The kissing polynomials and their Hankel determinants

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April 28, 2015

Abstract

We study a family of polynomials that are orthogonal with respect to the weight function $e^{i\omega x}$ in $[-1, 1]$, where $\omega \geq 0$. Since this weight function is complex-valued and, for large ω, highly oscillatory, many results in the classical theory of orthogonal polynomials do not apply. In particular, the polynomials need not exist for all values of the parameter ω, and, once they do, their roots lie in the complex plane. Our results are based on analysing the Hankel determinants of these polynomials, reformulated in terms of high-dimensional oscillatory integrals which are amenable to asymptotic analysis. This analysis yields existence of the even-degree polynomials for large values of ω, an asymptotic expansion of the polynomials in terms of rescaled Laguerre polynomials near ± 1 and a description of the intricate structure of the roots of the Hankel determinants in the complex plane.

This work is motivated by the design of efficient quadrature schemes for highly oscillatory integrals.

1 Introduction

Polynomials orthogonal with respect to the weight function $e^{i\omega x}$ in $[-1, 1]$ have been proposed in (Asheim, Deaño, Huybrechs & Wang 2014) as a means to derive complex Gaussian quadrature rules for highly oscillatory integrals. Though a wealth of observations were made regarding these polynomials, the focus in (Asheim et al. 2014) is on the numerical analysis of the quadrature error for such integrals. The current paper revolves around the analysis of the polynomials themselves, in the process confirming rigorously several of the observations that were made before.

We proceed by analysing the Hankel determinants associated with the polynomials as a function of the parameter ω. Though Hankel determinants are usually mentioned in the classical references for the theory of orthogonal polynomials (Szegő 1939, Gautschi 2004, Chihara 1978, Ismail 2005), and they are of interest in connection with integrable systems and random matrix theory, they are rarely a starting point for further analysis: insofar as classical orthogonal polynomials are concerned, it follows from standard results on the Hamburger moment problem that Hankel determinants are always positive and they throw little added insight on the underlying problem. The main reason for pursuing Hankel determinants in this paper is an old result of

Heine (Ismail 2005, §2.1) that they can be written as a multivariate integral, as shown in §3, which in the context of our oscillatory weight function becomes an oscillatory integral. The integral can be expanded asymptotically in negative powers of ω and this expansion is very revealing. Furthermore, several quantities of interest related to the orthogonal polynomials, including the polynomials themselves, can be formulated in terms of Hankel-like determinants.

To establish our notation, we consider monic orthogonal polynomials (OPs) $p_n^\omega(x)$ or $p_n^\omega(\,\cdot\,)$, formally defined as follows:

$$\int_{-1}^{1} p_n^\omega(x) x^k \mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x = 0, \qquad k = 0, 1, \ldots, n-1, \tag{1.1}$$

where $n \in \mathbb{Z}_+$ and $\omega \geq 0$. For simplicity of notation, in the sequel we omit the parameter ω and write $p_n(x)$ directly.

The standard construction of orthogonal polynomials with respect to a measure μ involves the Hankel matrices built from the moments of the measure. In the present case, $\mathrm{d}\mu(x) = \mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x$, let

$$\mu_n(\omega) = \int_{-1}^{1} x^n \mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x, \qquad n \in \mathbb{Z}_+$$

be the moments of the weight function and set

$$H_n = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix} \qquad \text{and} \qquad h_n = \det H_n, \quad n \in \mathbb{N}, \tag{1.2}$$

the nth *Hankel* matrix and determinant, respectively. It is well known that the polynomial $p_n(x)$ itself can also be written in terms of determinants as

$$p_n(x) = \frac{1}{h_{n-1}} \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_n & x \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} & x^n \end{bmatrix}, \tag{1.3}$$

see for instance (Ismail 2005, Chapter 2). Note that $p_n(x)$ exists if and only if $h_{n-1} \neq 0$. This is guaranteed whenever μ is a positive Borel measure, so for classical polynomials h_n is always strictly positive, and in that case any deeper study of the Hankel determinants is usually of little further consequence. In our setting, however, the study of the Hankel determinants, and in particular the study of their roots, is extremely insightful.

Note that, despite the fact that the weight function is complex–valued, the determinants h_n are real for all $n \in \mathbb{Z}_+$. This is so because

$$\mu_n(-\omega) = (-1)^n \mu_n(\omega), \qquad n \in \mathbb{Z}_+.$$

Thus, if we consider the Hankel matrix $H_n(-\omega)$, we can pull out a factor $(-1)^k$ from the k-th row and $(-1)^\ell$ from the ℓ-th column. As a consequence,

$$\overline{h_n(\omega)} = h_n(-\omega) = (-1)^{n(n-1)} h_n(\omega) = h_n(\omega),$$

and the result follows.

In §2 we sketch some properties of our orthogonal polynomials and in §3 we express their Hankel determinant as a multivariate integral. The heart of our investigation is §4, where we analyse Hankel determinants and their asymptotic expansion for $\omega \gg 1$. This is used in §5 to explain the kissing pattern of zeros, in §6 to investigate the highly nontrivial behaviour of orthogonal polynomials near the endpoints ± 1 and in §7 to shed light on the complex zeros of Hankel determinants. Finally, in §8 we extend the important fact that the existence and uniqueness of p_{2N}^ω is assured for $\omega \geq 0$ small enough or large enough to all $\omega \geq 0$.

The origins of 'kissing polynomials' are in highly oscillatory quadrature but, we believe, their behaviour and the crucial differences between them and conventional orthogonal polynomials are a matter of an independent mathematical interest. In this paper we consider the simplest possible case but we expect to return to this theme in future papers, investigating polynomials orthogonal with respect to the more general sesquilinear form,

$$\langle f_1, f_2 \rangle = \int_{-1}^{1} f_1(x) f_2(x) w(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \, \mathrm{d}x,$$

where w and g are given functions of suitable regularity. Preliminary results indicate a wealth of fascinating and beautiful behaviour.

2 Properties of the orthogonal polynomials

We recall few interesting properties and observations on the polynomials p_n from (Asheim et al. 2014). Firstly, they are symmetric with respect to the imaginary axis:

$$p_n(z) = (-1)^n \overline{p_n(-\bar{z})}, \qquad z \in \mathbb{C}. \tag{2.1}$$

Note that the map $z \to -\bar{z}$ represents a reflection with respect to the pure imaginary axis: if $z = x + \mathrm{i}y$ then $-\bar{z} = -x + \mathrm{i}y$.

Additionally, since the orthogonality relation (1.1) is non–Hermitian, the monic OPs satisfy a three term recurrence relation,

$$p_{n+1}(x) = (x - \alpha_n) p_n(x) - \beta_n p_{n-1}(x), \tag{2.2}$$

provided that three consecutive polynomials exist for a given value of ω. The initial values are taken as $p_{-1}(x) = 0$, $p_0(x) = 1$ and the coefficients α_n and β_n are complex valued.

The coefficients of the recurrence relation can be given in terms of Hankel determinants, namely,

$$\alpha_n = -\mathrm{i}\left(\frac{h_n'}{h_n} - \frac{h_{n-1}'}{h_{n-1}}\right), \qquad \beta_n = \frac{h_n h_{n-2}}{h_{n-1}^2}, \tag{2.3}$$

where h_n' indicates differentiation with respect to ω.

As a function of the parameter ω, the recurrence coefficients themselves satisfy the differential–difference equations

$$\begin{aligned} \alpha_n' &= \mathrm{i}(\beta_{n+1} - \beta_n) \\ \beta_n' &= \mathrm{i}\beta_n(\alpha_n - \alpha_{n-1}), \end{aligned} \tag{2.4}$$

This is nothing but a complex case of the classical Toda lattice equation subject to the Flaschka transformation, which is known to govern the deformation of the recurrence coefficients whenever the measure of orthogonality is a perturbation of a classical one with an exponential factor linear in the parameter, which is ω in our case. We refer the reader to (Ismail 2005, §2.8) for more details.

Another important consequence of the weight function not being positive is the fact that, even when $p_n(x)$ exists for some values of n and ω, its roots lie in the complex plane. They come in pairs of two, symmetric with respect to the imaginary axis, as a consequence of (2.1).

When $\omega = 0$, p_n is a multiple of the classical Legendre polynomial and the roots are real and inside the interval $[-1, 1]$. For increasing values of ω, they follow a trajectory in the upper half plane illustrated in the top panel of Figure 2.1. The trajectories corresponding to polynomials of consecutive even and odd degree touch at a discrete set of frequencies ω: the polynomials 'kiss'.

Regarding the asymptotic behaviour of the polynomials p_n, it is of interest to consider the case when one of both parameters, n and ω, are large. The large n asymptotics can be deduced using for instance the Deift–Zhou steepest descent method applied to the corresponding Riemann–Hilbert problem, as in (Deaño 2014) and references therein, see also the general monograph (Deift 2000). This analysis focuses in the case when n is large and ω is fixed or grows linearly with n, and provides existence of $p_n(z)$ for large enough n together with asymptotic behaviour for z in different regions of the complex plane.

The asymptotic analysis for fixed n and large ω is much less standard. It is conjectured in (Asheim et al. 2014) that the even-degree polynomials asymptotically behave like a product of Laguerre polynomials centered around the endpoints ± 1:

$$p_{2n}(x) \sim \left(\frac{\mathrm{i}}{\omega}\right)^{2n} \mathrm{L}_n(-\mathrm{i}\omega(x+1))\mathrm{L}_n(-\mathrm{i}\omega(x-1)), \qquad \omega \to \infty, \tag{2.5}$$

where L_n is the nth *Laguerre polynomial* with parameter $\alpha = 0$ (Szegő 1939). Thus, for large ω, it seems that the orthogonal polynomials of even degree become approximately a product of lower degree orthogonal polynomials. This conjecture also implies that the roots shown in Fig. 2.1 behave like $\pm 1 - \frac{c}{\mathrm{i}\omega}$, where c is a root of the Laguerre polynomial L_n.

3 Properties of Hankel determinants

3.1 An expression as a multivariate integral

We commence by revisiting an old result of Heine (Ismail 2005, Section 2.1), (Szegő 1939, pg. 27) which is fundamental to our analysis. Equally fundamental is the

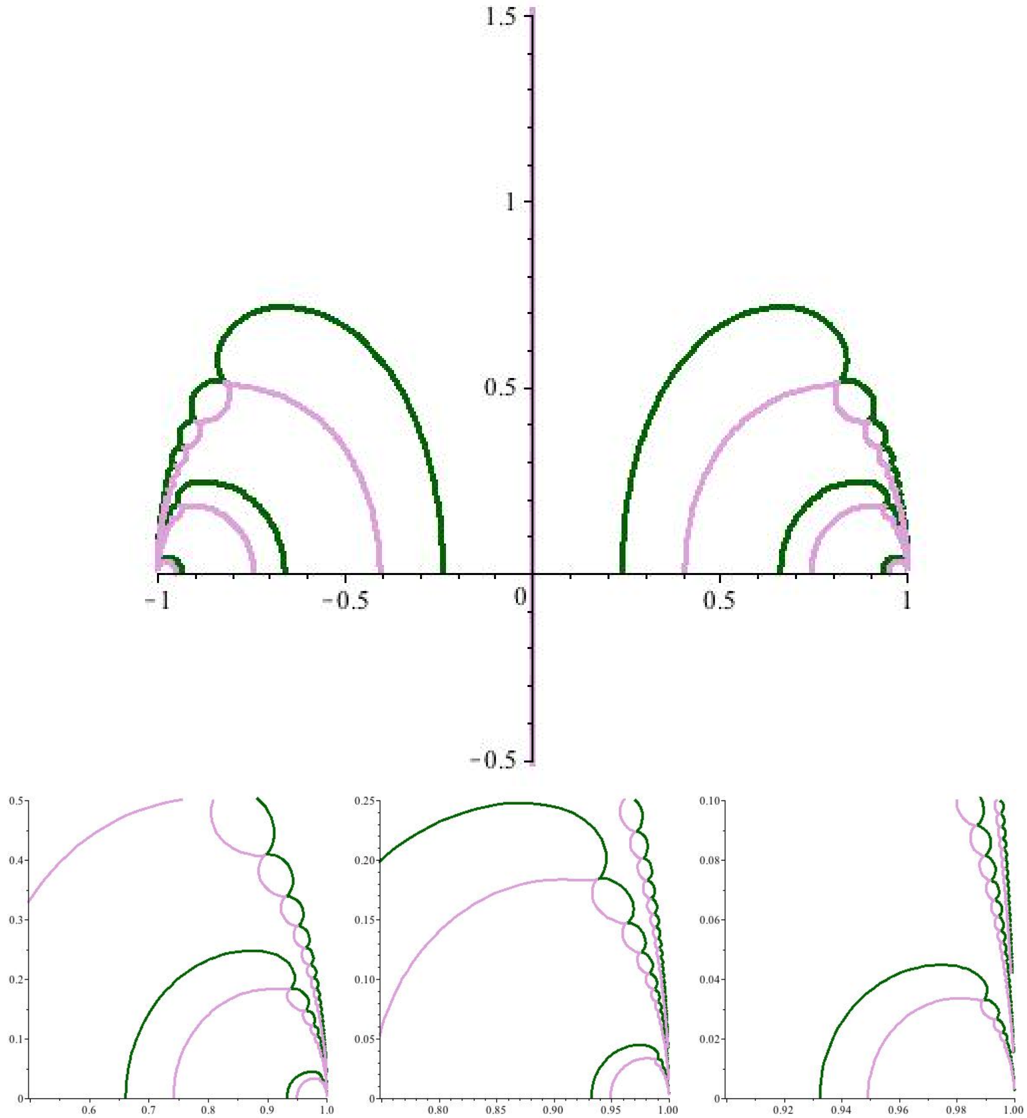

Figure 2.1: The zeros of p_6 (dark) and of p_7 (light) and, below, close-ups of the 'kissing' patterns near the right endpoint $+1$.

method of proof, which will be reused and generalised in the sequel.

Lemma 1 *For every $n \in \mathbb{Z}_+$ it is true that*

$$h_{n-1} = \frac{1}{n!} \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{0 \le k < \ell \le n-1} (x_\ell - x_k)^2 \mathrm{e}^{\mathrm{i}\omega(x_0+x_1+\cdots+x_{n-1})} \, \mathrm{d}x_{n-1} \cdots \mathrm{d}x_1 \, \mathrm{d}x_0. \tag{3.1}$$

Proof We write the determinant in the following form:

$$\begin{aligned}
h_{n-1} &= \det \begin{bmatrix} \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega x_0} \, \mathrm{d}x_0 & \int_{-1}^{1} x_1 \mathrm{e}^{\mathrm{i}\omega x_1} \, \mathrm{d}x_1 & \cdots & \int_{-1}^{1} x_{n-1}^{n-1} \mathrm{e}^{\mathrm{i}\omega x_{n-1}} \, \mathrm{d}x_{n-1} \\ \int_{-1}^{1} x_0 \mathrm{e}^{\mathrm{i}\omega x_0} \, \mathrm{d}x_0 & \int_{-1}^{1} x_1^2 \mathrm{e}^{\mathrm{i}\omega x_1} \, \mathrm{d}x_1 & \cdots & \int_{-1}^{1} x_{n-1}^{n} \mathrm{e}^{\mathrm{i}\omega x_{n-1}} \, \mathrm{d}x_{n-1} \\ \vdots & \vdots & & \vdots \\ \int_{-1}^{1} x_0^{n-1} \mathrm{e}^{\mathrm{i}\omega x_0} \, \mathrm{d}x_0 & \int_{-1}^{1} x_1^{n} \mathrm{e}^{\mathrm{i}\omega x_1} \, \mathrm{d}x_1 & \cdots & \int_{-1}^{1} x_{n-1}^{2n-2} \mathrm{e}^{\mathrm{i}\omega x_{n-1}} \, \mathrm{d}x_{n-1} \end{bmatrix} \\
&= \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} \det \begin{bmatrix} 1 & x_1 & \cdots & x_{n-1}^{n-1} \\ x_0 & x_1^2 & \cdots & x_{n-1}^{n} \\ \vdots & \vdots & & \vdots \\ x_0^{n-1} & x_1^{n} & \cdots & x_{n-1}^{2n-2} \end{bmatrix} \mathrm{e}^{\mathrm{i}\omega \boldsymbol{x}^\top \mathbf{1}} \, \mathrm{d}x_{n-1} \cdots \mathrm{d}x_1 \, \mathrm{d}x_0 \\
&= \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{k=0}^{n-1} x_k^k \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_{n-1} \\ \vdots & \vdots & & \vdots \\ x_0^{n-1} & x_1^{n-1} & \cdots & x_{n-1}^{n-1} \end{bmatrix} \mathrm{e}^{\mathrm{i}\omega \boldsymbol{x}^\top \mathbf{1}} \, \mathrm{d}x_{n-1} \cdots \mathrm{d}x_1 \, \mathrm{d}x_0 \\
&= \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{k=0}^{n-1} x_k^k \prod_{0 \le k < \ell \le n-1} (x_\ell - x_k) \, \mathrm{e}^{\mathrm{i}\omega \boldsymbol{x}^\top \mathbf{1}} \, \mathrm{d}x_{n-1} \cdots \mathrm{d}x_1 \, \mathrm{d}x_0,
\end{aligned}$$

using the well known formula for the determinant of a Vandermonde matrix. Let $\boldsymbol{\pi}$ be a permutation of $(0, 1, \ldots, n-1)$. Then, changing the order of integration,

$$h_{n-1} = (-1)^{\sigma(\boldsymbol{\pi})} \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{k=0}^{n-1} x_{\pi(k)}^k \prod_{0 \le k < \ell \le n-1} (x_\ell - x_k) \mathrm{e}^{\mathrm{i}\omega \boldsymbol{x}^\top \mathbf{1}} \, \mathrm{d}x_{n-1} \cdots \mathrm{d}x_1 \, \mathrm{d}x_0,$$

where $\sigma(\boldsymbol{\pi})$ is the sign of the permutation. Averaging over all $n!$ permutations,

$$h_{n-1} = \frac{1}{n!} \int_{-1}^{1} \cdots \int_{-1}^{1} g(x_0, \ldots, x_{n-1}) \prod_{0 \le k < \ell \le n-1} (x_\ell - x_k) \mathrm{e}^{\mathrm{i}\omega \boldsymbol{x}^\top \mathbf{1}} \, \mathrm{d}x_{n-1} \cdots \mathrm{d}x_1 \, \mathrm{d}x_0,$$

where

$$g(x_0, \ldots, x_{n-1}) = \sum_{\boldsymbol{\pi} \in \Pi_n} (-1)^{\sigma(\boldsymbol{\pi})} \prod_{k=0}^{n-1} x_{\pi(k)}^k$$

and Π_n is the set of all the permutations of $(0, 1, \ldots, n-1)$. And now comes the serendipitous step that we will use time and again in the sequel: the observation that

g is a determinant – specifically, a determinant of an $n \times n$ Vandermonde matrix. Therefore

$$g(x_0, \ldots, x_{n-1}) = \prod_{0 \le k < \ell \le n-1} (x_\ell - x_k) \tag{3.2}$$

and the proof of (3.1) is complete. □

3.2 The asymptotic expansion of h_n

The oscillatory integral (3.1) can be expanded asymptotically in inverse powers of ω. Indeed, the integrand has the canonical form $F(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\omega\varrho(\boldsymbol{x})}$ of a non-oscillatory function $F(\boldsymbol{x})$ multiplying an oscillatory exponential, with the so-called oscillator $\varrho(\boldsymbol{x})$ – in this case simply the linear function $x_0+x_1+\ldots+x_{n-1}$. It is well known how to derive such expansion, for example using repeated integration by parts (see, e.g., (Wong 2001)). This is straightforward in principle, but hampered by lengthy algebraic manipulations in our high-dimensional setting since (3.1) is an n-fold integral. In the following, we will use the multi-index notation of (Iserles & Nørsett 2006) to control the complexity:

$$\begin{aligned} &\int_{[-1,1]^n} F(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\omega\mathbf{1}^\top\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} \\ &\sim (-1)^n \sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+n}} \sum_{|\boldsymbol{k}|=m} \sum_{\boldsymbol{v}\in\mathcal{V}_n} (-1)^{s(\boldsymbol{v})}\mathrm{e}^{\mathrm{i}\omega\mathbf{1}^\top\boldsymbol{x}}\partial_{\boldsymbol{x}}^{\boldsymbol{k}}F(\boldsymbol{v}). \end{aligned} \tag{3.3}$$

Here, the notation $\mathbf{1}^\top\boldsymbol{x}$ is used for the linear function $x_0 + x_1 + \ldots + x_{n-1}$. We let $\mathcal{V}_n$ be the set of the 2^n vertices of the n-cube $[-1,1]^n$. The index function $s(\boldsymbol{v})$ of $\boldsymbol{v} \in \mathcal{V}_n$ is the number of -1 therein. For example, for $n = 3$ the index of the vertex $(-1,-1,1)$ would be 2.

Note that each term in the expansion, corresponding to some negative power of ω, consists of summing over all partial derivatives of a certain total order m over all possible vertices of the cube $[-1,1]^n$. One may think of these derivatives as originating from the integration by parts technique, and they are evaluated at the vertices because the endpoints of all univariate integrals involved are either $+1$ or -1 and in this case the integrand has no singularities or stationary points.

Specializing (3.3) to the integral (3.1) at hand, and using (3.2), we note that the non-oscillatory function

$$\frac{1}{n!} \prod_{0 \le k < \ell \le n-1} (x_\ell - x_k)^2, \qquad n \in \mathbb{Z}_+. \tag{3.4}$$

is a polynomial of total degree $(n-1)n$. This implies that expansion (3.3) terminates, as all derivatives vanish once $m \ge (n-1)n+1$. Since the expansion of h_{n-1} starts with ω^{-n}, because of Lemma 1, we expect it to have the form

$$h_{n-1} = \sum_{\ell=n}^{n^2} \frac{h_{n-1,\ell}}{\omega^\ell}. \tag{3.5}$$

The reason for the upper bound is that for the last significant value of $\ell = (n-1)n$, we have $\ell + n = n^2$.

By direct calculation we find the first few expansions

$$\begin{aligned}
h_0(\omega) &= \frac{2\sin\omega}{\omega}, \\
h_1(\omega) &= \frac{4}{\omega^2} + \frac{2(\cos 2\omega - 1)}{\omega^4}, \\
h_2(\omega) &= -\frac{32\sin\omega}{\omega^5} - \frac{64\cos\omega}{\omega^6} + \frac{96\sin\omega}{\omega^7} - \frac{32\sin^3\omega}{\omega^9}, \\
h_3(\omega) &= \frac{256}{\omega^8} + \frac{512(\cos 2\omega - 4)}{\omega^{10}} - \frac{3072\sin 2\omega}{\omega^{11}} - \frac{768(11\cos 2\omega - 2)}{\omega^{12}} + \frac{9216\sin 2\omega}{\omega^{13}} \\
&\quad + \frac{6912(\cos 2\omega - 1)}{\omega^{14}} + \frac{576(\cos 2\omega - 1)^2}{\omega^{16}}.
\end{aligned}$$

The upper bound in (3.5) is sharp. However, the leading powers in these expressions are substantially higher than predicted by (3.5) and the discrepancy becomes more pronounced as n increases. Instead of (3.5) we have

$$h_{n-1} = \sum_{\ell=\delta_n}^{n^2} \frac{h_{n-1,\ell}}{\omega^\ell}, \tag{3.6}$$

where

$$\delta_1 = 1, \quad \delta_2 = 2, \quad \delta_3 = 5, \quad \delta_4 = 8, \quad \delta_5 = 13, \quad \delta_6 = 18, \quad \delta_7 = 25.$$

Note that for even n the values h_{n-1,δ_n} in the examples above are positive constants: this implies the existence of even-degree polynomials for sufficiently large ω. The factor $\sin\omega$, however, appearing for odd n, demonstrates asymptotic non-existence of odd-degree polynomials, approximately at integer multiples of π.

In view of the fact that expansion (3.6) terminates, it is actually an exact and explicit expression for h_{n-1}. Though it may seem to have a high-order pole at $\omega = 0$, the singularity is removable, since from (3.1) it follows that h_{n-1} is an analytic function of ω.

The case $\omega = 0$ corresponds to the Legendre weight function and the corresponding Hankel determinants are all positive. Actually, as $\omega \to 0^+$, we have

$$\begin{aligned}
h_0(\omega) &= 2 - \frac{1}{3}\omega^2 + \frac{1}{60}\omega^4 + \mathcal{O}(\omega^6), \\
h_1(\omega) &= \frac{4}{3} - \frac{8}{45}\omega^2 + \frac{4}{315}\omega^4 + \mathcal{O}(\omega^6), \\
h_2(\omega) &= \frac{32}{135} - \frac{16}{525}\omega^2 + \frac{4}{2025}\omega^4 + \mathcal{O}(\omega^6), \\
h_3(\omega) &= \frac{256}{23625} - \frac{2048}{1488375}\omega^2 + \frac{1024}{11694375}\omega^4 + \mathcal{O}(\omega^6).
\end{aligned}$$

The quest for the leading order term in expansion (3.6) revolves around the study of the derivatives of the integrand at the vertices of the hypercube $[-1,1]^n$. In particular, it is clear from the explicit expression (3.4) that the integrand vanishes to some order whenever two coordinates x_ℓ and x_k coincide. This is the case at the vertices and, loosely speaking, the order is determined mostly by the difference between the number of $+1$s and -1s at the vertex.

4 Asymptotic analysis of a symmetric integral

It is convenient for later developments to aim for a slightly higher level of generality than in §3.2, before simplifying again to the case at hand. We set

$$F(\boldsymbol{x}) = f(\boldsymbol{x})g(\boldsymbol{x})^2, \tag{4.1}$$

where the function g is the Vandermonde determinant:

$$g(\boldsymbol{x}) = g(x_0, x_1, \ldots, x_{n-1}) = \prod_{0\le k\le \ell\le n-1}(x_\ell - x_k) = \sum_{\boldsymbol{\pi}}(-1)^{\sigma(\boldsymbol{\pi})}\prod_{k=0}^{n-1} x_k^{\pi(k)}, \tag{4.2}$$

and $\boldsymbol{\pi} \in \Pi_n$ is the set of all permutations of length n, acting for example on the n-tuple $(x_0, \ldots, x_{n-1})$. In the sequel we will also assume that the smooth function f is *symmetric in its arguments.*

Thus, we set out to study the n-fold integral

$$\begin{aligned}\boldsymbol{I}_n[f] &= \frac{1}{n!}\int_{-1}^{1}\cdots\int_{-1}^{1} F(x_0, \ldots, x_{n-1})\mathrm{e}^{\mathrm{i}\omega\boldsymbol{x}^\top\boldsymbol{1}}\,\mathrm{d}x_0\cdots\mathrm{d}x_{n-1}, \\ &= \frac{1}{n!}\int_{-1}^{1}\cdots\int_{-1}^{1} f(x_0, \ldots, x_{n-1})\prod_{0\le k<\ell\le n-1}(x_\ell - x_k)^2\mathrm{e}^{\mathrm{i}\omega\boldsymbol{x}^\top\boldsymbol{1}}\,\mathrm{d}x_0\cdots\mathrm{d}x_{n-1},\end{aligned} \tag{4.3}$$

The cases of primary interest in this paper are the following:

- $f \equiv 1$, since $\boldsymbol{I}_n[1]$ corresponds precisely to h_{n-1}, according to the integral (3.1),
- $f(\boldsymbol{x}) = \prod_{m=0}^{n-1}(x - x_m)$, that corresponds to the polynomial $h_{n-1}p_n^\omega(x)$. This follows from Heine's formula for the orthogonal polynomial:

$$p_n^\omega(x) = \frac{1}{n!\,h_{n-1}}\int_{-1}^{1}\cdots\int_{-1}^{1}\prod_{m=0}^{n-1}(x-x_m)\prod_{0\le k<\ell\le n-1}(x_\ell-x_k)^2\mathrm{e}^{\mathrm{i}\omega\boldsymbol{x}^\top\boldsymbol{1}}\,\mathrm{d}x_0\cdots\mathrm{d}x_{n-1}, \tag{4.4}$$

see, e.g., (Ismail 2005, Theorem 2.1.2).

Of course, once f is a polynomial, so is F, and in that case (3.3) terminates and is no longer an asymptotic expansion but an exact formula. In any case, we are faced with computing derivatives of $F(\boldsymbol{x})$ and, in particular, the first non-vanishing derivative of F at a vertex of the hypercube.

4.1 Vanishing derivatives of the function F

Everything is symmetric, hence we can assume that for a vertex with r $(-1)s$, i.e. $s(\boldsymbol{v}) = r$, we have

$$\boldsymbol{v} = (\overbrace{-1,\ldots,-1}^{r \text{ times}}, \overbrace{+1,\ldots,+1}^{n-r \text{ times}}). \tag{4.5}$$

Let us consider the following factorization:

$$\alpha_r(\boldsymbol{x}) = \prod_{0\le k<l\le r-1} (x_l - x_k), \qquad \text{and} \qquad \beta_{n,r}(\boldsymbol{x}) = \prod_{k=0}^{r-1}\prod_{l=r}^{n-1}(x_l - x_k), \tag{4.6}$$

then

$$F(\boldsymbol{x}) = f(\boldsymbol{x})\alpha_r^2(x_0,\ldots,x_{r-1})\alpha_{n-r}^2(x_r,\ldots,x_{n-1})\beta_{n,r}^2(x_0,\ldots,x_{n-1}). \tag{4.7}$$

We know by construction that $\beta_{n,r}$ does not vanish at a vertex, in fact $\beta_{n,r}(\boldsymbol{v}) = 2^{r(n-r)}$. We assume, at least for the time being, that $f(\boldsymbol{v})$ does not vanish either, a condition that is clearly satisfied in the cases enumerated before. Therefore, we need be concerned just with α_r at $-\mathbf{1}$ and α_{n-r} at $+\mathbf{1}$. Since $\alpha(\boldsymbol{x} + c\mathbf{1}) = \alpha(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$, $c \in \mathbb{R}$, because α only depends on the differences between elements of $\boldsymbol{x}$, it is sufficient to examine these expansions at $\boldsymbol{x} = \mathbf{0}$.

By the definition of a determinant,

$$\alpha_r(\boldsymbol{x}) = \mathrm{VDM}(x_0,\ldots,x_{r-1}) = \sum_{\pi\in\Pi_r} (-1)^{\sigma(\pi)} x_0^{\pi_0} x_1^{\pi_1} \ldots x_{r-1}^{\pi_{r-1}},$$

where Π_r is the set of permutations of length r and $\sigma(\pi)$ is the *sign* of π. We deduce that $\partial_{\boldsymbol{x}}^{\boldsymbol{k}}\alpha_r(\mathbf{0}) = 0$ unless $k = \pi \in \Pi_r$. In the latter case,

$$\partial_{\boldsymbol{x}}^{\boldsymbol{k}}\alpha_r(\mathbf{0}) = (-1)^{\sigma(\pi)} \prod_{j=0}^{r-1} \pi_j! = (-1)^{\sigma(\pi)}\mathrm{sf}(r-1),$$

where $\mathrm{sf}(m) = 0!1!\cdots m!$ is a *super-factorial.* In terms of the Barnes G function, see (DLMF 2014, §5.17), we have $\mathrm{sf}(m) = G(m+2)$.

Consequently, by Leibniz's formula,

$$\begin{aligned}
\partial_{\boldsymbol{x}}^{\boldsymbol{k}}\alpha_r^2(\mathbf{0}) &= \sum_{k_1+k_2=k} \prod_{i=0}^{r-1} \binom{k_{1,i}+k_{2,i}}{k_{1,i}} \partial_{\boldsymbol{x}}^{k_1}\alpha_r(\mathbf{0})\partial_{\boldsymbol{x}}^{k_2}\alpha_r(\mathbf{0}) \\
&= \sum_{\pi_1+\pi_2=k} \prod_{i=0}^{r-1} \binom{\pi_{1,i}+\pi_{2,i}}{\pi_{1,i}} (-1)^{\sigma(\pi_1)+\sigma(\pi_2)}\mathrm{sf}^2(r-1),
\end{aligned}$$

with $\pi_1, \pi_2 \in \Pi_r$ both permutations of length r, i.e. the only terms surviving in Leibniz's formula are those for which k_1 and k_2 are permutations.

Let

$$F_\alpha(\boldsymbol{x}) = \alpha_r(\boldsymbol{x})^2\alpha_{n-r}(\boldsymbol{x})^2. \tag{4.8}$$

Using the multi-index $\boldsymbol{k} = [k_0, \ldots, k_{n-1}]$, along with the definitions $\boldsymbol{k}^{[1]} = [k_0, \ldots, k_{r-1}]$ and $\boldsymbol{k}^{[2]} = [k_r, \ldots, k_{n-1}]$, we have

$$\partial_{\boldsymbol{x}}^{\boldsymbol{k}} F_\alpha(\boldsymbol{v}) = \partial_{\boldsymbol{x}}^{\boldsymbol{k}}[\alpha_r^2(-\mathbf{1})\alpha_{n-r}^2(+\mathbf{1})] = \partial_{\boldsymbol{x}}^{\boldsymbol{k}^{[1]}}[\alpha_r^2(\mathbf{0})]\, \partial_{\boldsymbol{x}}^{\boldsymbol{k}^{[2]}}[\alpha_{n-r}^2(\mathbf{0})]$$

This derivative is nonzero *only* for $\boldsymbol{k}^{[1]} = \pi_1^{[1]} + \pi_2^{[1]}$ and $\boldsymbol{k}^{[2]} = \pi_1^{[2]} + \pi_2^{[2]}$, where $\pi_i^{[1]} \in \Pi_r$ and $\pi_i^{[2]} \in \Pi_{n-r}$.

Combined with the above, we arrive at the expression

$$\begin{aligned} \partial_{\boldsymbol{x}}^{\boldsymbol{k}} F_\alpha(\boldsymbol{v}) = \mathrm{sf}^2(r-1)\mathrm{sf}^2(n-r-1) & \\ \times \sum_{\pi_1^{[1]}+\pi_2^{[1]}=\boldsymbol{k}^{[1]}} (-1)^{\sigma(\pi_1^{[1]})+\sigma(\pi_2^{[1]})} \prod_{i=0}^{r-1} \binom{\pi_{1,i}^{[1]}+\pi_{2,i}^{[1]}}{\pi_{1,i}^{[1]}} & \\ \times \sum_{\pi_1^{[2]}+\pi_2^{[2]}=\boldsymbol{k}^{[2]}} (-1)^{\sigma(\pi_1^{[2]})+\sigma(\pi_2^{[2]})} \prod_{i=0}^{n-r-1} \binom{\pi_{1,i}^{[2]}+\pi_{2,i}^{[2]}}{\pi_{1,i}^{[2]}} & . \end{aligned} \tag{4.9}$$

This expression is only semi-explicit and it is fairly difficult to proceed analytically with conditions of the form $\boldsymbol{k} = \pi_1 + \pi_2$. However, the expression is valid for any $\boldsymbol{k}$, and we are only interested in the derivative that corresponds to the leading order term in expansion (4.3). It turns out that, once all contributions are summed, such conditions drop out.

4.2 The leading order term

4.2.1 Vertices with minimal weight

We recall the definition of the function F in (4.1), and we aim for the leading order term in (4.3), which corresponds to the smallest $m = |\boldsymbol{k}|$ such that $\partial_{\boldsymbol{x}}^{\boldsymbol{k}} F(\boldsymbol{v})$ does not vanish.

It is clear from the preceding analysis that the number of (-1)s and $(+1)$s in $\boldsymbol{v}$ plays a crucial role. In particular, we call the difference (in absolute value) between the number of $(+1)$s and the number of (-1)s in a vertex the *weight* of that vertex. So far, we have been considering vertices with r $(-1)s$ and $n-r$ $(1)s$, so the weight is $|n-2r|$.

Note that for a derivative of order $\boldsymbol{k} = \pi_1^{[1]} + \pi_2^{[1]} + \pi_1^{[2]} + \pi_2^{[2]}$, where $\pi_i^{[1]} \in \Pi_r$ and $\pi_i^{[2]} \in \Pi_{n-r}$, we have

$$|\boldsymbol{k}| = (r-1)r + (n-r-1)(n-r), \tag{4.10}$$

since for $\pi \in \Pi_m$ we have $|\pi| = \frac{(m-1)m}{2}$.

It is straightforward to verify that $|\boldsymbol{k}|$ is minimal for vertices with minimal weight. This leads to $r = n/2$ for even n, and $r = (n+1)/2$ or $r = (n-1)/2$ for odd n. Consequently, we have to distinguish between these two cases.

4.2.2 The even case $n = 2N$

Let us examine the factors in expansion (4.3). There are $\binom{n}{r}$ vertices with r $(-1)s$, permutations of (4.5). Let us call this set $\mathcal{V}_{n,r}$. In this case, because $r = n/2$, we have $r = n - r = N$. For each vertex with minimal weight, i.e. for each $\boldsymbol{v} \in \mathcal{V}_{2N,N}$, we have $(-1)^{s(\boldsymbol{v})} = (-1)^r = (-1)^N$ and $\mathrm{e}^{\mathrm{i}\omega \boldsymbol{v}^T \mathbf{1}} = 1$. Furthermore, $\beta_{n,r}(\boldsymbol{v})^2 = 4^{r(n-r)} = 4^{N^2}$ and f was assumed to be a symmetric function in its variables, hence f is constant on $\mathcal{V}_{2N,N}$.

We have to sum over all derivatives of total order $|\boldsymbol{k}| = 2N(N-1)$. Since $|\boldsymbol{k}|$ is minimal, we have from Leibniz's formula that

$$\partial_{\boldsymbol{x}}^{\boldsymbol{k}} F(\boldsymbol{v}) = 4^{N^2} f(\boldsymbol{v}) \partial_{\boldsymbol{x}}^{\boldsymbol{k}} F_\alpha(\boldsymbol{v}),$$

where 4^{N^2} is the contribution of $\beta_{2N,N}^2$. Each possible $\boldsymbol{k}$ is reached by a combination of permutations of length N. From (4.9), we find that

$$\begin{aligned}\sum_{\boldsymbol{k}} \partial_{\boldsymbol{x}}^{\boldsymbol{k}} F_\alpha(\boldsymbol{v}) &= \mathrm{sf}^2(N-1)\,\mathrm{sf}^2(N-1) \\ &\quad \times \sum_{\pi_1^{[1]} \in \Pi_N} (-1)^{\sigma(\pi_1^{[1]})} \sum_{\pi_2^{[1]} \in \Pi_N} (-1)^{\sigma(\pi_2^{[1]})} \prod_{i=0}^{N-1} \binom{\pi_{1,i}^{[1]} + \pi_{2,i}^{[1]}}{\pi_{1,i}^{[1]}} \\ &\quad \times \sum_{\pi_1^{[2]} \in \Pi_N} (-1)^{\sigma(\pi_1^{[2]})} \sum_{\pi_2^{[2]} \in \Pi_N} (-1)^{\sigma(\pi_2^{[2]})} \prod_{i=0}^{N-1} \binom{\pi_{1,i}^{[2]} + \pi_{2,i}^{[2]}}{\pi_{1,i}^{[2]}}.\end{aligned}$$

Identifying a sum with a determinant and permuting rows,

$$\begin{aligned}\sum_{\pi_2 \in \Pi_s} (-1)^{\sigma(\pi_2)} \prod_{i=0}^{s-1} \binom{\pi_{1,i} + \pi_{2,i}}{\pi_{1,i}} &= \det(A^{[s]}_{\pi_{1,i},j})_{i,j=0,\ldots,s-1} \\ &= (-1)^{\sigma(\pi_1)} \det(A^{[s]}_{i,j})_{i,j=0,\ldots,s-1},\end{aligned}$$

where $A^{[s]}_{i,j} = \binom{i+j}{j}$, $i,j = 0,\ldots,s-1$.

It is easy to see that $\det A^{[s]} \equiv 1$. Indeed, it follows from the definition of $A^{[s]}_{i,j}$ that

$$A^{[s]}_{i,j} - A^{[s]}_{i,j-1} = A^{[s]}_{i-1,j}, \qquad i,j = 1,\ldots,s-1,$$

so subtracting the $(j-1)$st from the jth column we have

$$\det A^{[s]} = \det \begin{bmatrix} 1 & \mathbf{0}^\top \\ \mathbf{1} & A^{[s-1]} \end{bmatrix}.$$

Consequently, $\det A^{[s]} = \det A^{[s-1]}$, and by induction we have $\det A^{[s]} = 1$ for $s \geq 1$. Alternatively, this result follows from identifying $A^{[s]}$ as a classical Pascal matrix, see for instance (Edelman & Strang 2004).

Therefore,

$$\sum_{\pi_1^{[1]}\in\Pi_N}(-1)^{\sigma(\pi_1^{[1]})}\sum_{\pi_2^{[1]}\in\Pi_N}(-1)^{\sigma(\pi_2^{[1]})}\prod_{i=0}^{N-1}\begin{pmatrix}\pi_{1,i}^{[1]}+\pi_{2,i}^{[1]}\\ \pi_{1,i}^{[1]}\end{pmatrix}=N! \tag{4.11}$$

and the same holds for $\pi_1^{[2]}$ and $\pi_2^{[2]}$ instead of $\pi_1^{[1]}$ and $\pi_2^{[1]}$. Consequently,

$$\sum_{\boldsymbol{k}}\partial_{\boldsymbol{x}}^{\boldsymbol{k}}F_\alpha(\boldsymbol{v})=(N!)^2\mathrm{sf}(N-1)^4.$$

Assembling everything in formula (3.3), the term corresponding to $m=|\boldsymbol{k}|=2N(N-1)$ in the asymptotic expansion becomes

$$\frac{(-1)^N4^{N^2}}{(2N)!}\frac{1}{(-\mathrm{i}\omega)^{2N^2}}\begin{pmatrix}2N\\ N\end{pmatrix}(N!)^2\mathrm{sf}(N-1)^4=\frac{4^{N^2}}{\omega^{2N^2}}\mathrm{sf}(N-1)^4,$$

and therefore we arrive at the following result:

Proposition 2 *Let $n=2N$ be even, let $\boldsymbol{v}\in\mathcal{V}_{2N,N}$ be a vertex with weight 0 and let $f(\boldsymbol{x})$ be a symmetric function of its n arguments. If $f(\boldsymbol{v})\neq 0$, then*

$$\boldsymbol{I}_{2N}[f]=\frac{4^{N^2}}{\omega^{2N^2}}f(\boldsymbol{v})\mathrm{sf}(N-1)^4+\mathcal{O}\Big(\omega^{-2N^2-1}\Big).$$

4.2.3 The odd case $n=2N+1$

Similar considerations hold in the odd case. In this case, we may have $r=N$ and $n-r=N+1$ or viceversa, but these two cases are symmetric. They correspond to the sets $\mathcal{V}_{2N+1,N}$ and $\mathcal{V}_{2N+1,N+1}$ respectively.

For $\boldsymbol{v}\in\mathcal{V}_{2N+1,N}$, we have $(-1)^{s(\boldsymbol{v})}=(-1)^N$ and $\mathrm{e}^{\mathrm{i}\omega\boldsymbol{v}^T\boldsymbol{1}}=\mathrm{e}^{\mathrm{i}\omega}$. For $\boldsymbol{v}\in\mathcal{V}_{2N+1,N}$, we have $(-1)^{s(\boldsymbol{v})}=(-1)^{N+1}$ and $\mathrm{e}^{\mathrm{i}\omega\boldsymbol{v}^T\boldsymbol{1}}=\mathrm{e}^{-\mathrm{i}\omega}$. Furthermore, $\beta_{2N+1,N}(\boldsymbol{v})=4^{N(N+1)}$ and $m=|\boldsymbol{k}|=2N^2$ in both cases.

Proposition 3 *Let $n=2N+1$ be odd, let $\boldsymbol{v}_1\in\mathcal{V}_{2N+1,N}$ and $\boldsymbol{v}_2\in\mathcal{V}_{2N+1,N+1}$ be two vertices with weight 1 and let $f(\boldsymbol{x})$ be a symmetric function. If $f(\boldsymbol{v}_1)\neq 0$ and $f(\boldsymbol{v}_2)\neq 0$, then*

$$\begin{aligned}\boldsymbol{I}_{2N+1}[f]=&-\frac{\mathrm{i}(-1)^N}{\omega^{2N(N+1)+1}}4^{N(N+1)}\mathrm{sf}(N-1)^2\mathrm{sf}(N)^2[\mathrm{e}^{\mathrm{i}\omega}f(\boldsymbol{v}_1)-\mathrm{e}^{-\mathrm{i}\omega}f(\boldsymbol{v}_2)]\\&+\mathcal{O}\Big(\omega^{-2N(N+1)-2}\Big).\end{aligned}$$

If in addition f is an even function we have

$$\boldsymbol{I}_{2N+1}[f]=\frac{2(-1)^N}{\omega^{2N(N+1)+1}}4^{N(N+1)}\mathrm{sf}(N-1)^2\mathrm{sf}(N)^2f(\boldsymbol{v}_1)\sin\omega+\mathcal{O}\Big(\omega^{-2N(N+1)-2}\Big). \tag{4.12}$$

Using the symmetric integral in §4 with $f \equiv 1$, we have established in Proposition 2 that the Hankel determinant h_{2N-1} is strictly positive, for sufficiently large ω. Hence, the next even degree polynomial p_{2N} exists, because of (1.3). Proposition 3 shows that h_{2N} vanishes approximately at multiples of π. Hence, the odd degree polynomial p_{2N+1} does not exist for these values of ω. We examine this phenomenon in more detail in the next section.

5 The kissing pattern

The kissing pattern of polynomials of consecutive degrees, as illustrated in Fig. 2.1, results from degeneracy. The trajectories of the roots of an even-degree polynomial, say p_{2N}, and the next odd-degree polynomial p_{2N+1} tend to each other. These trajectories coincide at critical values of ω, the roots of h_{2N}. In that case, the lower-degree polynomial p_{2N} satisfies the orthogonality conditions of an orthogonal polynomial of degree $2N+1$.

5.1 Existence of the polynomials

We recall formula (1.3), that yields the monic nth degree orthogonal polynomial $p_n^\omega(x)$ in terms of determinants:

$$p_n(x) = \frac{1}{h_{n-1}} \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_n & x \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} & x^n \end{bmatrix}. \tag{5.1}$$

It is clear that $p_n(x)$ blows up exactly at the zeros of h_{n-1}, hence the importance of the analysis of the Hankel determinant.

Using a different normalization we can define a polynomial that always exists, regardless of the zeros of the Hankel determinant,

$$\tilde{p}_n(x) = h_{n-1} p_n(x). \tag{5.2}$$

This polynomial always exists but if $h_{n-1} = 0$ then it has degree less than n. From the theory of quasi-orthogonal polynomials, or formal orthogonal polynomials, it is known that the degree of $\tilde{p}_n$ equals the dimension of the largest leading non-singular principal submatrix of the Hankel matrix H_{n-1} (Brezinski 1980). This is equivalent to saying that the degree of $\tilde{p}_n$ is equal to the degree of the first existing polynomial p_k of lower degree $k \leq n$.

5.2 Degeneracy of $\tilde{p}_{2N+1}$

We study what happens as ω tends to a critical value $\hat{\omega}$. We assume that ω is sufficiently large, such that the even-degree polynomials exist but the odd-degree polynomial may not. Thus, we have $h_{2N}(\hat{\omega}) = 0$, for some $N > 0$, and from (4.12) we see that this is a simple root.

Recall the three-term recurrence relation (2.2) satisfied by the OPs:

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad n \in \mathbb{Z}_+, \tag{5.3}$$

where $p_{-1} = 0$ and $p_0 = 1$. Using expressions (2.3) we can rewrite this relation in terms of Hankel determinants,

$$\tilde{p}_{n+1}(x)h_{n-1}^2 = \left(h_n h_{n-1} x + \mathrm{i}\left(h_n' h_{n-1} - h_{n-1}' h_n\right)\right)\tilde{p}_n(x) - h_n^2 \tilde{p}_{n-1}(x). \tag{5.4}$$

For critical values of ω, the expression simplifies. Indeed, if $h_n = 0$ and $h_{n-1} \neq 0$, the relation becomes

$$\tilde{p}_{n+1}(x) = \mathrm{i}\frac{h_n'}{h_{n-1}}\tilde{p}_n(x), \tag{5.5}$$

i.e. $\tilde{p}_{n+1}$ is a scalar multiple of $\tilde{p}_n$. This means that, at zeros of h_{2N}, the polynomial $\tilde{p}_{2N+1}$ reduces to a constant multiple of the polynomial p_{2N} of lower degree. Hence, their zeros coincide and the trajectories of both polynomials 'kiss'. The monic polynomial p_{2N+1} blows up, as its $(2N+1)$th zero – the one on the imaginary axis – necessarily becomes infinite.

At this critical value $\hat{\omega}$, p_{2N} satisfies more orthogonality conditions than usual. The condition

$$\int_{-1}^{1} p_{2N}(x)p_{2N}(x)\mathrm{e}^{\mathrm{i}\hat{\omega}x}\,\mathrm{d}x = \frac{h_n}{h_{n-1}} = 0$$

implies that

$$\int_{-1}^{1} p_{2N}(x)x^k \mathrm{e}^{\mathrm{i}\hat{\omega}x}\,\mathrm{d}x = 0, \qquad k = 0, 1, \ldots, 2N.$$

Thus, the polynomial xp_{2N} still satisfies $2N$ orthogonality conditions. It is customary in the theory of quasi-orthogonal polynomials to in this case replace $\tilde{p}_{2N+1}$ by $x\tilde{p}_{2N}$, in order to obtain a complete basis for the space of polynomials.

Such completeness is not necessary for the sake of quadrature rules and we forego a more complete description. Instead, we focus on an aspect of the kissing pattern that is unique to the polynomials at hand: the kisses in Fig. 2.1 seemingly occur closer and closer to the endpoints ± 1 as ω increases. This behaviour is studied next.

6 Behaviour near the endpoints ± 1

The leading order term in the asymptotic expansion of the Hankel determinants h_{n-1} is very revealing, insofar as the existence of the polynomials is concerned. Yet, the expansion carries more information. It is conjectured and motivated in (Asheim et al. 2014) that the polynomials are approximately a multiple of Laguerre polynomials near the endpoints ± 1 – recall expression (2.5) – and this is the property we investigate next using Hankel determinants.

In other words, the observation is that for large ω the zeros of p_n behave like

$$\pm 1 - \frac{c}{\mathrm{i}\omega},$$

where c is a zero of the Laguerre polynomial $\mathrm{L}_{\lfloor n/2 \rfloor}$. In this section we intend to show that the leading order term in the expansion of $p_n\left(1-\frac{c}{\mathrm{i}\omega}\right)$ indeed vanishes if and only if $\mathrm{L}_{\lfloor n/2 \rfloor}(c)=0$.

Using Heine's formula (4.4) and the symmetric integral (4.3), we have

$$\tilde{p}_n\left(1-\frac{c}{\mathrm{i}\omega}\right)=\frac{1}{n!}I_n\left[\prod_{m=0}^{n-1}\left(1-\frac{c}{\mathrm{i}\omega}-x_m\right)\right].$$

Thus, we invoke the theory of §4 again and proceed by expanding $\boldsymbol{I}_n[f]$, where

$$f(x_0,\ldots,x_{n-1})=\prod_{m=0}^{n-1}\left(1-\frac{c}{\mathrm{i}\omega}-x_m\right)$$

is again a symmetric function, but now depending on ω.

6.1 The case $n=2N$

The analysis in section §4 needs minor modifications in view of the fact that the function f itself now depends on ω. The derivatives of $F=fg^2$ have an expansion in inverse powers of ω and we have to take this into account.

Recall that contributions to the asymptotic expansion (3.3) can be thought of as originating from vertices $\boldsymbol{v}\in\mathcal{V}_{n,r}$ with r (-1)s and $n-r$ $(+1)$s. Our first question is: which vertices contribute to the leading order term in the expansion of $I_n[f]$? Before, it was $\boldsymbol{v}\in\mathcal{V}_{2N,N}$. Here, in spite of the fact that f depends on ω, little changes. The leading order term still originates in vertices with minimal weight.

6.1.1 The leading order term

Consider a general vertex $\boldsymbol{v}\in\mathcal{V}_{2N,N+t}$, with $0\le t\le N$. Without loss of generality, and using the same multi-index notation as in the previous section, we can take $\boldsymbol{v}=(-\boldsymbol{1}^{[1]},\boldsymbol{1}^{[2]})$. Since f is linear in all its components, it is clear that $\partial_{\boldsymbol{x}}^{\boldsymbol{k}}f(\boldsymbol{v})=0$ unless $k_0,k_1,\ldots,k_{n-1}\in\{0,1\}$. Thus, suppose that $\boldsymbol{k}^{[1]}\in\mathbb{Z}_+^{N+t}$, $\boldsymbol{k}^{[2]}\in\mathbb{Z}_+^{N-t}$ such that $k_i^{[1]},k_i^{[2]}\in\{0,1\}$, $|\boldsymbol{k}^{[1]}|=\kappa_1$ and $|\boldsymbol{k}^{[2]}|=\kappa_2$. Then

$$\begin{aligned}\partial_{\boldsymbol{x}}^{(\boldsymbol{k}^{[1]},\boldsymbol{k}^{[2]})}f(\boldsymbol{v})&=(-1)^{\kappa_1+\kappa_2}\left(2-\frac{c}{\mathrm{i}\omega}\right)^{N+t-\kappa_1}\left(-\frac{c}{\mathrm{i}\omega}\right)^{N-t-\kappa_2}\\&=\frac{(-1)^{\kappa_1+\kappa_2}2^{N+t-\kappa_1}(-c)^{N-t-\kappa_2}}{(\mathrm{i}\omega)^{N-t-\kappa_2}}\left(1+\mathcal{O}(\omega^{-1})\right).\end{aligned}\tag{6.1}$$

Note the absence of symmetry here: the roles of κ_1 and κ_2 are not interchangeable because f focuses on the right endpoint $x=+1$.

There are three contributions to the leading order exponent of ω^{-1} in (3.3):

1. The dimension contributes $n=2N$.
2. The least order non-vanishing derivative of F_α, recall (4.8), consists of permutations of length $N+t$ and $N-t$ respectively. From (4.10), this contributes $|k|=2(N-1)N+2t^2$ to our exponent.

3. A derivative of degree $\kappa_1 + \kappa_2$ contributes $\mathcal{O}(\omega^{-\kappa_1-\kappa_2})$ to the ω^{-m-n} term in (3.3) and, from (6.1), additionally contributes $\mathcal{O}\big(\omega^{-N+t+\kappa_2}\big)$ – altogether $\mathcal{O}\big(\omega^{-N-\kappa_1+t}\big)$. We can choose $\kappa_1 = 0$ (i.e., all derivatives can be only with respect to the trailing $N-t$ components) and the contribution is $N-t$.

The total exponent is $2N^2 + N + 2t^2 - t$ and this, clearly, is minimised for $t = 0$. Though derivatives of f may contribute positive powers of ω, vertices with larger weight (i.e., larger t) contribute smaller powers of ω, and the latter effect is stronger. The resulting leading order behaviour, ω^{-2N^2-N}, is a factor ω^N smaller than that in Proposition 2 simply because that is the size of f at an endpoint with weight 0.

6.1.2 Analysis of the derivatives of $F = fg^2$

Let us examine the derivatives of $F = fg^2$ further. Corresponding to a vertex $\boldsymbol{v} \in \mathcal{V}_{2N,N}$, the leading order term must differentiate g^2 with permutations of length N. In addition, we may have derivatives of f with respect to the second set of variables. Denote by $\boldsymbol{p}_s \in \{0,1\}^N$ any vector such that $|\boldsymbol{p}_s| = s$, for $s \in \{0,\ldots,N\}$, then we compute the derivatives from the above formula:

$$\partial_{\boldsymbol{x}}^{(\boldsymbol{0},\boldsymbol{p}_s)} f(\boldsymbol{v}) = \frac{(-1)^s 2^N (-c)^{N-s}}{(\mathrm{i}\omega)^{N-s}} \left(1 + \mathcal{O}\big(\omega^{-1}\big)\right), \tag{6.2}$$

which corresponds to the choice $t = 0$, $\kappa_1 = 0$ and $\kappa_2 = s$ in (6.1). Derivatives of the former we have already analysed in §4. From (4.9), and remembering that $\beta^2_{n,N+t}(\boldsymbol{v}) = 2^{2(N+t)(N-t)}$, so $\beta^2_{n,N}(\boldsymbol{v}) = 2^{2N^2}$, we find that

$$\begin{aligned} \partial_{\boldsymbol{x}}^{(\boldsymbol{k}^{[1]},\boldsymbol{k}^{[2]})} g^2(\boldsymbol{v}) \quad = \quad & 2^{2N^2} \mathrm{sf}^4(N-1) \\ & \times \sum_{\pi_1+\pi_2=\boldsymbol{k}^{[1]}} (-1)^{\sigma(\pi_1)+\sigma(\pi_2)} \prod_{i=0}^{N-1} \begin{pmatrix} \pi_{1,i} + \pi_{2,i} \\ \pi_{1,i} \end{pmatrix} \\ & \times \sum_{\pi_3+\pi_4=\boldsymbol{k}^{[2]}} (-1)^{\sigma(\pi_3)+\sigma(\pi_4)} \prod_{i=0}^{N-1} \begin{pmatrix} \pi_{3,i} + \pi_{4,i} \\ \pi_{3,i} \end{pmatrix}. \end{aligned}$$

Note that the first sum in the previous formula is equal to $N!$, because of (4.11).

The increase of the order of the derivative of F in expansion (3.3) comes at a cost of a factor $(-\mathrm{i}\omega)^{-s}$. On the other hand, a higher order derivative of f yields a factor of $(\mathrm{i}\omega)^{-N+s}$, from (6.2). The product of these factors is $(-1)^s(\mathrm{i}\omega)^{-N}$ and has the same asymptotic size in ω for all s. Hence, we need to consider all s, $0 \le s \le N$.

Since derivatives of g^2 vanish unless the order of the derivative is a combination of permutations of length N, we need to consider all such combinations of permutations and all values of s. This leads to a sum of terms of the form

$$\partial_{\boldsymbol{x}}^{(\boldsymbol{k}^{[1]},\boldsymbol{k}^{[2]}+\boldsymbol{p}_s)} F(\boldsymbol{v}) = \prod_{p_{s,i}=1} (k_i^{[2]}+1) \partial_{\boldsymbol{x}}^{(\boldsymbol{k}^{[1]},\boldsymbol{k}^{[2]})} g^2(\boldsymbol{v}) \partial_{\boldsymbol{x}}^{(\boldsymbol{0},\boldsymbol{p}_s)} f(\boldsymbol{v}),$$

for $s \in \{0,1,\ldots,N\}$ and with $\boldsymbol{k}^{[1]}$ and $\boldsymbol{k}^{[2]}$ sums of two permutations in Π_N.

The sth term in this large sum is

$$\sum_{|\boldsymbol{p}_s|=s}\sum_{\boldsymbol{\pi}_1,\boldsymbol{\pi}_2,\boldsymbol{\pi}_3\boldsymbol{\pi}_4\in\Pi_N}\prod_{p_{s,i}=1}(k_i^{[2]}+1)\partial_{\boldsymbol{x}}^{(\boldsymbol{k}^{[1]},\boldsymbol{k}^{[2]})}g^2(\boldsymbol{v})\partial_{\boldsymbol{x}}^{(\boldsymbol{0},\boldsymbol{p}_s)}f(\boldsymbol{v})$$

$$= 2^{2N^2+N}\mathrm{sf}^4(N-1)(-1)^s(\mathrm{i}\omega)^{s-N}(-c)^{N-s}\sum_{|\boldsymbol{p}_s|=s}\sum_{\boldsymbol{\pi}_1,\boldsymbol{\pi}_2,\boldsymbol{\pi}_3\boldsymbol{\pi}_4\in\Pi_N}\prod_{p_{s,i}=1}(\pi_{3,i}+\pi_{4,i}+1)$$

$$\times(-1)^{\sigma(\pi_1)+\sigma(\pi_2)}\prod_{i=0}^{N-1}\binom{\pi_{1,i}+\pi_{2,i}}{\pi_{1,i}}(-1)^{\sigma(\pi_3)+\sigma(\pi_4)}\prod_{i=0}^{N-1}\binom{\pi_{3,i}+\pi_{4,i}}{\pi_{3,i}}$$

$$= 2^{2N^2+N}\mathrm{sf}^4(N-1)(-1)^s(\mathrm{i}\omega)^{s-N}(-c)^{N-s}N!$$

$$\times\sum_{|\boldsymbol{p}_s|=s}\sum_{\boldsymbol{\pi}_3,\boldsymbol{\pi}_4\in\Pi_N}(-1)^{\sigma(\pi_3)+\sigma(\pi_4)}\prod_{i=0}^{N-1}\frac{(\pi_{3,i}+\pi_{4,i}+p_{s,i})!}{\pi_{3,i}!\pi_{4,i}!}.$$

In the last computation, we have used (4.11) and the fact that

$$\sum_{\boldsymbol{\pi}_3,\boldsymbol{\pi}_4\in\Pi_N}\prod_{p_{s,i}=1}(\pi_{3,i}+\pi_{4,i}+1)(-1)^{\sigma(\pi_3)+\sigma(\pi_4)}\prod_{i=0}^{N-1}\binom{\pi_{3,i}+\pi_{4,i}}{\pi_{3,i}}$$

$$=\sum_{\boldsymbol{\pi}_3,\boldsymbol{\pi}_4\in\Pi_N}(-1)^{\sigma(\pi_3)+\sigma(\pi_4)}\prod_{i=0}^{N-1}\frac{(\pi_{3,i}+\pi_{4,i}+p_{s,i})!}{\pi_{3,i}!\pi_{4,i}!}.$$

In this last sum, every $\boldsymbol{p}_s$ consists of s ones and $N-s$ zeros, hence there are $\binom{N}{s}$ such vectors. Note that each gives exactly the same result, because everything else in the relevant expression is constructed from two permutations. Therefore, we might just consider

$$\boldsymbol{p}_s^\star=(\overbrace{0,\ldots,0}^{N-s\text{ times}},\overbrace{1,\ldots,1}^{s\text{ times}})$$

$\binom{N}{s}$ times. Therefore

$$\sum_{|\boldsymbol{p}_s|=s}\sum_{\boldsymbol{\pi}_3,\boldsymbol{\pi}_4\in\Pi_N}(-1)^{\sigma(\boldsymbol{\pi}_3)+\sigma(\boldsymbol{\pi}_4)}\prod_{i=0}^{N-1}\frac{(\pi_{3,i}+\pi_{4,i}+p_{s,i})!}{\pi_{3,i}!\pi_{4,i}!}$$

$$=\binom{N}{s}\sum_{\boldsymbol{\pi}_3,\boldsymbol{\pi}_4\in\Pi_N}(-1)^{\sigma(\boldsymbol{\pi}_3)+\sigma(\boldsymbol{\pi}_4)}\prod_{i=0}^{N-1}\frac{(\pi_{3,i}+\pi_{4,i}+p^\star_{s,i})!}{\pi_{3,i}!\pi_{4,i}!}$$

$$=\binom{N}{s}\sum_{\boldsymbol{\pi}\in\Pi_N}(-1)^{\sigma(\boldsymbol{\pi})}\det\mathcal{E}^{[N,s]}(\boldsymbol{\pi}),$$

where

$$\mathcal{E}_{i,j}^{[N,s]}(\boldsymbol{\pi})=\begin{cases}\dfrac{(\pi_i+j)!}{\pi_i!j!}, & i=0,\ldots,N-s-1,\\[2ex] \dfrac{(\pi_i+j+1)!}{\pi_i!j!}, & i=N-s,\ldots,N-1,\end{cases}\qquad j=0,\ldots,N-1. \tag{6.3}$$

Proposition 4 *Let* $\boldsymbol{\pi} = \boldsymbol{\pi}^{[1]} + \boldsymbol{\pi}^{[2]}$*, with*

$$\boldsymbol{\pi}^{[1]} = (\pi_0, \ldots, \pi_{N-s-1}), \qquad \boldsymbol{\pi}^{[2]} = (\pi_{N-s}, \ldots, \pi_{N-1}). \tag{6.4}$$

If $\boldsymbol{\pi}^{[1]} \in \Pi_{N-s}$ *and* $\boldsymbol{\pi}^{[2]} \in N - s + \Pi_s$*, then*

$$(-1)^{\sigma(\boldsymbol{\pi})} \det \mathcal{E}^{[N,s]}(\boldsymbol{\pi}) = \frac{N!^2}{s!(N-s)!^2}, \tag{6.5}$$

otherwise $\det \mathcal{E}^{[N,s]}(\boldsymbol{\pi}) = 0$.

Proof Suppose first that the hypothesis on $\boldsymbol{\pi}^{[1]}$ and $\boldsymbol{\pi}^{[2]}$ holds. We deduce, rearranging rows separately in the first $N - s$ and the last s rows of $\mathcal{E}^{[N,s]}(\boldsymbol{\pi})$, that $(-1)^{\sigma(\boldsymbol{\pi})} \det \mathcal{E}^{[N,s]}(\boldsymbol{\pi}) = \det \mathcal{E}^{[N,s]}(0, 1, \ldots, N-1)$. Since

$$\mathcal{E}^{[N,s]}_{i,j}(0, 1, \ldots, N-1) = \begin{cases} \binom{i+j}{i}, & i = 0, \ldots, N-s-1, \\ (i+1)\binom{i+j+1}{i+1}, & i = N-s, \ldots, N-1, \end{cases}$$

and $j = 0, \ldots, N-1$, once we extract a factor of $i+1$ from rows $i = N-s, \ldots, N-1$, the outcome is

$$\mathcal{E}^{[N,s]}(0, 1, \ldots, N-1) = \frac{N!}{(N-s)!} \mathcal{C}^{[N,s]}(0, 1, \ldots, N-1),$$

where

$$\mathcal{C}^{[N,s]}_{i,j}(0, 1, \ldots, N-1) = \begin{cases} \binom{i+j}{i}, & i = 0, \ldots, N-s-1, \\ \binom{i+j+1}{i+1}, & i = N-s, \ldots, N-1, \end{cases} \qquad j = 0, \ldots, N-1.$$

The matrix $\mathcal{C}$ is somewhat easier to manipulate. First, note that $\mathcal{C}^{[N,0]}$ equals the Pascal matrix $A^{[N]}$ used in the analysis in §4.2.2, hence $\det \mathcal{C}^{[N,0]} = 1$.

In case $s = 1$, easy calculation with binomial numbers confirms that

$$\mathcal{C}^{[N,1]}_{i,j} - \mathcal{C}^{[N,1]}_{i,j-1} = \mathcal{C}^{[N,1]}_{i-1,j}, \qquad j = 0, \ldots, N-2$$

and

$$\mathcal{C}^{[N,1]}_{i,N-1} - \mathcal{C}^{[N]}_{i,N-2} = \binom{N+i-1}{i-1} + \binom{N+i-2}{i-1}.$$

Therefore

$$\det \mathcal{C}^{[N,1]} = \det \begin{bmatrix} 1 & \mathbf{0}^\top \\ \mathbf{1} & \mathcal{C}^{[N-1,1]} + \mathcal{C}^{[N-1,0]} \end{bmatrix},$$

where both matrices in the lower right block differ in only one column. This leads to

$$\det \mathcal{C}^{[N,1]} = \det \mathcal{C}^{[N-1,1]} + \det \mathcal{C}^{[N-1,0]} = \det \mathcal{C}^{[N-1]} + 1$$

and we deduce that $\det \mathcal{C}^{[N,1]} = N$.

Let us generalize the above to larger s. Note that

$$\mathcal{C}^{[N,s]}_{i,j} - \mathcal{C}^{[N,s]}_{i,j-1} = \begin{cases} \mathcal{C}^{[N-1,s]}_{i-1,j} = \mathcal{C}^{[N,s-1]}_{i-1,j}, & j = 0,\ldots,N-s-1, \\ \mathcal{C}^{[N-1,s]}_{i-1,N-s} + \mathcal{C}^{[N-1,s-1]}_{i-1,N-s}, & j = N-s, \\ \mathcal{C}^{[N-1,s]}_{i-1,j} = \mathcal{C}^{[N,s-1]}_{i-1,j}, & j = N-s+1,\ldots,N-1, \end{cases}$$

An argument identical to the one we have used before shows that

$$\det \mathcal{C}^{[N,s]} = \det \mathcal{C}^{[N-1,s]} + \det \mathcal{C}^{[N-1,s-1]}, \qquad s = 0,1,\ldots,N.$$

In case $N = s$, the same reasoning leads to $\det \mathcal{C}^{[N,N]} = \det \mathcal{C}^{[N,0]} = 1$. Induction then shows that $\det \mathcal{C}^{[N,s]} = \binom{N}{s}$, and (6.5) must be true.

Suppose now that $\boldsymbol{\pi}$ is not isomorphic to $\Pi_{N-s} \oplus (N-s+\Pi_s)$, which makes sense only when $s \in \{1,\ldots,N-1\}$. Then there exists (at least) one integer $r \geq N-s$ such that $\pi_r \in \boldsymbol{\pi}^{[2]}$ and $\pi_r = N-s-i$ for some $i \geq 1$. Among those, we choose r so that π_r is minimum, and we take t to be the index such that $\pi_t = \pi_r + 1$. Then either $\pi_t \in \boldsymbol{\pi}^{[1]}$ or $\pi_t \in \boldsymbol{\pi}^{[2]}$. In the first case, using (6.3), the r-th row of $\mathcal{E}^{[N,s]}(\boldsymbol{\pi})$ is

$$\frac{(N-s-i+j+1)!}{(N-s-i)!j!}, \qquad j = 0,\ldots,N-1,$$

and the t-th row of is

$$\frac{(N-s-i+j+1)!}{(N-s-i+1)!j!}, \qquad j = 0,\ldots,N-1.$$

Since the t-th row is a scalar multiple of the r-th row, the determinant vanishes and the proposition is true. If $\pi_r \in \boldsymbol{\pi}^{[2]}$, then we repeat the previous reasoning with the index t and the index u such that $\pi_u = \pi_t + 1$. We continue this process and at some point we must find two indices with the property above, since it cannot happen that all permutations in $\boldsymbol{\pi}^{[1]}$ take smaller values than those in $\boldsymbol{\pi}^{[2]}$. □

6.1.3 Asymptotic behaviour of p_{2N}

Let us assemble everything. Since there are precisely $s!(N-s)!$ permutations isomorphic to $\Pi_{N-s} \oplus (N-s+\Pi_s)$ (hence, which produce nonzero determinants in the proposition), we obtain

$$\sum_{|\boldsymbol{p}_s|=s} \sum_{\boldsymbol{\pi}_3,\boldsymbol{\pi}_4 \in \Pi_N} (-1)^{\sigma(\boldsymbol{\pi}_3)+\sigma(\boldsymbol{\pi}_4)} \prod_{i=0}^{N-1} \frac{(\pi_{3,i}+\pi_{4,i}+p_{s,i})!}{\pi_{3,i}!\pi_{4,i}!} = \frac{N!^3}{s!(N-s)!^2}$$

for $s = 0,\ldots,N$.

The contribution of $\boldsymbol{v}$ includes the derivatives derived above, as well as an additional factor $(-\mathrm{i}\omega)^{-s}$ for each s arising from the $\mathcal{O}(\omega^{-m-n})$ term in (3.3), and the factor $(-1)^{\sigma(v)} = (-1)^N$. This totals, after some manipulations,

$$\sum_{s=0}^{N}(-1)^N(-\mathrm{i}\omega)^{-s}2^{2N^2+N}\mathrm{sf}(N-1)^4(-1)^s(\mathrm{i}\omega)^{s-N}(-c)^{N-s}\frac{N!^4}{s!(N-s)!^2}$$
$$= (-\mathrm{i}\omega)^{-N}2^{2N^2+N}\mathrm{sf}(N-1)^4N!^3\sum_{s=0}^{N}\frac{1}{s!}\binom{N}{s}(-c)^s = (-\mathrm{i}\omega)^{-N}2^{2N^2+N}\frac{\mathrm{sf}(N)^4}{N!}\mathrm{L}_N(c),$$

where the latter simplification follows from the known explicit expression

$$\mathrm{L}_N(c) = \sum_{s=0}^{N}\frac{1}{s!}\binom{N}{s}(-c)^s. \tag{6.6}$$

See for instance (Chihara 1978, Chapter V) or (Szegő 1939, Chapter 5).

There are $\binom{2N}{N}$ vertices in $\mathcal{V}_{2N,N}$, hence the leading term in the expansion of $\tilde{p}_{2N}(1-c/(\mathrm{i}\omega))$ is

$$\frac{2^{2N^2+N}\mathrm{sf}(N)^4}{(-\mathrm{i}\omega)^{2N^2}(-\mathrm{i}\omega)^N N!^3}\mathrm{L}_N(c) = (-\mathrm{i})^N\frac{2^{2N^2+N}\mathrm{sf}(N)^4}{\omega^{2N^2+N}N!^3}\mathrm{L}_N(c).$$

Finally, we can divide by the leading term of h_{2N-1}, recall Proposition 2,

$$\frac{4^{N^2}\mathrm{sf}(N-1)^4}{\omega^{2N^2}},$$

to obtain the leading term of the monic polynomial:

$$p_{2N}\left(1-\frac{c}{\mathrm{i}\omega}\right) = \frac{(-2\mathrm{i})^N N!}{\omega^N}\mathrm{L}_N(c)\left(1+\mathcal{O}(\omega^{-1})\right), \qquad \omega\to\infty.$$

This is precisely the result we wanted to show.

6.2 The case $n = 2N+1$

Now, we consider a general vertex $\boldsymbol{v}\in\mathcal{V}_{2N+1,N+t}$. As before, $\partial_{\boldsymbol{x}}^{\boldsymbol{k}}f(\boldsymbol{v}) = 0$ unless the multi–indices are such that $\boldsymbol{k}^{[1]}\in\{0,1\}^{N+t}$, $\boldsymbol{k}^{[2]}\in\{0,1\}^{N+1-t}$, with $|\boldsymbol{k}^{[1]}| = \kappa_1$ and $|\boldsymbol{k}^{[2]}| = \kappa_2$. In this case, we have

$$\partial_{\boldsymbol{x}}^{(\boldsymbol{k}^{[1]},\boldsymbol{k}^{[2]})}f(\boldsymbol{v}) = \frac{(-1)^{\kappa_1+\kappa_2}2^{N+t-\kappa_1}(-c)^{N+1-t-\kappa_2}}{(\mathrm{i}\omega)^{N+1-t-\kappa_2}}\left(1+\mathcal{O}(\omega^{-1})\right). \tag{6.7}$$

The contributions to the leading term are as follows:

1. The dimension contributes $2N+1$.

2. The least order non-vanishing derivative of F_α, that consists of permutations of length $N+t$ and $N+1-t$. This amounts to $|\boldsymbol{k}| = 2N^2+2t(t-1)$.

3. By similar reason as for $n = 2N$, from (6.7), the leading power is $\kappa_1 + \kappa_2$ (the degree of the derivative) plus $N+1-t-\kappa_2$ (the above contribution) – altogether $N + 1 - t + \kappa_1$. Since we wish to minimise this, we need to take $\kappa_1 = 0$ and the contribution is $N - t + 1$.

The total exponent is therefore $2N^2 + 3N + 2 + t(2t - 3)$, which is minimised for $t \in \{-N, \ldots, N+1\}$ in the case $t = 1$. The exponent is then equal to $(N+1)(2N+1)$. Therefore, we need consider just vertices in $\mathcal{V}_{2N+1,N+1}$.

As before, derivatives of f in the term $F = fg^2$ may have order $s \geq 0$. It is important to observe that the range of s is unchanged, i.e. $0 \leq s \leq N$, since it is constrained by the number of $(+1)$s in the vertex $v \in \mathcal{V}_{2N+1,N+1}$ which is N as before. Going through similar computations, the identity (6.6) quickly resurfaces in the leading order term.

This leads to the following result.

Theorem 5 *For $\omega \gg 1$ the zeros of $p^{\omega}_{2N}(\,\cdot\,)$ are $\pm 1 + \mathrm{i}c^{[N]}_k/\omega + \mathcal{O}(\omega^{-2})$, where $c^{[N]}_1, \ldots, c^{[N]}_N > 0$ are zeros of the Nth Laguerre polynomial. The same is true for p^{ω}_{2N+1}, except for a single zero on the pure imaginary axis.*

Proof We have shown the asymptotic agreement with the roots of the Laguerre polynomial. The single zero on the pure imaginary axis follows at once from the symmetry of p_n with respect to the pure imaginary axis. □

7 Roots of h_n in the complex plane

In this section we investigate in more detail the zeros of the Hankel determinants in the complex plane. First of all, we recall that $h_n(\omega)$ is real for $\omega \in \mathbb{R}$, so all complex zeros must come in conjugate pairs. Also, since $h_n(-\omega) = h_n(\omega)$, we can restrict ourselves to the first quadrant of the complex plane.

Figures 7.1 and 7.2 show these zeros for different values of n. They follow very regular and symmetric patterns reminiscent of onion peels, that we intend to explain in this section, at least for large values of ω. These patterns result from a delicate balancing act, in which algebraic powers of ω^{-1} become comparable in size to decaying complex exponentials of the form $\mathrm{e}^{\mathrm{i}\omega c z}$, $c > 0$, in the upper half of the complex plane. We revisit the asymptotic analysis of §4, this time taking complex exponential factors into into account.

As before, we need to distinguish two cases, corresponding to even and odd values of n.

7.1 The odd case: roots of h_{2N-1}

We commence from $h_{2N-1} = \boldsymbol{I}_{2N}[1]$. The contribution to h_{2N-1} in the asymptotic expansion (3.3) corresponds to the layers $\mathcal{V}_{2N,N\pm t}$, $t = 0, \ldots, N$, with the previous notation. We quantify the contribution of each layer with two numbers:

a) a complex exponential factor: each vertex $\boldsymbol{v} \in \mathcal{V}_{2N,N\pm t}$ contributes $\mathrm{e}^{\mathrm{i}\omega \boldsymbol{v}^\top \mathbf{1}} = \mathrm{e}^{\mp 2\mathrm{i}\omega t}$;

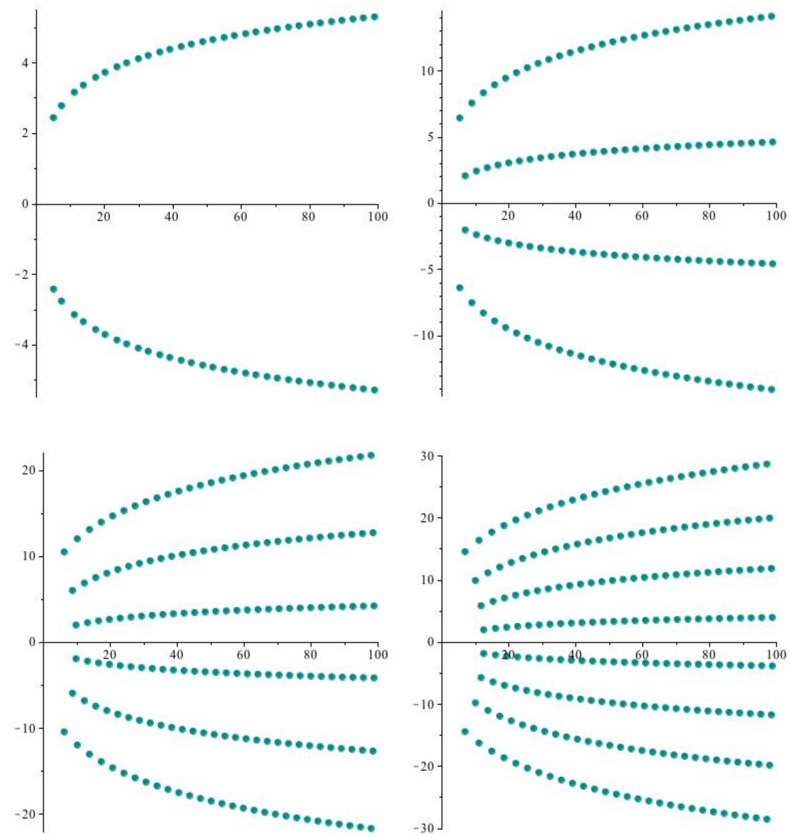

Figure 7.1: The zeros of h_1, h_3, h_5 and h_7 in the complex plane.

b) the *leading* power of ω 'originating' in vertices in the layer. As computed before, recall (4.10), that contributes a total power $2N^2 - 2N + 2t^2$. Therefore, in the end we obtain $\omega^{-2N-2(N^2-N+t^2)} = \omega^{-2N^2-2t^2}$.

In other words, each layer $\mathcal{V}_{2N,N\pm t}$ contributes

$$P_{2N,\pm t}(\omega) := \frac{\mathrm{e}^{\mp 2\mathrm{i}\omega t}}{\omega^{2N^2+2t^2}} \left[c_{2N,t} + \mathcal{O}\big(\omega^{-1}\big)\right]. \tag{7.1}$$

We now need to choose a plus or a minus sign: $\mathrm{e}^{\mathrm{i}\omega}$ is large for $\operatorname{Im}\omega < 0$ and $\mathrm{e}^{-\mathrm{i}\omega}$ for $\operatorname{Im}\omega > 0$, so we choose the latter, since our goal is to balance it with a positive power of ω. We thus assume that $\mathrm{e}^{-\mathrm{i}\omega} = \mathcal{O}(\omega^p)$ for some $p > 0$. The immediate implication is that

$$P_{2N,t}(\omega) = \mathcal{O}\Big(\omega^{-2N^2-2t^2+2tp}\Big).$$

The idea is now to choose $t_1 < t_2$ in $\{0, 1, \dots, N\}$ such that P_{2N,t_1} and P_{2N,t_2} are of the same order of magnitude *and* the remaining $P_{2N,t}$ are of smaller order of magnitude.

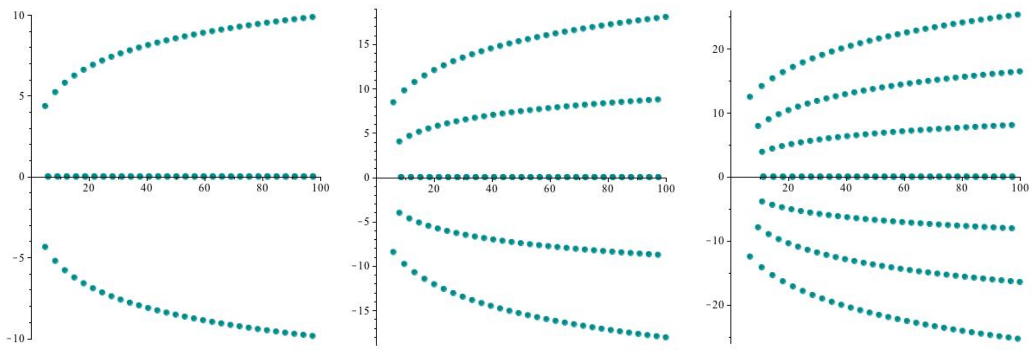

Figure 7.2: The zeros of $h_2(\omega), h_4(\omega)$ and $h_6(\omega)$ in the complex plane.

The first requirement means that

$$-2N^2 - 2t_1^2 + 2t_1 p = -2N^2 - 2t_2^2 + 2t_2 p,$$

therefore $p = t_1 + t_2$, and then

$$P_{2N,t_1}, P_{2N,t_2} = \mathcal{O}\Big(\omega^{-2N^2+2t_1t_2}\Big).$$

The second requirement is that

$$-2N^2 - 2t^2 + 2t(t_1+t_2) < -2N^2 + 2t_1t_2,$$

therefore $(t-t_1)(t-t_2) > 0$ for all $t \neq t_1, t_2$.

Let $t_1 = k$, $t_2 = k+1$, where $k = 0, 1, \ldots, N-1$. Then $p = 2k+1$,

$$P_{2N,k}, P_{2N,k+1} = \mathcal{O}\Big(\omega^{-2N^2+2k^2+2k}\Big),$$
$$P_{2N,t} = \mathcal{O}\Big(\omega^{-2N^2-2t^2+4tk+2t}\Big), \qquad t \neq k, k+1.$$

It is trivial to verify that indeed $(t-k)(t-k-1) > 0$ for all $t \neq k, k+1$, so all is well. Moreover, these are all possible such choices. For suppose that there exist t_1, t_2 such that $t_1 < t_2 - 1$. Then $(t-t_1)(t-t_2) < 0$ for $t = t_1 + 1$ and we reach a contradiction. Thus, we have exactly the right number of N different choices, which correspond exactly to a separate 'onion peel' in the lower-right quadrant in the Figure 7.1.

Thus, we choose $k \in \{0, 1, \ldots, N-1\}$ and extract just the kth and $(k+1)$st terms,

$$\begin{aligned} h_{2N-1,k}(\omega) &= c_{2N,k}\frac{\mathrm{e}^{-2\mathrm{i}\omega k}}{\omega^{2N^2+2k^2}} + c_{2N,k+1}\frac{\mathrm{e}^{-2\mathrm{i}\omega(k+1)}}{\omega^{2N^2+2(k+1)^2}} \\ &= \frac{\mathrm{e}^{-2\mathrm{i}\omega k}}{\omega^{2N^2+2k^2}}\left(c_{2N,k} + c_{2N,k+1}\frac{\mathrm{e}^{-2\mathrm{i}\omega}}{\omega^{2(2k+1)}}\right). \end{aligned}$$

Setting $h_{2N-1,k}(\omega)$ to zero, we obtain an asymptotic expression for the kth 'onion peel'. This leads to the equation

$$\omega^{2(2k+1)} e^{2i\omega} \approx -\frac{c_{2N,k+1}}{c_{2N,k}}.$$

We compute the coefficients $c_{N,k}$ later on, in order to arrive at explicit expressions and rule out the case $c_{2N,k} = 0$.

Taking roots,

$$\omega e^{i\omega/(2k+1)} \approx \left(-\frac{c_{2N,k+1}}{c_{2N,k}}\right)^{1/(4k+2)} e^{\pi i\ell/(2k+1)}, \qquad \ell = 0, 1, \ldots, 4k+1.$$

Therefore

$$\frac{i\omega}{2k+1} e^{i\omega/(2k+1)} \approx \frac{i}{2k+1} \left(-\frac{c_{2N,k+1}}{c_{2N,k}}\right)^{1/(4k+2)} e^{\pi i\ell/(2k+1)}, \qquad \ell = 0, 1, \ldots, 4k+1$$

and we deduce that

$$\omega \approx -(2k+1) i W\left(\frac{i}{2k+1} \left(-\frac{c_{2N,k+1}}{c_{2N,k}}\right)^{1/(4k+2)} e^{\pi i\ell/(2k+1)}\right), \qquad \ell = 0, 1, \ldots, 4k+1, \tag{7.2}$$

in terms of the Lambert W function, see for instance (Olver, Lozier, Boisvert & Clark 2010, §4.13).

7.2 The even case: roots of h_{2N}

We revisit the work of the last subsection, and now we have the layers $\mathcal{V}_{2N+1,N-t}$ and $\mathcal{V}_{2N+1,N+1+t}$, $t = 0, \ldots, N$. The exponential factor is $e^{(2t+1)i\omega}$ in the first case, and $e^{-(2t+1)i\omega}$ in the second one.

For reasons of symmetry, it is enough to look at one of these cases. Since we are interested in the upper-right quadrant, we choose the second case. Computing as before, the leading power of ω is $2N^2 + 2t^2 + 2t$. Adding the dimension $n = 2N + 1$, we deduce that the contribution of $\mathcal{V}_{2N+1,N+1+t}$ is

$$P_{2N+1,t}(\omega) := \frac{e^{-(2t+1)i\omega}}{\omega^{2N^2+2N+1+2t^2+2t}} \left[c_{2N+1,t} + \mathcal{O}\big(\omega^{-1}\big)\right], \qquad r = 0, \ldots, N-1. \tag{7.3}$$

Assuming again that $e^{-i\omega} = \mathcal{O}(\omega^p)$ for some $p > 0$, we have

$$P_{2N+1,N+1+t} = \mathcal{O}\Big(\omega^{-2N^2-2N-1-2t^2-2t+2pt+p}\Big).$$

We again choose $t_1 < t_2$ according to the two rules. This leads to $p = t_1 + t_2 + 1$, and

$$P_{2N+1,N+1+t_1}, P_{2N+1,N+1+t_2} = \mathcal{O}\Big(\omega^{-2N^2-2N+t_1+t_2+2t_1t_2}\Big),$$
$$P_{2N+1,N+1+t} = \mathcal{O}\Big(e^{-i\omega}\omega^{-2N^2-2N+t_1+t_2-2t^2+2t(t_1+t_2)}\Big).$$

In particular, the second requirement reduces again to $(t-t_1)(t-t_2)>0$ for all $t\neq t_1,t_2$. It follows at once that $t_1=k$, $t_2=k+1$ for some $k\in\{0,1,\ldots,N-1\}$, otherwise the inequality fails for t_1+1.

So, all that remains is to check if the above choice of consecutive t_1,t_2 works and indeed, trivially, it does: either $t<t_1,t_2$ or $t>t_1,t_2$ and in each case the inequality works. In other words, $p=2k+2$ and we investigate the zeros of

$$\begin{aligned} h_{2N+1,k}(\omega) &= c_{2N+1,k}\frac{\mathrm{e}^{-(2k+1)\mathrm{i}\omega}}{\omega^{2N^2+2N+1+2k^2+2k}}+c_{2N+1,k+1}\frac{\mathrm{e}^{-(2k+3)\mathrm{i}\omega}}{\omega^{2N^2+2N+1+2k^2+6k+4}} \\ &= \frac{\mathrm{e}^{-(2k+1)\mathrm{i}\omega}}{\omega^{2N^2+2N+1+2k^2+2k}}\left(c_{2N+1,k}+c_{2N+1,k+1}\frac{\mathrm{e}^{-2\mathrm{i}\omega}}{\omega^{4k+4}}\right). \end{aligned}$$

Therefore

$$\begin{aligned} &\omega^{4k+4}\mathrm{e}^{2\mathrm{i}\omega}\approx-\frac{c_{2N+1,k+1}}{c_{2N+1,k}} \\ \Longrightarrow\qquad &\omega\mathrm{e}^{\mathrm{i}\omega/(2k+2)}\approx\left(-\frac{c_{2N+1,k+1}}{c_{2N+1,k}}\right)^{1/(4k+4)}\mathrm{e}^{\pi\mathrm{i}\ell/(2k+2)},\quad \ell=0,\ldots,4k+3 \end{aligned}$$

and we deduce that

$$\omega\approx-2\mathrm{i}(k+1)\mathrm{W}\left(\frac{\mathrm{i}}{2(k+1)}\left(-\frac{c_{2N+1,k+1}}{c_{2N+1,k}}\right)^{1/(4k+4)}\mathrm{e}^{\pi\mathrm{i}\ell/(2k+2)}\right),\quad \ell=0,\ldots,4k+3, \tag{7.4}$$

again in terms of solutions of the Lambert W function. All this explains, at least in an asymptotic sense, the 'onion peel' structure of zeros of h_n.

7.3 The coefficients $c_{n,k}$

The coefficients $c_{n,k}$ are defined in (7.1) and (7.3) as the contribution of the layers of vertices $\mathcal{V}_{2N,N+k}$ or $\mathcal{V}_{2N+1,N+1+k}$.

Elaborate calculations, similar to the ones in §4, lead to the following results. For odd $n=2N-1$, it is the case that

$$c_{2N,k}=4^{N^2-k^2}[\mathrm{sf}(N-k-1)\mathrm{sf}(N+k-1)]^2,$$

and consequently, see (7.2),

$$\left(-\frac{c_{2N,k+1}}{c_{2N,k}}\right)^{1/(4k+2)}=\frac{\mathrm{e}^{\pi\mathrm{i}/(4k+2)}}{2}\left[\frac{(N+k)!}{(N-k-1)!}\right]^{1/(2k+1)},\qquad k=0,\ldots,N-1.$$

In the even case, we have

$$c_{2N+1,k}=\mathrm{i}(-1)^{N+k}4^{(N-k)(N+k+1)}[\mathrm{sf}(N+k)\mathrm{sf}(N-k-1)]^2,$$

consequently, see (7.4),

$$\left(-\frac{c_{2N+1,k+1}}{c_{2N+1,k}}\right)^{1/(4k+4)}=\frac{\mathrm{e}^{\pi\mathrm{i}/(4k+4)}}{2}\left[\frac{(N+k+1)!}{(N-k-1)!}\right]^{1/(2k+2)}.$$

8 Existence for all ω

Finally, we take a different approach to show existence of the even-degree polynomials for all values of ω. We are assured of existence for small ω, since the $\omega = 0$ limit yields Legendre polynomials, and for large ω, from the preceding asymptotic analysis. We fill in the gap in between using elementary arguments, which are not asymptotic in nature.

We recall from (Asheim et al. 2014) that

$$\frac{\partial p_n^\omega(x)}{\partial \omega} = -\mathrm{i}\beta_n p_{n-1}^\omega(x),$$

where β_n is a coefficient from the three term recurrence relation (2.2). This expression holds for all values of ω for which p_n and p_{n-1} exist. Upon substitution of $p_n = \tilde{p}_n/h_{n-1}$, and using (2.3), we find that

$$\frac{\partial \tilde{p}_n^\omega(x)}{\partial \omega} h_{n-1} = h'_{n-1}\tilde{p}_n^\omega(x) - \mathrm{i}h_n\tilde{p}_{n-1}^\omega(x). \tag{8.5}$$

By continuity, this expression holds for all ω. Differentiating with respect to ω again leads to

$$\begin{aligned}\frac{\partial^2 \tilde{p}_n^\omega(x)}{\partial \omega^2} h_{n-1} + \frac{\partial \tilde{p}_n^\omega(x)}{\partial \omega} h'_{n-1} = h''_{n-1}\tilde{p}_n^\omega(x) + h'_{n-1}\frac{\partial \tilde{p}_n^\omega(x)}{\partial \omega} \\ - \mathrm{i}h'_n\tilde{p}_{n-1}^\omega(x) - \mathrm{i}h_n\frac{\partial \tilde{p}_{n-1}^\omega(x)}{\partial \omega}\end{aligned} \tag{8.6}$$

Recall also the three-term recurrence relation, written in terms of Hankel determinants in (5.4). Another important identity, although known (cf. for instance (Bleher & Its 2005, Section 2)) is presented for the sake of completeness in the next lemma.

Lemma 6 *It is true that*

$$h''_n h_n - (h'_n)^2 = -h_{n-1}h_{n+1}, \qquad n \geq 1. \tag{8.7}$$

Proof This identity is closely related to the Toda differential equation, and it is of great importance in integrable systems and random matrix theory, see for instance (Bleher & Its 2005, Section 2), (Bleher 2011, Proposition 18.1) or (Bleher & Liechty 2013, Theorem 1.4.2). For completeness, we present here a brief proof in the current setting, which uses the connection between the Hankel determinant and the partition function of a certain random matrix ensemble.

We revisit the Heine formula for h_{n-1}, see (3.1), and we note that using elementary operations, we can replace the row $(x_0^i, x_1^i, \ldots, x_{n-1}^i)$ for $i = 1, 2, \ldots, n-1$ in the Vandermonde determinants by $(p_i(x_0), p_i(x_1), \ldots, p_i(x_{n-1}))$, provided that all $p_i(x)$ exist. This leads to an alternative formula for the Hankel determinant:

$$h_{n-1} = \prod_{j=0}^{n-1} \kappa_j, \qquad \kappa_j = \int_{-1}^{1} p_j(x)^2 \mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x. \tag{8.8}$$

Next, we observe that

$$\kappa_j' = \mathrm{i}\int_{-1}^{1} x p_j(x)^2 \mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x = \mathrm{i}\alpha_j\kappa_j, \tag{8.9}$$

using the recurrence relation (2.2) and orthogonality. As a consequence,

$$h_{n-1}' = \sum_{\ell=0}^{n-1}\left(\prod_{j\neq\ell}\kappa_j\right)\kappa_\ell' = \sum_{\ell=0}^{n-1}\left(\prod_{j\neq\ell}\kappa_j\right)\mathrm{i}\alpha_\ell\kappa_\ell = \mathrm{i}h_{n-1}\sum_{\ell=0}^{n-1}\alpha_\ell. \tag{8.10}$$

We differentiate again, bearing in mind that $\alpha_\ell' = \mathrm{i}(\beta_{\ell+1}-\beta_\ell)$, see (2.4):

$$h_{n-1}'' = \mathrm{i}h_{n-1}'\sum_{\ell=0}^{n-1}\alpha_\ell - h_{n-1}\sum_{\ell=0}^{n-1}(\beta_{\ell+1}-\beta_\ell) = \frac{(h_{n-1}')^2}{h_{n-1}} - h_{n-1}\beta_n = \frac{(h_{n-1}')^2}{h_{n-1}} - \frac{h_n h_{n-2}}{h_{n-1}},$$

where we have used the second equation in (2.3) and telescoped the second sum, taking $\beta_0 = 0$. Multiplying throughout by h_{n-1} and shifting the index $n-1 \mapsto n$, we obtain the result. □

We note that (8.7) can be rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left(\log h_{n-1}\right) = -\beta_n,$$

in terms of the recurrence coefficient β_n.

We first show that no two consecutive Hankel determinants can vanish simultaneously.

Lemma 7 *There is no $n \geq 1$ and $\omega^* > 0$ such that*

$$h_{n-1}(\omega^*) = h_n(\omega^*) = 0.$$

Proof Assume that $h_n = h_{n-1} = 0$ for some value $\omega = \omega^*$. Then by (8.7) we have $h_n' = 0$ and so h_n has a double root. Since both terms in the left hand side of (8.7) have a double root, so must the right hand side and this implies that either $h_{n+1} = 0$ as well, or that h_{n-1} has a double root.

In the latter case, two consecutive Hankel determinants have a double root. In the former case this happens too. Indeed, in this case we have $h_n = h_n' = h_{n-1} = h_{n+1} = 0$. We can reformulate (8.7) as

$$h_{n+1}''h_{n+1} - (h_{n+1}')^2 = -h_n h_{n+2}, \qquad n \geq 0.$$

It follows that $h_{n+1}' = 0$, i.e. both h_n and h_{n+1} have a double root.

It remains to rule out two consecutive double roots. Let's assume they are h_n and h_{n+1}. In that case, the right hand side of (5.4) vanishes at $\omega = \omega^*$ but the left hand side does not, since $\tilde{p}_n$ does not vanish identically, unless also $h_{n-1} = 0$. Subsequently, we can deduce from another reformulation of (5.4) that $h_{n-1}' = 0$ too. Continuing this reasoning leads to a chain of double roots and all Hankel determinants vanishing down to $n = 0$, which leads to contradiction. □

It follows immediately that one can not have $h_n = h_n' = 0$ either, since by (8.7) at least one of h_{n-1} or h_{n+1} has to vanish too. We can also exclude $h_n = h_{n+2} = 0$ as follows.

The lemma below is of an independent interest.

Lemma 8 *There is no $n \geq 0$ and $\omega^* > 0$ such that*

$$h_n(\omega^*) = h_{n+2}(\omega^*) = 0.$$

Proof The result is true by direct computation for $n = 0$ and $n = 1$. Let us assume it is true up to $n - 1$, and assume that $h_n(\omega^*) = 0$ for some $\omega^* > 0$. We intend to show that $h_{n+2}(\omega^*) \neq 0$.

We know that $h_{n-1} \neq 0$ by Lemma 7 and that $h_{n-2} \neq 0$ by our inductive assumption. It follows from (2.3) that α_{n-1} is analytic at ω^*. It also follows from (2.3) that $\beta_n(\omega^*) = 0$. Since $h_n' \neq 0$ and $h_{n-2} \neq 0$, this root of β_n is simple.

We reformulate the differential-difference equations (2.4) as

$$\begin{aligned}\beta_{n+1} &= -\mathrm{i}\alpha_n' + \beta_n,\\ \alpha_{n+1} &= -\mathrm{i}\frac{\beta_{n+1}'}{\beta_n} + \alpha_n.\end{aligned}$$

Plugging in a Taylor series of α_{n-1} and β_n around $\omega = \omega^*$ and using the above recusions shows, after straightforward computation, that α_n and α_{n+1} have a simple pole, β_{n+1} has a double pole, β_{n+2} has a simple root and α_{n+2} is analytic at ω^*. Using the expressions

$$\alpha_n = \frac{\langle xp_n, p_n\rangle}{\langle p_n, p_n\rangle}, \qquad \text{and} \qquad \beta_n = \frac{\langle p_n, p_n\rangle}{\langle p_{n-1}, p_{n-1}\rangle}$$

this implies that $\langle p_{n+1}, p_{n+1}\rangle$ has a simple pole and $\langle p_{n+2}, p_{n+2}\rangle \neq 0$. The latter in turn implies that $h_{n+2}(\omega^*) \neq 0$. □

Theorem 9 *The monic orthogonal polynomial p_{2N}^ω exists and is unique for all values of $\omega \geq 0$.*

Proof We will use induction and the symmetry of the even-degree polynomials to prove their existence. The constant polynomial $p_0^\omega \equiv 1$ exists for all ω. Assuming that $p_{2N-2}^\omega(z)$ exists, we set out to show that $p_{2N}^\omega(z)$ exists as well. Furthermore, for technical reasons in the proof, we also assume inductively that $p_{2N-2}^\omega(z)$ has no roots on the imaginary axis and we intend to show the same property for $p_{2N}^\omega(z)$.

It follows from (1.3) that $p_{2N}^\omega(z)$ is a meromorphic function of ω for each z. Its only singularities are poles at isolated values of ω, in which case at least one of its roots must tend to infinity when approaching any such critical value. It follows from the symmetry (2.1) with respect to the imaginary axis that all roots either come in pairs, or they lie on the imaginary axis. Thus, there can not be a single root moving to infinity, unless along the imaginary axis. We first rule out this case, so that we can conclude that roots must always tend to infinity in symmetric pairs. This implies that $\tilde{p}_{2N}^\omega$ has degree $2N - 2$, which in turn implies that $h_{2N} = h_{2N-1} = 0$. However, two consecutive Hankel determinants can not vanish simultaneously and this is a contradiction.

We formulate the proof in terms of $\tilde{p}_{2N}^{\omega}$, which exists for all ω. At $\omega = 0$, the polynomial $\tilde{p}_{2N}^{0}(z)$ is a (scaled) Legendre polynomial of even degree $2N$, which does not vanish at $z = 0$, i.e. it has no roots on the imaginary axis. The only way $\tilde{p}_{2N}^{\omega}$ can have roots on the imaginary axis for larger ω is from the intersection of two (or more) symmetric trajectories of roots crossing the imaginary axis. In that case, $\tilde{p}_{2N}^{\omega}$ has at least a double root on the imaginary axis when the trajectories intersect.

Assume first there are precisely two symmetric trajectories and let them be $\xi(\omega)$ and $\theta(\omega)$, for ω in a neighbourhood of a critical value ω^* at which point they cross the imaginary axis, and let ω^* be the smallest such value. In other words,

$$\tilde{p}_{2N}^{\omega}(\xi(\omega)) = \tilde{p}_{2N}^{\omega}(\theta(\omega)) = 0,$$

and, without loss of generality,

$$\operatorname{Re}\xi(\omega) < 0 < \operatorname{Re}\theta(\omega), \qquad \omega < \omega^*,$$

and finally

$$\xi(\omega^*) = \theta(\omega^*) = \mathrm{i}x^*,$$

for some $x^* \in \mathbb{R}$. Furthermore, in order for the roots to remain on the imaginary axis afterwards, it is an elementary exercise that they must hit the imaginary axis perpendicularly. This implies that $\xi'(\omega^*) = \theta'(\omega^*) = 0$.

Let $q(\omega) = \tilde{p}_{2N}^{\omega}(\xi(\omega))$. Since $q(\omega^*) = q'(\omega^*) = 0$, we have that

$$q'(\omega^*) = \frac{\partial \tilde{p}_{2N}^{\omega^*}}{\partial z}(\mathrm{i}x^*)\xi'(\omega^*) + \frac{\partial \tilde{p}_{2N}^{\omega^*}}{\partial \omega}(\mathrm{i}x^*) = 0.$$

Because $\partial \tilde{p}_{2N}^{\omega^*}/\partial z = 0$ at the double root, this leads to $\partial \tilde{p}_{2N}^{\omega^*}(\mathrm{i}x^*)/\partial\omega = 0$. From (8.5), we find that

$$h_{2N}(\omega^*)\tilde{p}_{2N-1}^{\omega^*}(\mathrm{i}x^*) = 0.$$

There are now two possibilities: $h_{2N}(\omega^*) = 0$ or $\tilde{p}_{2N-1}^{\omega^*}$ vanishes at $\mathrm{i}x^*$.

Consider the former case first. In this case we have $h_{2N} = 0$, and therefore $h_{2N-1} \neq 0$, and $\tilde{p}_{2N}^{\omega^*}$ vanishes at $\mathrm{i}x^*$ along with its derivatives in z and in ω. It follows from (8.6) that

$$\frac{\partial^2 \tilde{p}_{2N}^{\omega^*}(\mathrm{i}x^*)}{\partial \omega^2} h_{2N-1} = -\mathrm{i}h'_{2N}\tilde{p}_{2N-1}^{\omega^*}(\mathrm{i}x^*). \tag{8.11}$$

On the other hand, the second order derivative of $q(\omega)$, evaluated at ω^* is

$$0 = q''(\omega^*) = \frac{\partial^2 \tilde{p}_{2N}^{\omega^*}}{\partial z^2}(\mathrm{i}x^*)\xi'^2(\omega^*) + \frac{\partial \tilde{p}_{2N}^{\omega^*}(\mathrm{i}x^*)}{\partial z}\xi''(\omega^*) + \frac{\partial^2 \tilde{p}_{2N}^{\omega^*}(\mathrm{i}x^*)}{\partial \omega^2}.$$

Since the first two terms in the right hand side vanish, so does the third. Given that $h_{2N-1} \neq 0$ and $h'_{2N} \neq 0$ in (8.11), this yields $\tilde{p}_{2N-1}^{\omega^*}(ix^*) = 0$.

Thus, we are left with the latter case, where we have $\tilde{p}_{2N-1}^{\omega^*}(\mathrm{i}x^*) = 0$, in addition to $\tilde{p}_{2N}^{\omega^*}(\mathrm{i}x^*) = 0$. From the recurrence relation (5.4) we deduce that also $h_{2N-1}(\omega^*)p_{2N-2}^{\omega^*}(\mathrm{i}x^*) = 0$. The polynomial p_{2N-2}^{ω} cannot vanish on the imaginary axis by the inductive assumption. Thus, we must have $h_{2N-1}(\omega^*) = 0$. In case $h_N = 0$ this is a contradiction. In the other case, there is still a contradiction as the only possible singularity of p^{ω}_{2N} is for two symmetric roots to tend to infinity at ω^*, because p_{2N} does not have roots on the imaginary axis for $\omega < \omega^*$ by our choice of ω^*.

Finally, we still have to consider the case where an even number of trajectories cross the imaginary axis at a point $\mathrm{i}x^*$. This case is more degenerate. By similar reasoning, it follows that $\partial \tilde{p}^{\omega^*}_n/\partial\omega$ vanishes at least to third order at $\mathrm{i}x^*$. From (8.5) it follows that $\tilde{p}^{\omega^*}_{n-1}$ must vanish at least to second order at this point, since h_n can have at most a simple root. From the recurrence relation $\tilde{p}^{\omega^*}_{n-2}$ must vanish at least to first order on the imaginary axis, which is a contradiction. $\square$

Acknowledgements

The financial support from projects MTM2012-36732-C03-01 and MTM2012-34787 from the Spanish Ministry of Economy and Competitivity (A. Deaño), and FWO Flanders project G.0641.11N (A. Deaño and D. Huybrechs) is gratefully acknowledged.

We wish to record our thanks to a long list of colleagues with whom we have discussed this work, in particular to Andreas Asheim, Percy Deift, Nele Lejon (KU Leuven) and Marcus Webb (University of Cambridge).

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