\mathbb{Z}}\boldsymbol{a}_m\mathrm{e}^{\mathrm{i}m\omega t}, \tag{3.7}$$

$$\mathbf{0} = \boldsymbol{g}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})$$

while for $r = 1$ we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \left(\boldsymbol{p}'_{1,m} + \mathrm{i}m\boldsymbol{p}_{2,m}\right) \mathrm{e}^{\mathrm{i}m\omega t} &= \sum_{m\in\mathbb{Z}} \boldsymbol{f}_{1,0}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{q}_{1,m}\mathrm{e}^{\mathrm{i}m\omega t} \\ &+ \sum_{m\in\mathbb{Z}} \boldsymbol{f}_{1,1}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{p}_{1,m}\mathrm{e}^{\mathrm{i}m\omega t}, \\ \mathbf{0} = \sum_{m\in\mathbb{Z}} \boldsymbol{g}_{1,0}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{q}_{1,m}\mathrm{e}^{\mathrm{i}m\omega t} &+ \sum_{m\in\mathbb{Z}} \boldsymbol{g}_{1,1}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{p}_{1,m}\mathrm{e}^{\mathrm{i}m\omega t} + \sum_{m\in\mathbb{Z}} \boldsymbol{b}_m\mathrm{e}^{\mathrm{i}m\omega t}\end{aligned} \tag{3.8}$$

and for $r \geq 2$

$$\begin{aligned}\sum_{m=-\infty}^{\infty} \left(\boldsymbol{p}'_{r,m} + \mathrm{i}m\boldsymbol{p}_{r+1,m}\right) \mathrm{e}^{\mathrm{i}m\omega t} &= \sum_{n=1}^{r}\sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{m\in\mathbb{Z}} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,k,r}} \theta_{n,k,r}(\boldsymbol{\ell}) \\ \sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,m}} \boldsymbol{f}_{n,k}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})&\left[\boldsymbol{p}_{\ell_1,\sigma_1}, \cdots, \boldsymbol{p}_{\ell_k,\sigma_k}\right]\left[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}}, \cdots, \boldsymbol{q}_{\ell_n,\sigma_n}\right] \mathrm{e}^{\mathrm{i}m\omega t}, \\ \mathbf{0} = \sum_{n=1}^{r}\sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{m\in\mathbb{Z}} &\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,k,r}} \theta_{n,k,r}(\boldsymbol{\ell}) \sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,m}} \boldsymbol{g}_{n,k}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0}) \\ \left[\boldsymbol{p}_{\ell_1,\sigma_1}, \cdots, \boldsymbol{p}_{\ell_k,\sigma_k}\right]&\left[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}}, \cdots, \boldsymbol{q}_{\ell_n,\sigma_n}\right] \mathrm{e}^{\mathrm{i}m\omega t}.\end{aligned} \tag{3.9}$$

Similarly to [3, 4], we have the initial conditions

$$\boldsymbol{x}(0) = \boldsymbol{p}_{0,0}(0) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} \boldsymbol{p}_{r,m}(0)\mathrm{e}^{\mathrm{i}m\omega t} = \boldsymbol{x}_0.$$

Therefore, we impose the initial conditions

$$\boldsymbol{p}_{0,0}(0) = \boldsymbol{x}_0, \qquad \boldsymbol{p}_{r,0}(0) = -\sum_{m\neq 0} \boldsymbol{p}_{r,m}(0), \quad r \in \mathbb{N}. \tag{3.10}$$

3.2 The first few terms of r

For $r = 0$ (3.7) yields the *non-oscillatory* DAE

$$\begin{aligned}\boldsymbol{p}'_{0,0} &= \boldsymbol{f}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0}) + \boldsymbol{a}_0, \quad t \geq 0, \qquad \boldsymbol{p}_{0,0}(0) = \boldsymbol{x}_0, \\ \mathbf{0} &= \boldsymbol{g}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0}),\end{aligned}$$

and the recursion

$$\boldsymbol{p}_{1,m} = \frac{\boldsymbol{a}_m}{\mathrm{i}m}, \qquad m \neq 0.$$

In the case of $r = 1$, for $m = 0$ we obtain from (3.8) a non-oscillatory DAE

$$\begin{aligned} \boldsymbol{p}_{1,0}' &= \boldsymbol{f}_{1,0}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{q}_{1,0} + \boldsymbol{f}_{1,1}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{p}_{1,0}, \\ \boldsymbol{0} &= \boldsymbol{g}_{1,0}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{q}_{1,0} + \boldsymbol{g}_{1,1}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{p}_{1,0} + \boldsymbol{b}_0, \end{aligned} \tag{3.11}$$

as well as the recursions

$$\begin{aligned} \boldsymbol{p}_{2,m} &= \frac{1}{\mathrm{i}m}\left(-\boldsymbol{p}_{1,m}' + \boldsymbol{f}_{1,0}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{q}_{1,m} + \boldsymbol{f}_{1,1}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{p}_{1,m}\right), \\ \boldsymbol{q}_{1,m} &= \boldsymbol{g}_{1,0}^{-1}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\left(-\boldsymbol{g}_{1,1}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{p}_{1,m} - \boldsymbol{b}_m\right) \end{aligned}$$

(the Jacobian matrix $\boldsymbol{g}_{1,0}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})$ is, by the index-1 assumption, invertible).

The next scale is $r = 2$ whereby, after elementary algebra, we obtain for each $m \in \mathbb{Z}$

$$\begin{aligned} &\boldsymbol{p}_{2,m}' + \mathrm{i}m\boldsymbol{p}_{3,m} = \boldsymbol{f}_{1,0}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{q}_{2,m} + \boldsymbol{f}_{1,1}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{p}_{2,m} \\ &\quad + \frac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=m \\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{f}_{2,0}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})[\boldsymbol{q}_{1,\sigma_1}, \boldsymbol{q}_{1,\sigma_2}] + \sum_{\substack{\sigma_1+\sigma_2=m \\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{f}_{2,1}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})[\boldsymbol{p}_{1,\sigma_1}, \boldsymbol{q}_{1,\sigma_2}] \\ &\quad + \frac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=m \\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{f}_{2,2}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})[\boldsymbol{p}_{1,\sigma_1}, \boldsymbol{p}_{1,\sigma_2}] \\ &\boldsymbol{0} = \boldsymbol{g}_{1,0}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{q}_{2,m} + \boldsymbol{g}_{1,1}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{p}_{2,m} \\ &\quad + \frac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=m \\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{g}_{2,0}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})[\boldsymbol{q}_{1,\sigma_1}, \boldsymbol{q}_{1,\sigma_2}] + \sum_{\substack{\sigma_1+\sigma_2=m \\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{g}_{2,1}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})[\boldsymbol{p}_{1,\sigma_1}, \boldsymbol{q}_{1,\sigma_2}] \\ &\quad + \frac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=m \\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{g}_{2,2}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})[\boldsymbol{p}_{1,\sigma_1}, \boldsymbol{p}_{1,\sigma_2}]. \end{aligned}$$

Letting $m = 0$, we derive a non-oscillatory DAE

$$\begin{aligned} \boldsymbol{p}_{2,0}' &= \boldsymbol{f}_{1,0}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{q}_{2,0} + \boldsymbol{f}_{1,1}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})\boldsymbol{p}_{2,0} \\ &\quad + \frac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=0 \\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{f}_{2,0}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})[\boldsymbol{q}_{1,\sigma_1}, \boldsymbol{q}_{1,\sigma_2}] + \sum_{\substack{\sigma_1+\sigma_2=0 \\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{f}_{2,1}(\boldsymbol{p}_{0,0}, \boldsymbol{q}_{0,0})[\boldsymbol{p}_{1,\sigma_1}, \boldsymbol{q}_{1,\sigma_2}] \end{aligned}$$

$$+\tfrac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=0\\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{f}_{2,2}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})[\boldsymbol{p}_{1,\sigma_1},\boldsymbol{p}_{1,\sigma_2}],$$

$$\boldsymbol{0} = \boldsymbol{g}_{1,0}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\boldsymbol{q}_{2,0} + \boldsymbol{g}_{1,1}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\boldsymbol{p}_{2,0} + \tfrac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=0\\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{g}_{2,0}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})[\boldsymbol{q}_{1,\sigma_1},\boldsymbol{q}_{1,\sigma_2}]$$

$$+\sum_{\substack{\sigma_1+\sigma_2=0\\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{g}_{2,1}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})[\boldsymbol{p}_{1,\sigma_1},\boldsymbol{q}_{1,\sigma_2}] + \tfrac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=0\\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{g}_{2,2}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})[\boldsymbol{p}_{1,\sigma_1},\boldsymbol{p}_{1,\sigma_2}]$$

with the initial condition

$$\boldsymbol{p}_{2,0}(0) = -\sum_{m\neq 0} \boldsymbol{p}_{2,m}(0).$$

When $m \neq 0$, we derive the recursions

$$\boldsymbol{p}_{3,m} = \frac{1}{\mathrm{i}m}\Bigg\{ -\boldsymbol{p}'_{2,m} + \boldsymbol{f}_{1,0}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\boldsymbol{q}_{2,m} + \boldsymbol{f}_{1,1}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\boldsymbol{p}_{2,m}$$

$$+\tfrac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=m\\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{f}_{2,0}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})[\boldsymbol{q}_{1,\sigma_1},\boldsymbol{q}_{1,\sigma_2}] + \sum_{\substack{\sigma_1+\sigma_2=m\\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{f}_{2,1}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})[\boldsymbol{p}_{1,\sigma_1},\boldsymbol{q}_{1,\sigma_2}]$$

$$+\tfrac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=m\\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{f}_{2,2}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})[\boldsymbol{p}_{1,\sigma_1},\boldsymbol{p}_{1,\sigma_2}] \Bigg\}$$

$$\boldsymbol{q}_{2,m} = -\boldsymbol{g}^{-1}_{1,0}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\Bigg\{\boldsymbol{g}_{1,1}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\boldsymbol{p}_{2,m} + \tfrac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=m\\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{g}_{2,0}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})[\boldsymbol{q}_{1,\sigma_1},\boldsymbol{q}_{1,\sigma_2}]$$

$$+\sum_{\substack{\sigma_1+\sigma_2=m\\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{g}_{2,1}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})[\boldsymbol{p}_{1,\sigma_1},\boldsymbol{q}_{1,\sigma_2}] + \tfrac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=m\\ \sigma_1,\sigma_2\in\mathbb{Z}}} \boldsymbol{g}_{2,2}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})[\boldsymbol{p}_{1,\sigma_1},\boldsymbol{p}_{1,\sigma_2}]\Bigg\}.$$

3.3 The general expansion

Proceeding with full generality, we can easily convert the $r = 2$ example of the last subsection into a general rule. Thus, for any $r \geq 2$ in (3.9), the outcome is

$$\sum_{m\in\mathbb{Z}} \left(\boldsymbol{p}'_{r,m} + \mathrm{i}m\boldsymbol{p}_{r+1,m}\right) \mathrm{e}^{\mathrm{i}m\omega t} = \sum_{n=1}^{r}\sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{m\in\mathbb{Z}} \sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,k,r}} \theta_{n,k,r}(\boldsymbol{\ell})$$

$$\sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,m}} \boldsymbol{f}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\big[\boldsymbol{p}_{\ell_1,\sigma_1},\cdots,\boldsymbol{p}_{\ell_k,\sigma_k}\big]\Big[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}},\cdots,\boldsymbol{q}_{\ell_n,\sigma_n}\Big]\mathrm{e}^{\mathrm{i}m\omega t},$$

$$\mathbf{0} = \sum_{n=1}^{r}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_{m\in\mathbb{Z}}\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,k,r}}\theta_{n,k,r}(\boldsymbol{\ell})\sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,m}}\boldsymbol{g}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})$$
$$\big[\boldsymbol{p}_{\ell_1,\sigma_1},\cdots,\boldsymbol{p}_{\ell_k,\sigma_k}\big]\Big[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}},\cdots,\boldsymbol{q}_{\ell_n,\sigma_n}\Big]\mathrm{e}^{\mathrm{i}m\omega t}.$$

As before, we separate frequencies. For $m=0$ it again obtain the *non-oscillatory DAE*

$$\boldsymbol{p}'_{r,0} = \sum_{n=1}^{r}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,k,r}}\theta_{n,k,r}(\boldsymbol{\ell})\sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,0}}\boldsymbol{f}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})$$
$$\big[\boldsymbol{p}_{\ell_1,\sigma_1},\cdots,\boldsymbol{p}_{\ell_k,\sigma_k}\big]\Big[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}},\cdots,\boldsymbol{q}_{\ell_n,\sigma_n}\Big],$$
$$\mathbf{0} = \sum_{n=1}^{r}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,r,k}}\theta_{n,k,r}(\boldsymbol{\ell})\sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,0}}\boldsymbol{g}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})$$
$$\big[\boldsymbol{p}_{\ell_1,\sigma_1},\cdots,\boldsymbol{p}_{\ell_k,\sigma_k}\big]\Big[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}},\cdots,\boldsymbol{q}_{\ell_n,\sigma_n}\Big],$$

accompanied by the initial conditions $\boldsymbol{p}_{r,0}(0) = -\sum_{m\neq 0}\boldsymbol{p}_{r,m}(0)$ to solve for $\boldsymbol{p}_{r,0}$ and $\boldsymbol{q}_{r,0}$. When $m\neq 0$ we have recursive formulæ for $\boldsymbol{p}_{r+1,m}$ and $\boldsymbol{q}_{r,m}$,

$$\boldsymbol{p}_{r+1,m} = \frac{1}{\mathrm{i}m}\Bigg\{-\boldsymbol{p}'_{r,m} + \boldsymbol{f}_{1,0}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\boldsymbol{q}_{r,m} + \boldsymbol{f}_{1,1}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\boldsymbol{p}_{r,m}$$
$$+\sum_{n=2}^{r}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,k,r}}\theta_{n,k,r}(\boldsymbol{\ell})\sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,m}}\boldsymbol{f}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})$$
$$\big[\boldsymbol{p}_{\ell_1,\sigma_1},\cdots,\boldsymbol{p}_{\ell_k,\sigma_k}\big]\;\Big[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}},\cdots,\boldsymbol{q}_{\ell_n,\sigma_n}\Big]\Bigg\},$$
$$\boldsymbol{q}_{r,m} = -\boldsymbol{g}_{1,0}^{-1}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\Bigg\{\boldsymbol{g}_{1,1}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})\boldsymbol{p}_{r,m} + \sum_{n=2}^{r}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}$$
$$\sum_{\boldsymbol{\ell}\in\mathbb{I}_{n,k,r}}\theta_{n,k,r}(\boldsymbol{\ell})\sum_{\boldsymbol{\sigma}\in\mathbb{J}_{n,m}}\boldsymbol{g}_{n,k}(\boldsymbol{p}_{0,0},\boldsymbol{q}_{0,0})$$
$$\big[\boldsymbol{p}_{\ell_1,\sigma_1},\cdots,\boldsymbol{p}_{\ell_k,\sigma_k}\big]\;\Big[\boldsymbol{q}_{\ell_{k+1},\sigma_{k+1}},\cdots,\boldsymbol{q}_{\ell_n,\sigma_n}\Big]\Bigg\}.$$

3.4 An example

To illustrate the above formulæ, consider the nonlinear circuit in Fig. 3.1. Its behaviour is governed by the equations

$$\begin{aligned} C_{\text{in}}\frac{\mathrm{d}v(t)}{\mathrm{d}t}+\frac{v(t)}{R}+i(t)+S_{\text{inj}}\tanh\left(\frac{G_{\text{n}}}{S_{\text{inj}}}v(t)\right)&=h(t)+e_1(t),\\ L\frac{\mathrm{d}i(t)}{\mathrm{d}t}-v(t)&=0, \\ e(t)-Rh(t)-v(t)&=0, \qquad t\geq 0, \end{aligned} \tag{3.12}$$

where $I_{\text{nl}}=S_{\text{inj}}\tanh\left(G_{\text{n}}v(t)/S_{\text{inj}}\right)$.

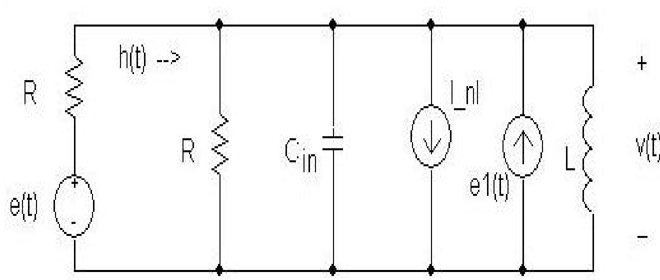

Figure 3.1: The nonlinear circuit (3.12)

The circuit DAEs may be rewritten in the form of (1.1)

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}&=\boldsymbol{f}(\boldsymbol{x},y(t))+\frac{1}{C_{\text{in}}}e_1(t)\\ 0&=g(\boldsymbol{x},y(t))+e(t), \end{aligned}$$

where $\boldsymbol{x}(t)=\begin{bmatrix}v(t) & i(t)\end{bmatrix}^\top$, $y(t)=h(t)$ and

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x},y)&=\begin{bmatrix}-\left[v(t)/R+i(t)+S_{\text{inj}}\tanh\left(\frac{G_{\text{n}}}{S_{\text{inj}}}v(t)\right)-h(t)\right]/C_{\text{in}}\\ v(t)/L\end{bmatrix},\\ g(\boldsymbol{x},y)&=-v(t)-Rh(t), \qquad e_1(t)=\mathrm{e}^{\mathrm{i}\omega t},\\ e(t)&=\frac{A_{\text{inj}}\sin(2\pi f_\omega t)}{2\pi f_\omega}=\frac{1}{\omega}\frac{A_{\text{inj}}}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\omega t}-\mathrm{e}^{-\mathrm{i}\omega t}\right), \end{aligned}$$

with $\omega=2\pi f_\omega$ and $\boldsymbol{x}(0)=[1 \quad 0]^\top$. In addition, $\boldsymbol{a}_1(t)=[1/C_{\text{in}} \quad 0]^\top$, $b_1(t)=\frac{A_{\text{inj}}}{2\mathrm{i}}$ and $b_{-1}(t)=-\frac{A_{\text{inj}}}{2\mathrm{i}}$. The source of nonlinearity is the term

$\tanh(G_n v(t)/S_{inj})$. The DAE does not have a known analytical solution, hence we have used a reference solution, employing the MAPLE DAE routine procedure with the very high error tolerances `AbsErr` $= 10^{-10}$, `RelErr` $= 10^{-10}$. For the numerical simulations we have set $L = 10^{-1}$, $R = 10$, $C_{in} = 0.2533$, $A_{inj} = 10$, $S_{inj} = 1/R$ and $G_n = -1.1/R$. Our asymptotic expansion is of the form

$$\boldsymbol{x}(t) = \begin{bmatrix} v(t) \\ i(t) \end{bmatrix} = \boldsymbol{p}_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} \boldsymbol{p}_{r,m}(t) e^{im\omega t},$$
$$h(t) = q_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} q_{r,m}(t) e^{im\omega t}.$$

The purpose of this example is to illustrate the computation of the first terms in these expansions.

In the case of $r = 0$, the non-oscillatory equations for $\boldsymbol{p}_{0,0}$ and $q_{0,0}$ are

$$\boldsymbol{p}'_{0,0}(t) = \boldsymbol{f}(\boldsymbol{p}_{0,0}, q_{0,0}) + \boldsymbol{a}_0(t), \quad t \geq 0, \qquad \boldsymbol{p}_{0,0}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\top},$$
$$0 = g(\boldsymbol{p}_{0,0}, q_{0,0}),$$

and the recurrence is $\boldsymbol{p}_{1,m} = \boldsymbol{a}_m(t)/(im)$. Since $\boldsymbol{a}_m = \boldsymbol{0}$ unless $m = 1$, the only nonzero term of this kind is $\boldsymbol{p}_{1,1} = [-i/C_{in} \;\; 0]^{\top}$.

Since

$$\boldsymbol{f}_{1,0} = \begin{bmatrix} 1/C_{in} \\ 0 \end{bmatrix}, \qquad \boldsymbol{g}_{1,0} = -R, \qquad \boldsymbol{g}_{1,1} = [-1 \;\; 0],$$
$$\boldsymbol{f}_{1,1} = \begin{bmatrix} -\left\{1/R + G_n\left[1 - \tanh^2\left(G_n \boldsymbol{p}^1_{0,0}(t)/S_{inj}\right)\right]\right\}/C_{in} & 1/C_{in} \\ 1/L & 0 \end{bmatrix},$$

where $\boldsymbol{p}_{0,0} = [\boldsymbol{p}^1_{0,0} \;\; \boldsymbol{p}^2_{0,0}]^{\top}$, the leading terms $\boldsymbol{p}_{1,0}$ and $q_{1,0}$ based on our expansion are computed by the non-oscillatory DAEs

$$\boldsymbol{p}'_{1,0}(t) = \boldsymbol{f}_{1,0}(\boldsymbol{p}_{0,0}, q_{0,0}) q_{1,0} + \boldsymbol{f}_{1,1}(\boldsymbol{p}_{0,0}, q_{0,0}) \boldsymbol{p}_{1,0}, \; t \geq 0, \quad \boldsymbol{p}_{1,0}(0) = -\boldsymbol{p}_{1,1}(0),$$
$$0 = -R q_{1,0} + [-1 \;\; 0] \boldsymbol{p}_{1,0}.$$

For $r = 1$ and $m \neq 0$

$$q_{1,m} = -\frac{1}{R}\left\{[1 \;\; 0] \boldsymbol{p}_{1,m} - b_m\right\},$$

$$\boldsymbol{p}_{2,m} = \frac{1}{\mathrm{i}m}\left\{\left[\frac{1}{C_{\mathrm{in}}} \;\; 0\right]^{\top} q_{1,m} + \boldsymbol{f}_{1,1}(\boldsymbol{p}_{0,0}, q_{0,0})\boldsymbol{p}_{1,m}\right\}.$$

Since

$$\begin{aligned}
\boldsymbol{f}_{2,0} &= \boldsymbol{f}_{2,1} = \boldsymbol{0}, \qquad \boldsymbol{g}_{2,0} = \boldsymbol{g}_{2,1} = \boldsymbol{g}_{2,2} = \boldsymbol{0},\\
\boldsymbol{f}_{2,2} &= \begin{bmatrix} \frac{2G_{\mathrm{n}}^2}{C_{\mathrm{in}}S_{\mathrm{inj}}} \tanh\left(\frac{G_{\mathrm{n}}}{S_{\mathrm{inj}}}\boldsymbol{p}_{0,0}^1(t)\right)\left[1 - \tanh^2\left(\frac{G_{\mathrm{n}}}{S_{\mathrm{inj}}}\boldsymbol{p}_{0,0}^1(t)\right)\right] & 0\\ 0 & 0 \end{bmatrix},
\end{aligned}$$

the DAE for $r = 2$ is

$$\begin{aligned}
\boldsymbol{p}_{2,0}' &= \boldsymbol{f}_{1,0}(\boldsymbol{p}_{0,0}, q_{0,0})q_{2,0} + \boldsymbol{f}_{1,1}(\boldsymbol{p}_{0,0}, q_{0,0})\boldsymbol{p}_{2,0} + \frac{1}{2}\boldsymbol{f}_{2,2}(\boldsymbol{p}_{0,0}, q_{0,0})\boldsymbol{p}_{1,0}^2,\\
0 &= -Rq_{2,0} + \begin{bmatrix} -1 & 0 \end{bmatrix}\boldsymbol{p}_{2,0}.
\end{aligned}$$

The initial condition is

$$\boldsymbol{p}_{2,0}(0) = \begin{bmatrix} -\left\{2 - A_{\mathrm{inj}}C_{\mathrm{in}} + RG_{\mathrm{n}}\left[1 - \tanh^2\left(\frac{G_{\mathrm{n}}}{S_{\mathrm{inj}}}\right)\right]\right\}/(C_{\mathrm{in}}^2 R)\\ 1/(C_{\mathrm{in}}L) \end{bmatrix}.$$

Moreover,

$$q_{2,m} = \frac{1}{R}[-1 \;\; 0\,]\,\boldsymbol{p}_{2,m}, \qquad m \neq 0.$$

Figures 3.2–4 display the errors e_s and ϵ_s, where

$$\begin{aligned}
e_0 &= \left|\boldsymbol{x}(t) - \boldsymbol{p}_{0,0}\right|,\\
e_1 &= \left|\boldsymbol{x}(t) - \boldsymbol{p}_{0,0} - \frac{1}{\omega}(\boldsymbol{p}_{1,0} + \boldsymbol{p}_{1,1}\mathrm{e}^{\mathrm{i}\omega t})\right|,\\
e_2 &= \left|\boldsymbol{x}(t) - \boldsymbol{p}_{0,0} - \frac{1}{\omega}(\boldsymbol{p}_{1,0} + \boldsymbol{p}_{1,1}\mathrm{e}^{\mathrm{i}\omega t}) - \frac{1}{\omega^2}(\boldsymbol{p}_{2,0} + \boldsymbol{p}_{2,1}\mathrm{e}^{\mathrm{i}\omega t} + \boldsymbol{p}_{2,-1}\mathrm{e}^{-\mathrm{i}\omega t})\right|
\end{aligned}$$

for $\boldsymbol{x}(t) = [v(t) \;\; i(t)]^{\top}$ and

$$\begin{aligned}
\epsilon_0 &= \left|h(t) - q_{0,0}\right|,\\
\epsilon_1 &= \left|h(t) - q_{0,0} - \frac{1}{\omega}(q_{1,0} + q_{1,1}\mathrm{e}^{\mathrm{i}\omega t} + q_{1,-1}\mathrm{e}^{-\mathrm{i}\omega t})\right|,\\
\epsilon_2 &= \Big|h(t) - q_{0,0} - \frac{1}{\omega}(q_{1,0} + q_{1,1}\mathrm{e}^{\mathrm{i}\omega t} + q_{1,-1}\mathrm{e}^{-\mathrm{i}\omega t})\\
&\qquad - \frac{1}{\omega^2}(q_{2,0} + q_{2,1}\mathrm{e}^{\mathrm{i}\omega t} + q_{2,-1}\mathrm{e}^{-\mathrm{i}\omega t})\Big|
\end{aligned}$$

for $h(t)$ for the cases $\omega = 200\pi$ and $\omega = 2000\pi$. Consistent with our theory, the magnitude of the errors reduces as ω increases.

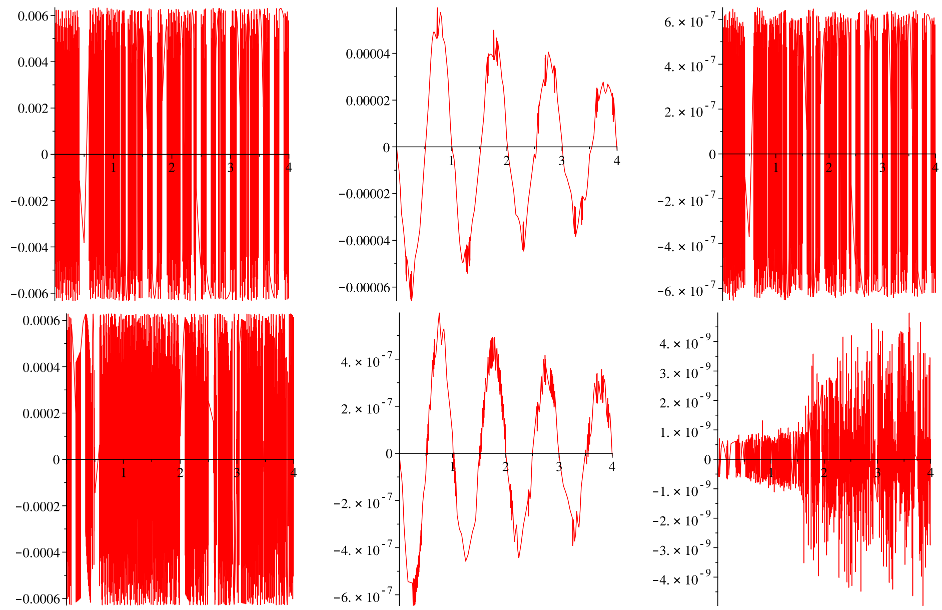

Figure 3.2: The errors for $v(t)$ for $\omega = 200\pi$ (top row) and $\omega = 2000\pi$ (bottom row): from the left e_0, e_1 and e_2.

4 Conclusions

Differential equations with highly oscillatory forcing terms are ubiquitous in many applications, since the forcing term corresponds to a high-frequency input into a dynamical system. Their solution by standard numerical methods is radically restricted by the requirement that the step size should scale as the reciprocal of the largest frequency. This explains recent interest in computation which combines numerical and asymptotic insight, e.g. [1, 3, 4, 9]. This has been focussed on ordinary and partial differential equations, inclusive of the multi-frequency case [2], with a single paper on delay-differential equations [5]. However, many realistic problems occurring in circuit simulation require the solution of DAEs: this is the first paper addressing the solution of DAE systems with high-frequency input using asymptotic-numerical techniques. We have demonstrated that the solution of such equations can be approximated to an exceedingly high precision using solely *non-oscillatory* computations – whether the solution of DAEs with no forcing terms or straightforward recursion. The oscillation is intro-

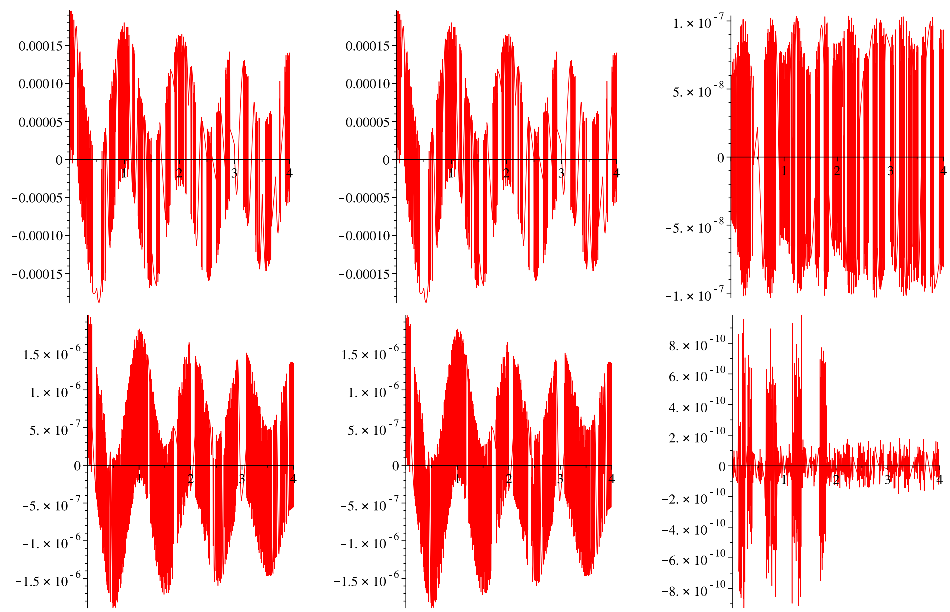

Figure 3.3: The errors for $i(t)$ for $\omega = 200\pi$ (top row) and $\omega = 2000\pi$ (bottom row): from the left e_0, e_1 and e_2.

duced into the computed solution only once the non-oscillatory ingredients are synthesised in a simple manner. Paradoxically (unless one is familiar with an asymptotic 'frame of mind'), the precision for fixed computational cost increases with frequency.

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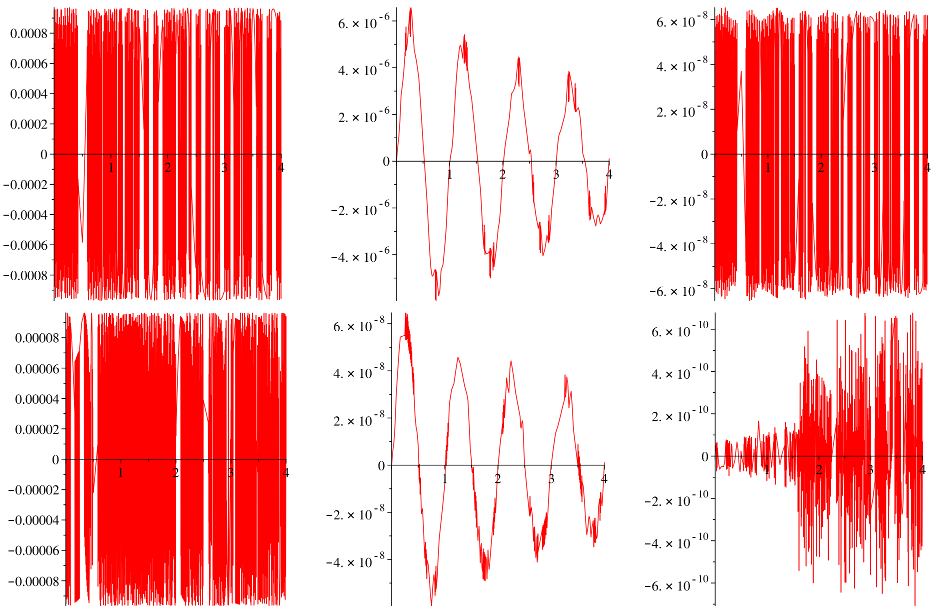

Figure 3.4: The errors for $h(t)$ for $\omega = 200\pi$ (top row) and $\omega = 2000\pi$ (bottom row): from the left ϵ_0, ϵ_1 and ϵ_2

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