Quadrature methods for highly oscillatory singular integrals

Jing GAO[*,†]
School of Mathematics and Statistics, Xi'an Jiaotong University
E-mail: jgao@xjtu.edu.cn

Marissa CONDON
School of Electronic Engineering, Dublin City University
E-mail: marissa.condon@dcu.ie

Arieh ISERLES
DAMTP, Centre for Mathematical Sciences, University of Cambridge
E-mail: ai10@cam.ac.uk

Abstract

We study asymptotic expansions, Filon-type methods and complex-valued Gaussian quadrature for highly oscillatory integrals with power-law and logarithmic singularities. We show that the asymptotic behaviour of the integral depends on the integrand and its derivatives at the singular point of the integrand, the stationary points and the endpoints of the integral. A truncated asymptotic expansion achieves an error that decays faster for increasing frequency. Based on the asymptotic analysis, a Filon-type method is constructed to approximate the integral. Unlike the asymptotic method, the Filon method achieves high accuracy for both small ω and large ω. The complex-valued quadrature involves interpolation at the zeros of polynomials orthogonal to a complex weight function. Numerical results indicate that the complex-valued Gaussian quadrature achieves the highest accuracy when the three methods are compared.

2010 Mathematics Subject Classification: 65D32, 41A55.

Keywords: Numerical quadrature, Singular highly oscillatory integral, Asymptotic analysis, Filon-type method, Complex-valued Gaussian quadrature.

1 Introduction

We shall consider two kinds of highly oscillatory singular integrals as follows

$$I[f]=\int_0^b x^{-\alpha}f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \tag{1.1}$$

and

$$I[f]=\int_0^b \log x f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x, \tag{1.2}$$

[*]The corresponding author.

[†]The work is supported by the Projects of International Cooperation and Exchanges NSFC-RS (Grant No. 11511130052) and the Key Science and Technology Program of Shaanxi Province of China (Grant No. 2016GY-080).

where, f and g are sufficiently smooth functions, $|\omega|$ is large and $\alpha \in (0,1)$.

Such integrals occur, for example, in the computation of integral equations with singular kernels and are ubiquitous in electromagnetic calculations [17, 18]. Our placement of the singularity at the origin is for convenience only, e.g.

$$\int_a^b |x-c|^{-\alpha} f(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x$$
$$= \int_0^{c-a} x^{-\alpha} f(c-x) \mathrm{e}^{\mathrm{i}\omega g(c-x)} \mathrm{d}x + \int_0^{b-c} x^{-\alpha} f(c+x) \mathrm{e}^{\mathrm{i}\omega g(c+x)} \mathrm{d}x,$$

where $c \in (a,b)$.

We note in passing that the computation of singular highly oscillatory integrals has been already considered in literature [1, 8, 9, 15, 19]. The purpose of this paper is to bring to bear on this problem the full power of modern machinery for the computation of smooth highly oscillatory integrals.

Before we embark on our analysis, it is interesting to investigate what can be achieved by a simple change of variable in (1.1), $x = t^{1/(1-\alpha)}$. This results in

$$I[f] = \frac{1}{1-\alpha} \int_0^{b^{1-\alpha}} f\left(t^{1/(1-\alpha)}\right) \mathrm{e}^{\mathrm{i}\omega g(t^{1/(1-\alpha)})} \mathrm{d}t.$$

In principle, we could have asymptotically expanded an integral in the above form except that, unless $1/(1-\alpha)$ is an integer, this may introduce new singularities at the origin and the outcome is probably more complicated than our subsequent analysis for (1.1). However, if $\alpha = (m-1)/m$, $m \geq 2$, then

$$I[f] = m \int_0^{b^{1/m}} f(t^m) \mathrm{e}^{\mathrm{i}\omega g(t^m)} \mathrm{d}t.$$

In other words, a weak singularity at the origin is equivalent to a stationary point subject to the above change of variables. This can easily be seen for the simple oscillator $g(x) = x$: the 'price' of eliminating the singularity is a more complicated oscillator, $g(t^m)$, with an $(m-1)$-fold stationary point at the origin.

For a general value of $\alpha \in (0,1)$, however, the setting is more complicated. In the case of an oscillator $g(x) = x^p$, $p \in \mathbb{N}$, where it is known that

$$\int_0^b f(x) \mathrm{e}^{\mathrm{i}\omega x^p} \mathrm{d}x \sim O(\omega^{-1/p}),$$

it is possible to prove directly from the transformed integral using the asymptotic approach of [13] that

$$\int_0^b x^{-a} f(x) \mathrm{e}^{\mathrm{i}\omega x^p} \mathrm{d}x \sim O(\omega^{-(1-a)/p}).$$

We do not pursue this approach further because asymptotic expansions and numerical methods based on the original formulation (1.1) are in our experience simpler.

Insofar as the integral (1.1) is concerned, in the case of a fractional $\alpha < 0$ we have a proper integral and the Filon-type approach of [13] is applicable, but only up to a point, since the integrand therein is not $C^\infty[0,b]$ and we cannot expand asymptotically beyond $\lfloor -\alpha \rfloor$. However, once we can deal with the singular case, the case of $\alpha < 0$ can be accommodated.

The logarithmic oscillatory integral features in electromagnetic shielding problems [9] and acoustic scattering at high frequencies[4]. The traditional boundary element method for solving high-frequency acoustic scattering problems suffers from excessive computational costs since polynomial basis functions cannot describe the characteristics of the oscillatory solution accurately at high frequencies. To remedy this, there exist some high-frequency discretization methods, such as the hybrid numerical-asymptotic boundary element methods [4] and the partition of unity boundary element methods (PU-BEM) [21]. These methods employ basis functions with oscillatory characteristics to approximate the oscillatory solution. Another method is the Filon-Clenshaw-Curtis method in [7]. In particular, consider PU-BEM method for a boundary integral equation,

$$\frac{u(\mathrm{p})}{2}+\int_\Gamma\left[\frac{\partial G(\mathrm{p},\mathrm{q})}{\partial \mathrm{n}(\mathrm{q})}-\alpha G(\mathrm{p},\mathrm{q})\right]u(\mathrm{q})\mathrm{d}\Gamma_{\mathrm{q}}=\int_\Gamma \beta G(\mathrm{p},\mathrm{q})\mathrm{d}\Gamma(q)+u^i(\mathrm{p}), \tag{1.3}$$

where G is the Green's function, Γ is the scatterer boundary and u^i is the incident plane wave. The source point is $\mathrm{p}\in\Gamma$. The Robin boundary condition

$$\frac{\partial u}{\partial \mathrm{n}}=\alpha u+\beta,$$

is assumed. The boundary line Γ is partitioned into N_e elements and each element e is mapped onto the parametric space $\xi_e\in[-1,1)$. The Partition of Unity expansion for the potential at a point q on each element is

$$u(\xi_e(\mathrm{q}))=\sum_{j=1}^{J}\sum_{m=1}^{M}N_j(\xi_e)\mathrm{e}^{\mathrm{i}\omega \mathrm{d}_m\cdot\mathrm{q}}A_{ejm}, \tag{1.4}$$

where J is the number of nodes on the element, N_j is the usual Lagrangian polynomial shape function for node j, d_m is the unit vector describing the direction of propagation of the m-th plane wave in a set of M plane waves. Applying (1.4) to discretize (1.3), we obtain several oscillatory integrals e.g.

$$-\frac{\mathrm{i}\alpha}{4}\int_{-1}^{1}H_0^{(1)}(\omega\mathrm{r})N_j(\xi)\mathrm{e}^{\mathrm{i}\omega\mathrm{d}_m\cdot\mathrm{q}(\xi_e)}\bar{J}(\xi_e)\mathrm{d}\xi_e.$$

The numerical steepest descent method was successfully applied for this integral in [10] to reduce the computational effort. To design an effective quadrature method, we are more interested in the asymptotic analysis of this kind of integral. Note that as the distance r goes to zero, the Hankel function behaves as

$$H_0^{(1)}(\omega\mathrm{r})\sim\frac{2\mathrm{i}}{\pi}\log(\omega\mathrm{r}),$$

referenced from the website `http://dlmf.nist.gov/10.7`. This results in the requirement for asymptotic analysis of a highly oscillatory integral with a logarithmic singularity.

Integrals of the form

$$\int_0^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x,\qquad \omega\gg 1,$$

where f and g are sufficiently smooth functions, appear in a very wide range of applications and their computation by conventional quadrature methods is exceedingly expensive and inefficient: essentially, the number of quadrature points must be $O(\omega)$ [5] and this is prohibitive for large ω. Unlike traditional quadrature methods, based upon local Taylor expansions, a new type of highly oscillatory quadrature algorithms – asymptotic expansions and Filon-type methods introduced by Iserles and Nørsett [12, 13], Levin methods [16, 20] and the numerical steepest descent methods of Huybrechs and Vandewalle [11] – excel in the presence of high oscillation. All such methods are based upon asymptotic expansions and their accuracy scales like $O(\omega^{-p-1})$ for some $p \geq 1$. In other words, their precision *increases* with growing frequency. Furthermore, the complex-valued Gaussian quadrature studied in [2, 6] can obtain an optimal asymptotic order. The existing theory of highly oscillatory quadrature, however, does not extend to the presence of singular integrands. Therefore, it is necessary to study the asymptotic properties of highly oscillatory singular integrals.

The theme of this paper is the development of efficient quadrature schemes for integrals of the form of (1.1) and (1.2). This is not a straightforward generalisation of standard theory. All modern highly oscillatory quadrature methods, whether Filon-type, Levin-type, computational stationary phase or their combinations, are based upon an asymptotic expansion of the solution in inverse (but not necessarily integer) powers of ω. Although, as we demonstrate in the sequel, this can be extended to the current setting, the extension is nontrivial because the weak singularity at the origin interacts with other aspects of the asymptotic expansion.

We explore three typical quadrature schemes for highly oscillatory integration. Their asymptotic properties and related expansions are formulated in Section 2. Based on this analysis, Filon methods are constructed in Section 3. Both sections are accompanied by relevant numerical examples. Section 4 displays the numerical results of the complex-valued Gaussian quadrature.

Before we commence with our asymptotic analysis, we first define the generalised moments of $I[f]$ in (1.1) and (1.2)

$$\mu_j(\alpha, \omega) = \int_0^b x^{j-\alpha} \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x,$$
$$\nu_j(\omega) = \int_0^b x^j \log x \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x.$$

2 Asymptotic analysis of highly oscillatory integrals with power-law and logarithmic singularities

In this section we are concerned with the asymptotic analysis of highly oscillatory power-law integrals (1.1) and logarithmic integrals (1.2). Given that the presence of the stationary point determines the result of asymptotic analysis, we will present the theory first without and subsequently with stationary points.

2.1 Asymptotic analysis of power-law singularity

We commence with the case $g'(x) \neq 0$ for the integral (1.1) before extending to more complicated oscillation with stationary points. In contrast to the non-singular oscillatory integral [13], our first step is to examine the behaviour of the moment function $\mu(\alpha, \omega)$ which constitutes an important part for the asymptotic analysis. Firstly, the moment function is bounded,

$$|\mu_0(\alpha;\omega)| = \left|\int_0^b x^{-\alpha}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x\right| \le \int_0^b x^{-\alpha}\mathrm{d}x = \frac{b^{1-\alpha}}{1-\alpha}.$$

To get a sharper upper bound, we separate the interval of integration into

$$\int_0^b x^{-\alpha}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x = \int_0^\epsilon x^{-\alpha}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x + \int_\epsilon^b x^{-\alpha}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x,$$

where a small number $\epsilon > 0$ will be set momentarily. Different choices of ϵ determine different upper bounds and in the next lemma we choose an optimal value of ϵ to get the least upper bound.

Lemma 1. *Given $\omega \gg 1$ and $g'(x) \neq 0$, $x \in [0,b]$, the zeroth moment function $\mu_0(\alpha;\omega)$ satisfies*

$$|\mu_0(\alpha;\omega)| = \left|\int_0^b x^{-\alpha}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x\right| \sim O(\omega^{-(1-\alpha)}).$$

Proof. Assume a small number $\epsilon > 0$. We write the moment in the form

$$\int_0^b x^{-\alpha}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x = \int_0^\epsilon x^{-\alpha}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x + \int_\epsilon^b x^{-\alpha}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x.$$

The first integral on the right side is $O\left(\epsilon^{1-\alpha}\right)$. The remaining integral is non-singular and can be calculated using integration by parts,

$$\begin{aligned}\int_\epsilon^b x^{-\alpha}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x &= \frac{1}{\mathrm{i}\omega}\left[\frac{b^{-\alpha}}{g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)} - \frac{\epsilon^{-\alpha}}{g'(\epsilon)}\mathrm{e}^{\mathrm{i}\omega g(\epsilon)}\right] \\ &\quad - \frac{1}{\mathrm{i}\omega}\int_\epsilon^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{-\alpha}}{g'(x)}\right)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \sim O\left(\omega^{-1}\epsilon^{-\alpha}\right).\end{aligned}$$

Together then, the integral is bounded by $O\left(\epsilon^{1-\alpha}\right) + O\left(\omega^{-1}\epsilon^{-\alpha}\right)$. We take $\epsilon = \omega^{-1}$ to get the desired result. □

It directly follows from Lemma 1 that

$$\left|\int_0^b x^{-\alpha}f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x\right| \le C_1\omega^{-(1-\alpha)}, \tag{2.1}$$

where f is a sufficiently smooth function and the constant C_1 is related to the upper bound of f.

Based on (2.1), we deduce the following theorem.

Theorem 2. *Assume that $g'(x) \neq 0$. Then for $s \in \mathbb{N}$ and $\omega \gg 1$, the first $2s$ terms of the asymptotic expansion of $I[f]$ are*

$$Q_s^A[f] \sim \mu_0(\alpha;\omega)\sum_{k=0}^{s-1}\frac{1}{(-i\omega)^k}\sigma_k[f](0) - \sum_{k=1}^{s}\frac{1}{(-i\omega)^k}\frac{\sigma_{k-1}[f](b) - \sigma_{k-1}[f](0)}{b^\alpha g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)}$$

for every smooth function f, where

$$\begin{aligned}\sigma_0[f](x) &= f(x), \\ \sigma_{k+1}[f](x) &= x^\alpha\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sigma_k[f](x) - \sigma_k[f](0)}{x^\alpha g'(x)}, \qquad k \ge 0.\end{aligned}$$

The corresponding truncation error is

$$I[f] - Q_s^A[f] \sim O\left(\omega^{-s-(1-\alpha)}\right). \tag{2.2}$$

Proof. Firstly, use subtraction to remove the singularity of $x^{-\alpha}$,

$$\begin{aligned}
&\int_0^b x^{-\alpha} f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \\
&= f(0)\int_0^b x^{-\alpha}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x + \int_0^b x^{-\alpha}[f(x)-f(0)]\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \\
&= f(0)\mu_0(\alpha;\omega) + \int_0^b x^{-\alpha}[f(x)-f(0)]\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \\
&= f(0)\mu_0(\alpha;\omega) + I_2,
\end{aligned}$$

where $\mu_0(\alpha;\omega)$ is bounded in (2.1) and

$$I_2 = \int_0^b x^{-\alpha}[f(x)-f(0)]\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x.$$

The first term in the expansion of I_2 is determined from

$$\begin{aligned}
I_2 &= \int_0^b x^{-\alpha}[f(x)-f(0)]\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x = \frac{1}{\mathrm{i}\omega}\int_0^b \frac{f(x)-f(0)}{x^\alpha g'(x)}\mathrm{d}\mathrm{e}^{\mathrm{i}\omega g(x)} \\
&= \frac{1}{\mathrm{i}\omega}\frac{f(b)-f(0)}{b^\alpha g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)} - \frac{1}{\mathrm{i}\omega}\left[\lim_{x\to 0}\frac{f(x)-f(0)}{x^\alpha g'(x)}\right]\mathrm{e}^{\mathrm{i}\omega g(0)} \\
&\quad - \frac{1}{\mathrm{i}\omega}\int_0^b \frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)-f(0)}{x^\alpha g'(x)}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \\
&= \frac{1}{\mathrm{i}\omega}\frac{f(b)-f(0)}{b^\alpha g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)} \\
&\quad - \frac{1}{\mathrm{i}\omega}\int_0^b x^{-\alpha}\frac{f'(x)g'(x)-[f(x)-f(0)][\frac{\alpha g'(x)}{x}+g''(x)]}{g'^2(x)}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x
\end{aligned}$$

provided that the function $\frac{f(x)-f(0)}{x^\alpha g'(x)}$ is continuous and that $\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x^\alpha g'(x)} = 0$. Since

$$\frac{\sigma_{k-1}(x)-\sigma_{k-1}(0)}{x^\alpha g'(x)}|_{x=0} = \lim_{x\to 0}\frac{\sigma_{k-1}(x)-\sigma_{k-1}(0)}{x^\alpha g'(x)} = 0,$$

we deduce the remaining terms using integration by parts,

$$\begin{aligned}
I[f] &\sim \mu_0(\alpha;\omega)\sum_{k=0}^{s-1}\frac{1}{(-\mathrm{i}\omega)^k}\sigma_k[f](0) - \sum_{k=1}^{s}\frac{1}{(-\mathrm{i}\omega)^k}\frac{\sigma_{k-1}[f](b)-\sigma_{k-1}[f](0)}{b^\alpha g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)} \\
&\quad + \frac{1}{(-\mathrm{i}\omega)^s}\int_0^b x^{-\alpha}\sigma_s[f](x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x.
\end{aligned}$$

The result (2.2) follows. □

In particular, in the most important case of the Fourier oscillator $g(x)=x$ and $\omega \gg 1$, we have

$$\int_0^b x^{-\alpha}f(x)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}x \sim \mu_0(\alpha;\omega)\sum_{k=0}^{s-1}\frac{1}{(-\mathrm{i}\omega)^k}\sigma_k[f](0) - \sum_{k=1}^{s}\frac{1}{(-\mathrm{i}\omega)^k}\frac{\sigma_{k-1}(b)-\sigma_{k-1}(0)}{b^\alpha}\mathrm{e}^{\mathrm{i}\omega b},$$

where

$$\sigma_k(x) = \sum_{j=0}^{\infty}\frac{(j+k-\alpha)\cdots(j+2-\alpha)(j+1-\alpha)}{(j+k)!}f^{(j+k)}(0)x^j,$$

$$\sigma_k(0) = \frac{(k-\alpha)\cdots(2-\alpha)(1-\alpha)}{k!} f^{(k)}(0).$$

Also in the case of a stationary point, the key to asymptotic analysis is repeated integration by parts. Once the function $g'(x) = 0$ at one or more points in $[0, b]$, the character of the asymptotic analysis depends on the stationary points, the endpoints and the order of the singularity. Assume for simplicity that the integral (1.1) possesses just a single stationary point at $\xi \in [0, b]$. The extension to the case of more stationary points is fairly straightforward. In addition, we only explore the asymptotic analysis for $\xi = 0$ since in the case $\xi \neq 0$ the integration can be re-arranged in this form. To start with, we analyse the asymptotic order of the moment $\mu_j(\alpha;\omega)$ based on the method of stationary phase [3, p. 279].

Lemma 3. *Suppose that $\omega \gg 1$ and $g'(x)$ has an r order stationary point at the point 0, that is $g'(0) = \cdots = g^{(r)}(0) = 0$, $g^{(r+1)}(0) \neq 0$, then*

$$|\mu_j(\alpha;\omega)| = \left| \int_0^b x^{j-\alpha} \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x \right| \sim O\left(\omega^{-\min\left(\frac{j+1-\alpha}{r+1},1\right)}\right). \tag{2.3}$$

Proof. We separate μ_j into two terms:

$$\int_0^b x^{j-\alpha} \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x = \int_0^\infty - \int_b^\infty .$$

Using integration by parts, the second integral on the right behaves as

$$\int_b^\infty x^{j-\alpha} \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x \sim O\left(\omega^{-1}\right),$$

since it is a proper integral without a stationary point in the interval $[b, \infty]$.

To obtain the essential behaviour of the first integral on the right, we expand the function $g(x)$ in a Taylor series

$$g(x) \sim \sum_{m=0}^{r} \frac{g^{(m)}(0)}{m!} x^m + \frac{g^{(r+1)}(x)}{(r+1)!} x^{r+1} + \cdots = \frac{g^{(r+1)}(\xi)}{(r+1)!} x^{r+1}$$

for some $\xi \in [0, x]$, since $g^{(m)}(0) = 0$, $m = 0, 1, \cdots, r$, and $g^{(r+1)}(0) \neq 0$. This gives

$$\int_0^\infty x^{j-\alpha} \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x \sim \int_0^\infty x^{j-\alpha} \mathrm{e}^{\mathrm{i}\omega \frac{g^{(r+1)}(\xi)}{(r+1)!} x^{r+1}} \mathrm{d}x.$$

Then we rotate the contour of integration from the real-x axis by an angle $\frac{\pi}{2(r+1)}$ if $g^{(r+1)}(\xi) > 0$ and make the substitution

$$x = \mathrm{e}^{\frac{\mathrm{i}\pi}{2(r+1)}} \left[\frac{(r+1)! u}{\omega g^{(r+1)}(\xi)} \right]^{1/(r+1)},$$

where u is real (we rotate by an angle $\frac{-\pi}{2(r+1)}$ if $g^{(r+1)}(\xi) < 0$ with $x = \mathrm{e}^{\frac{-\mathrm{i}\pi}{2(r+1)}} \left[\frac{(r+1)! u}{\omega |g^{(r+1)}(\xi)|} \right]^{1/(r+1)}$). This yields

$$\int_0^\infty x^{j-\alpha} \mathrm{e}^{\mathrm{i}\omega \frac{g^{(r+1)}(\xi)}{(r+1)!} x^{r+1}} \mathrm{d}x \sim C \omega^{-\frac{j+1-\alpha}{r+1}} \int_0^\infty e^{-u} u^{\frac{1}{r+1}-1} \mathrm{d}u$$

$$\sim C\omega^{-\frac{j+1-\alpha}{r+1}}\Gamma\left(\frac{1}{r+1}\right)\sim O\left(\omega^{-\frac{j+1-\alpha}{r+1}}\right).$$

Comparing the two orders, it follows that

$$|\mu_j(\alpha;\omega)|\sim O\left(\max\left(\omega^{-\frac{j+1-\alpha}{r+1}},\omega^{-1}\right)\right).$$

Thus, the exponent $\min\left(\frac{j+1-\alpha}{r+1},1\right)$ is the right one. This completes the proof of (2.3). □

Based on Lemma 3, it follows that

$$\left|\int_0^b x^{-\alpha}f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x\right|\le C_2\omega^{-\frac{1-\alpha}{r+1}}, \tag{2.4}$$

where the constant C_2 is related to the bound of the smooth function f as $\omega\to\infty$.

We commence from the simple case $r=1$ and next progress to a general $r\ge 1$.

Theorem 4. *Assume that $g'(0)=0$, $g''(0)\neq 0$ and $g'(x)\neq 0$, $x\in(0,b]$. For every smooth function f and $\omega\gg 0$, it is true that*

$$\begin{aligned}I[f]\ \sim\ &\mu_0(\alpha;\omega)\sum_{k=0}^{s-1}\frac{\rho_k[f](0)}{(-\mathrm{i}\omega)^k}+\mu_1(\alpha;\omega)\sum_{k=0}^{s-1}\frac{\rho_k'[f](0)}{(-\mathrm{i}\omega)^k}\\&-\sum_{k=1}^{s}\frac{1}{(-\mathrm{i}\omega)^k}\frac{\rho_{k-1}[f](b)-\rho_{k-1}[f](0)-\rho_{k-1}'[f](0)b}{b^\alpha g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)}\end{aligned} \tag{2.5}$$

with the error of $O\left(\omega^{-s-(1-\alpha)/2}\right)$, where

$$\begin{aligned}&\rho_0[f](x)=f(x),\\&\rho_k[f](x)=x^\alpha\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\rho_{k-1}[f](x)-\rho_{k-1}[f](0)-\rho_{k-1}'[f](0)x}{x^\alpha g'(x)}\right],\quad k\ge 1.\end{aligned}$$

When $s\to\infty$ we obtain the asymptotic expansion of $I[f]$.

Proof. Subtracting out the singularity, it is true that

$$\begin{aligned}I[f]&=\int_0^b x^{-\alpha}f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x\\&=\int_0^b x^{-\alpha}[f(x)-f(0)-f'(0)x+f(0)+f'(0)x]\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x\\&=f(0)\int_0^b x^{-\alpha}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x+f'(0)\int_0^b x^{1-\alpha}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x\\&\quad+\int_0^b x^{-\alpha}[f(x)-f(0)-f'(0)x]\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x.\end{aligned}$$

Insofar as the third integral is concerned, we use integration by parts to deduce that

$$\begin{aligned}&\int_0^b x^{-\alpha}[f(x)-f(0)-f'(0)x]\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x\\&=\frac{1}{\mathrm{i}\omega}\int_0^b\frac{f(x)-f(0)-f'(0)x}{x^\alpha g'(x)}\mathrm{d}\mathrm{e}^{\mathrm{i}\omega g(x)}\\&=\frac{1}{\mathrm{i}\omega}\left[\frac{f(b)-f(0)-f'(0)b}{b^\alpha g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)}-\lim_{x\to 0}\frac{f(x)-f(0)-f'(0)x}{x^\alpha g'(x)}\mathrm{e}^{\mathrm{i}\omega g(0)}\right]\end{aligned}$$

$$- \int_0^b x^{-\alpha}\rho_1[f](x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x,$$

where

$$\begin{aligned}
\rho_0[f](x) &= f(x),\\
\rho_1[f](x) &= x^\alpha \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\rho_0[f](x)-\rho_0[f](0)-\rho_0'[f](0)x}{x^\alpha g'(x)}\right],\\
&\lim_{x\to 0}\frac{f(x)-f(0)-f'(0)x}{x^\alpha g'(x)} = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I[f] &= \rho_0[f](0)\mu_0(\alpha;\omega) + \rho_0'[f](0)\mu_1(\alpha;\omega)\\
&\quad + \frac{1}{\mathrm{i}\omega}\frac{\rho_0[f](b)-\rho_0[f](0)-\rho_0'[f](0)b}{b^\alpha g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)} - \frac{1}{\mathrm{i}\omega}\int_0^b x^{-\alpha}\rho_1[f](x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x.
\end{aligned}$$

Similarly, the general form can be obtained by induction

$$\begin{aligned}
I[f] \sim \mu_0(\alpha;\omega)&\sum_{k=0}^{s-1}\frac{\rho_k[f](0)}{(-\mathrm{i}\omega)^k} + \mu_1(\alpha;\omega)\sum_{k=0}^{s-1}\frac{\rho_k'[f](0)}{(-\mathrm{i}\omega)^k}\\
&-\sum_{k=1}^{s}\frac{1}{(-\mathrm{i}\omega)^k}\frac{\rho_{k-1}[f](b)-\rho_{k-1}[f](0)-\rho_{k-1}'[f](0)b}{b^\alpha g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)}\\
&+\frac{1}{(-\mathrm{i}\omega)^s}\int_0^b x^{-\alpha}\rho_s[f](x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x,
\end{aligned}$$

with

$$\rho_k[f](x) = x^\alpha\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\rho_{k-1}[f](x)-\rho_{k-1}[f](0)-\rho_{k-1}'[f](0)x}{x^\alpha g'(x)}\right].$$

Using (2.4), we deduce the proof of (2.5). □

Integrals with higher-order stationary points are important and common in applications [22]. The asymptotic expansion in Theorem 4 can be readily extended to the case of $g(x)$ having a single stationary point of order $r \geq 1$ at 0, in other words

$$0 = g(0) = g'(0) = \cdots = g^{(r)}(0) = 0, \qquad g^{(r+1)}(0) \neq 0$$

and $g'(x) \neq 0$ for $x \in (0, b]$. If $g(0) \neq 0$, we transform the integral into the form

$$I[f] = \int_0^b x^{-\alpha}f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x = \mathrm{e}^{\mathrm{i}\omega g(\xi)}\int_0^b x^{-\alpha}f(x)\mathrm{e}^{\mathrm{i}\omega[g(x)-g(\xi)]}\mathrm{d}x.$$

Consequently, setting

$$\begin{aligned}
\rho_0[f](x) &= f(x),\\
\rho_{k+1}[f](x) &= x^\alpha\frac{\mathrm{d}}{\mathrm{d}x}\frac{\rho_k[f](x) - \sum_{j=0}^{r}\frac{\rho_k^{(j)}[f](0)}{j!}x^j}{x^\alpha g'(x)}, \qquad k \geq 0,
\end{aligned}$$

we obtain the general form for the asymptotic expansion,

$$I[f] \sim \sum_{j=0}^{r} \frac{\mu_j(\alpha;\omega)}{j!} \sum_{k=0}^{s-1} \frac{\rho_k^{(j)}[f](0)}{(-i\omega)^k} - \sum_{k=1}^{s} \frac{\mathrm{e}^{\mathrm{i}\omega g(b)}}{(-i\omega)^k} \frac{\rho_{k-1}[f](b) - \sum_{j=0}^{r} \frac{\rho_{k-1}^{(j)}[f](0)}{j!} b^j}{b^\alpha g'(b)}, \tag{2.6}$$

the remainder term being $O\left(\omega^{-(s+\frac{1-\alpha}{r+1})}\right)$.

As we have shown in our analysis, the method $Q^{A,s}[f]$ can reduce the asymptotic error very effectively indeed when $\omega \to \infty$. However, it does not work for a small ω. We consider the integral without stationary points

$$\int_0^1 x^{-\alpha} f(x) \mathrm{e}^{\mathrm{i}\omega x} \mathrm{d}x \tag{2.7}$$

where $f(x) = (1+x)^{-1}$ and $\alpha = \frac{1}{2}$, which also will be considered as the test example without stationary points for the Filon method and complex-valued Gaussian quadrature, which will be considered in the sequel. The truncated expansion for $s = 1, 2, 3$ are

$$\begin{aligned}
Q^{A,1}[f](\omega) &= \mu_0(\alpha,\omega) + \frac{\mathrm{i}}{2\omega}\mathrm{e}^{\mathrm{i}\omega}, \\
&\quad \text{originating in } f(0), f(1); \\
Q^{A,2}[f](\omega) &= Q^{A,1}[f](\omega) + \mu_0(\alpha,\omega)\frac{-\mathrm{i}}{2\omega} + \frac{\mathrm{e}^{\mathrm{i}\omega}}{2\omega^2}, \\
&\quad \text{originating in } f(0), f'(0), f(1), f'(1); \\
Q^{A,3}[f](\omega) &= Q^{A,2}[f](\omega) + \mu_0(\alpha,\omega)\frac{-3}{4\omega^2} - \frac{7\mathrm{i}}{8\omega^3}\mathrm{e}^{\mathrm{i}\omega}, \\
&\quad \text{originating in } f(0), f'(0), f''(0), f(1), f'(1), f''(0),
\end{aligned}$$

and their asymptotic error is $O\left(\omega^{-s-\frac{1}{2}}\right)$. In Fig. 2.1 we depict the magnitude of the error, $\log\left|Q^{A,s}[f] - I[f]\right|$, $s = 1, 2, 3$. As we mentioned, the error blows up near $\omega = 0$ and deceases rapidly when $\omega > 10$. In addition, it is noted that inclusion of more terms in the expansion produces a better error order.

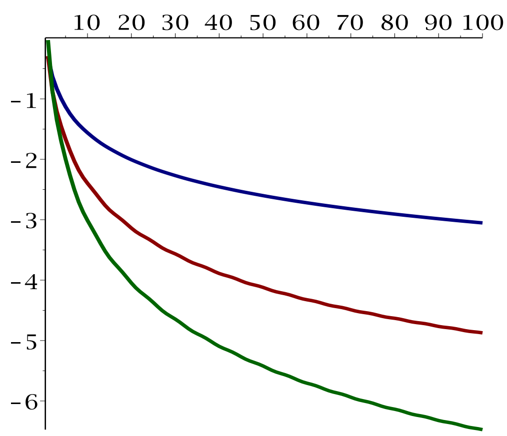

Figure 2.1: The error $\log\left|Q^{A,s}[f] - I[f]\right|$ with $x^{-\frac{1}{2}}$ and $g(x) = x$. The colours are navy blue (the top), dark red (the middle) and dark green (the bottom) for $s = 1, 2, 3$.

We now consider the same example but with an order-1 stationary point. I.e. calculate the integral

$$\int_0^1 x^{-\alpha} f(x) \mathrm{e}^{\mathrm{i}\omega x^2} \mathrm{d}x \tag{2.8}$$

where $f(x) = (1+x)^{-1}$, $\alpha = \frac{1}{2}$. Throughout the whole paper, this integral will be calculated as an example with a stationary point to illustrate the performance of different numerical quadratures. We truncate the expansion in Theorem 4 as $Q^{A,s}[f]$, $s = 1, 2, 3$ with the asymptotic error $O\left(\omega^{-s-\frac{1}{4}}\right)$,

$$\begin{aligned}
Q^{A,1}[f](\omega) &= \mu_0(\alpha,\omega) - \mu_1(\alpha,\omega) - \frac{\mathrm{i}}{4\omega}\mathrm{e}^{\mathrm{i}\omega}, \\
&\quad \text{originating in } f(1), f^{(j)}(0), \quad j = 0, 1; \\
Q^{A,2}[f](\omega) &= Q^{A,1}[f](\omega) + \frac{\mathrm{i}}{4\omega}\mu_0(\alpha,\omega) - \frac{3\mathrm{i}}{4\omega}\mu_1(\alpha,\omega) + \frac{\mathrm{e}^{\mathrm{i}\omega}}{4\omega^2}, \\
&\quad \text{originating in } f(1), f'(1), f^{(j)}(0), \quad j = 0, \cdots, 3; \\
Q^{A,3}[f](\omega) &= Q^{A,2}[f](\omega) + \frac{-5}{16\omega^2}\mu_0(\alpha,\omega) + \frac{21}{16\omega^2}\mu_1(\alpha,\omega) + \frac{31\mathrm{i}}{64\omega^3}\mathrm{e}^{\mathrm{i}\omega}, \\
&\quad \text{originating in } f(1), f'(1), f''(1), f^{(j)}(0), \quad j = 0, \cdots, 5.
\end{aligned}$$

Fig. 2.2 demonstrates the error order of the asymptotic expansion for different terms. Similar error behaviour has been observed in Figs 2.1 and 2.2 and stationary point does not deteriorate the performance of the asymptotic method, provided that we subtract the singularity originating in both the stationary point and singular points.

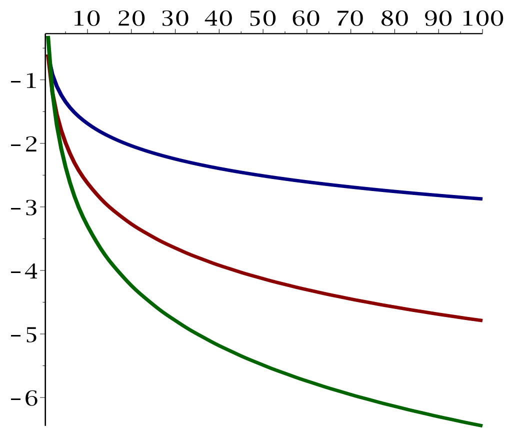

Figure 2.2: The error $\log\left|Q^{A,s}[f] - I[f]\right|$ with $x^{-\frac{1}{2}}$ and $g(x) = x^2$, for $s = 1$ (navy blue, the top), 2 (dark red, the middle), 3 (dark green, the bottom).

2.2 Asymptotic analysis for logarithmic singularity

Our second instance of a singular oscillatory integral originates in the logarithmic singularity (1.2). Our analysis is similar to the case of power-law singularity, hence we present it with greater brevity. As before, we commence with the case $g' \neq 0$, $x \in [0, 1]$. In that case

$$\begin{aligned}
I[f] &= \int_0^b \log x f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \\
&= \int_0^b \log x\,[f(x) - f(0)]\,\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x + f(0)\int_0^b \log x \mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \\
&= \int_0^b \log x\,[f(x) - f(0)]\,\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x + f(0)\nu_0,
\end{aligned}$$

where

$$\nu_k = \int_0^b x^k \log x \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x, \qquad k \geq 0.$$

On the interval $[0,b]$, the function $\log x\,[f(x) - f(0)]$ is non-singular and analytic. Define

$$\begin{aligned}
\sigma_0[f](x) &= f(x), \\
\sigma_1[f](x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sigma_1[f](x) - \sigma_0[f](0)}{g'(x)}\right], \\
&\vdots \\
\sigma_{k+1}[f](x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sigma_k[f](x) - \sigma_k[f](0)}{g'(x)}\right].
\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}
I[f] &= \int_0^b \log x\,[f(x) - f(0)]\,\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x + f(0)\nu_0 \\
&= f(0)\nu_0 + \frac{1}{\mathrm{i}\omega}\int_0^b \log x \frac{(f(x) - f(0))}{g'(x)} \mathrm{d}\mathrm{e}^{\mathrm{i}\omega g(x)} \\
&= f(0)\nu_0 + \frac{1}{\mathrm{i}\omega}\left[\log x \frac{f(x) - f(0)}{g'(x)}\right]\mathrm{e}^{\mathrm{i}\omega g(x)}\bigg|_0^b - \frac{1}{\mathrm{i}\omega}\int_0^b \frac{f(x) - f(0)}{xg'(x)}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \\
&\quad - \frac{1}{\mathrm{i}\omega}\int_0^b \log x \sigma_1[f](x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x.
\end{aligned}$$

Iterating the procedure results in the asymptotic expansion

$$\begin{aligned}
I[f] &= \nu_0(\omega)\sum_{k=0}^{s-1}\frac{\sigma_k[f](0)}{(-\mathrm{i}\omega)^k} - \sum_{k=1}^{s}\frac{e^{\mathrm{i}\omega g(b)}}{(-\mathrm{i}\omega)^k}\frac{\log b\,(\sigma_{k-1}[f](b) - \sigma_{k-1}[f](0))}{g'(b)} \\
&\quad + \sum_{k=1}^{s}\frac{1}{(-\mathrm{i}\omega)^k}\int_0^b \frac{\sigma_{k-1}[f](x) - \sigma_{k-1}[f](0)}{xg'(x)}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \\
&\quad + \frac{1}{(-\mathrm{i}\omega)^s}\int_0^b \log x \sigma_s[f](x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x, \qquad (2.9)
\end{aligned}$$

provided that

$$\lim_{x\to 0}\left[\log x \frac{(\sigma_{k-1}[f](x) - \sigma_{k-1}[f](0))}{g'(x)}\right] = 0.$$

Note that there are non-singular highly oscillatory integrals which appear in the expansion. Before determining the asymptotic order of (2.9), we need to estimate first the behaviour of the zeroth moment. The following lemma clarifies this issue.

Lemma 5. *Given $\omega \gg 1$ and $g'(x) \neq 0$, $x \in [0,b]$, the function $\nu_0(\omega)$ satisfies*

$$|\nu_0(\omega)| = \left|\int_0^b \log x \mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x\right| \sim O\left(\omega^{-1}\log\omega\right).$$

Proof. Consider a small positive number $\epsilon \to 0$ and separate this integral into two parts

$$\int_0^b \log x \mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x = \int_0^\epsilon + \int_\epsilon^b,$$

where

$$\int_\epsilon^b \log x \mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \sim O\left(\omega^{-1}\log\epsilon\right)$$

and

$$\left|\int_0^\epsilon \log x \mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x\right| \le -\int_0^\epsilon \log x \mathrm{d}x \sim O\left(\epsilon\log\epsilon\right).$$

Comparing both error bounds, an optimal upper bound is $O\left(\omega^{-1}\log\omega\right)$, letting $\epsilon = \omega^{-1}$. □

It follows immediately that

$$\int_0^b \log x f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \sim O\left(\omega^{-1}\log\omega\right).$$

For the expansion (2.9), truncating after the first s terms, the error is $O\big(\omega^{-(s+1)}\log\omega\big)$. Note however that the nonsingular oscillatory integrals in the asymptotic expansion (2.9) must be calculated with an error which is consistent with the above asymptotic decay. We denote the number of the truncation terms in the nonsingular oscillatory integral by L_k which depends on the index k. These oscillatory integrals without logarithmic integrands are expanded by the asymptotic expansion in [13] as

$$\begin{aligned}
&\int_0^b \frac{\sigma_{k-1}[f](x)-\sigma_{k-1}[f](0)}{xg'(x)}\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \\
&\sim -\sum_{\ell=0}^{L_k-1}\frac{1}{(-\mathrm{i}\omega)^{\ell+1}}\left[\frac{\gamma_{k-1,\ell}[f](b)}{g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)} - \frac{\gamma_{k-1,\ell}[f](0)}{g'(0)}\mathrm{e}^{\mathrm{i}\omega g(0)}\right]
\end{aligned} \tag{2.10}$$

with a truncation error of $O(\omega^{-L_k-1})$, where

$$\begin{aligned}
\gamma_{k-1,0}[f](x) &= \frac{\sigma_{k-1}[f](x)-\sigma_{k-1}[f](0)}{xg'(x)}, \\
\gamma_{k-1,1}[f](x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\gamma_{k-1,0}[f](x)}{g'(x)}, \\
&\vdots \\
\gamma_{k-1,\ell}[f](x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\gamma_{k-1,\ell-1}[f](x)}{g'(x)}.
\end{aligned}$$

Inserting (2.10) into the expansion (2.9) yields the complete asymptotic expansion,

$$\begin{aligned}
I[f] \sim{}& \nu_0(\omega)\sum_{k=0}^{s-1}\frac{\sigma_k[f](0)}{(-\mathrm{i}\omega)^k} - \sum_{k=1}^{s}\frac{e^{\mathrm{i}\omega g(b)}}{(-\mathrm{i}\omega)^k}\frac{\log b\left(\sigma_{k-1}[f](b)-\sigma_{k-1}[f](0)\right)}{g'(b)} \\
&-\sum_{k=1}^{s}\frac{1}{(-\mathrm{i}\omega)^k}\sum_{\ell=0}^{L_k-1}\frac{1}{(-\mathrm{i}\omega)^{\ell+1}}\left[\frac{\gamma_{k-1,\ell}[f](b)}{g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)} - \frac{\gamma_{k-1,\ell}[f](0)}{g'(0)}\mathrm{e}^{\mathrm{i}\omega g(0)}\right] \\
&-\sum_{k=1}^{s}\frac{1}{(-\mathrm{i}\omega)^k}O\left(\omega^{-(L_k+1)}\right) + O\left(\omega^{-(s+1)}\log\omega\right).
\end{aligned} \tag{2.11}$$

To ensure an error of $O(\omega^{-s-1}\log\omega)$, the index L_k must satisfy the inequality

$$L_k \ge s-k.$$

Next, we intend to derive the asymptotic analysis for the more complicated oscillatory case with stationary points. If the stationary point is $\xi \neq 0$, we can separate the integral into two parts

$$\int_0^b \log x f(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x = \int_0^{\xi/2} + \int_{\xi/2}^b . \tag{2.12}$$

The first integral on the right-hand side of (2.12) can be approximated by the formula (2.9). The asymptotic analysis for the second integral on the right-hand side of (2.12) can be found in [13]. Therefore, in the following subsection, we only concentrate on the situation in which 0 is the stationary point,

$$g(0) = g'(0) = g^{(2)}(0) = \cdots = g^{(r)}(0) = 0, \quad g^{(r+1)}(0) \neq 0.$$

Let us assume that the moments ν_j can be calculated explicitly. Since the function f is an analytic function, we use the subtraction technique again to remove the logarithmic singularity,

$$\begin{aligned} I[f] &= \int_0^b \log x f(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x \\ &= \sum_{j=0}^{r} \frac{f^j(0)}{j!} \nu_j(\omega) + \int_0^b \log x \left[f(x) - \sum_{j=0}^{r} \frac{f^j(0)}{j!} x^j \right] \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x. \end{aligned}$$

Therefore, integrating by parts, we have

$$\begin{aligned} \frac{1}{\mathrm{i}\omega} \int_0^b \log x \frac{f(x) - \sum\limits_{j=0}^{r} \frac{f^j(0)}{j!} x^j}{g'(x)} \mathrm{d}\mathrm{e}^{\mathrm{i}\omega g(x)} &= \frac{1}{\mathrm{i}\omega} \left[\log x \frac{f(x) - \sum\limits_{j=0}^{r} \frac{f^j(0)}{j!} x^j}{g'(x)} \mathrm{e}^{\mathrm{i}\omega g(x)} \right] \Bigg|_0^b \\ - \frac{1}{\mathrm{i}\omega} \int_0^b \frac{\mathrm{d}}{\mathrm{d}x} \left[\log x \frac{f(x) - \sum\limits_{j=0}^{r} \frac{f^j(0)}{j!} x^j}{g'(x)} \right] \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x. \end{aligned}$$

Iterating this process results in an asymptotic expansion,

$$\begin{aligned} I[f] \sim & \sum_{k=0}^{s-1} \sum_{j=0}^{r} \frac{\nu_j}{j!} \frac{\rho_k^{(j)}[f](0)}{(-\mathrm{i}\omega)^k} - \sum_{k=1}^{s} \frac{\mathrm{e}^{\mathrm{i}\omega g(b)}}{(-\mathrm{i}\omega)^k} \frac{\log b}{g'(b)} \left[\rho_{k-1}[f](b) - \sum_{j=0}^{r} \frac{\rho_{k-1}^{(j)}[f](0)}{j!} b^j \right] \\ & + \sum_{k=1}^{s} \frac{1}{(-\mathrm{i}\omega)^k} \int_0^b \frac{\rho_{k-1}[f](x) - \sum\limits_{j=0}^{r} \frac{\rho_{k-1}^{(j)}[f](0)}{j!} x^j}{x g'(x)} \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x + \frac{1}{(-\mathrm{i}\omega)^s} I[\rho_s[f](x)], \end{aligned} \tag{2.13}$$

where

$$\rho_0[f](x) = f(x), \qquad \rho_k[f](x) = \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{\rho_{k-1}[f](x) - \sum\limits_{j=0}^{r} \frac{\rho_{k-1}^{(j)}[f](0)}{j!} x^j}{g'(x)} \right], \quad k \geq 1,$$

and

$$\lim_{x\to 0} \frac{\rho_k[f](x) - \sum\limits_{j=0}^{r} \frac{\rho_k^j[f](0)}{j!} x^j}{g'(x)} = 0.$$

Prior to determining the error involved in truncating the expansion, we need to examine the behaviour of the generalized moments ν_j.

Lemma 6. *Let g satisfy $g(0) = g'(0) = \cdots = g^{(r)}(0) = 0$, $g^{(r+1)}(0) \neq 0$. Then*

$$|\nu_j(\omega)| = \left| \int_0^b x^j \log x \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x \right| \sim O\Big(\omega^{-\frac{j+1}{r+1}} \log \omega\Big), \qquad j \leq r, \tag{2.14}$$

$$|\nu_j(\omega)| \sim O\big(\omega^{-1}\big). \qquad j > r, \tag{2.15}$$

Proof. Firstly, since $g(x)$ is an analytic function, it may be written as a Taylor series about $x = 0$,

$$g(x) = \sum_{m=0}^{r} \frac{g^{(m)}(0)}{m!} x^m + \frac{g^{(r+1)}(\tau)}{(r+1)!} x^{r+1} = \frac{g^{(r+1)}(\tau)}{(r+1)!} x^{r+1},$$

where $\tau \in [0, x]$. Substitute this series into the formula for ν_j.

Secondly, it is required to prove (2.14). Similarly to the proof of Lemma 3, assume a small positive number $\epsilon > 0$ and decompose the integral of ν_j into two parts

$$\nu_j(\omega) = \int_0^\epsilon + \int_\epsilon^b,$$

where

$$\int_\epsilon^b x^j \log x \mathrm{e}^{\mathrm{i}\omega \frac{g^{(r+1)}(\tau)}{(r+1)!} x^{r+1}} \mathrm{d}x \sim O\left(\omega^{-1} \epsilon^{j-r} \log \epsilon\right),$$

since x^{r+1} is a non-singular function in $[\epsilon, b]$. For the first integral on the right, we deduce that

$$\begin{aligned} \left| \int_0^\epsilon x^j \log x \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x \right| &\leq \int_0^\epsilon x^j \log x \mathrm{d}x = \int_0^\epsilon x^j \mathrm{d}(x \log x - x) \\ &\sim O\big(\epsilon^{j+1} \log \epsilon\big). \end{aligned}$$

Comparing the two error bounds, we determine the error bound to be $O\left(\omega^{-\frac{j+1}{r+1}} \log \omega\right)$ by equating $\omega^{-1} \epsilon^{j-r} = \epsilon^{j+1}$. This implies that $\epsilon = \omega^{-\frac{1}{r+1}}$.

When $j > r$, the formula (2.15) can be derived by integration by parts

$$\begin{aligned} \nu_j(\omega) &= \int_0^b x^j \log x \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x = \frac{1}{\mathrm{i}\omega} \int_0^b \frac{x^j \log x}{g'(x)} \mathrm{d}\left(\mathrm{e}^{\mathrm{i}\omega g(x)}\right) \\ &= \frac{1}{\mathrm{i}\omega} \left[\frac{x^j \log x}{g'(x)} \mathrm{e}^{\mathrm{i}\omega g(x)} \right] \Bigg|_0^b - O\left(\omega^{-\frac{3}{2}}\right) \sim O\left(\omega^{-1}\right). \end{aligned}$$

$\square$

From Lemma 6, the asymptotic order of the expansion in (2.13) is $O\Big(\omega^{-\left(s+\frac{1}{r+1}\right)}\log\omega\Big)$. Hence, the oscillatory integrals in (2.13) without a singularity must be expanded with higher asymptotic order. Setting

$$
\begin{aligned}
\eta_{k-1,0}[f](x) &= \frac{\rho_{k-1}[f](x) - \sum\limits_{j=0}^{r}\frac{\rho_{k-1}^{(j)}[f](0)}{j!}x^j}{xg'(x)},\\
\eta_{k-1,1}[f](x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\eta_{k-1,0}[f](x) - \sum\limits_{n=0}^{r-1}\frac{\eta_{k-1,0}^{(n)}[f](0)}{n!}x^n}{g'(x)}\right],\\
&\vdots\\
\eta_{k-1,\ell+1}[f](x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\eta_{k-1,\ell}[f](x) - \sum\limits_{n=0}^{r-1}\frac{\eta_{k-1,\ell}^{(n)}[f](0)}{n!}x^n}{g'(x)}\right],
\end{aligned}
$$

we have

$$
\begin{aligned}
\int_0^b \eta_{k-1,0}[f](x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x = &\sum_{\ell=0}^{L_k-1}\frac{1}{(-\mathrm{i}\omega)^\ell}\sum_{n=0}^{r-1}\frac{\mu_n(0,\omega)}{n!}\eta_{k-1,\ell}^{(n)}[f](0)\\
&-\sum_{\ell=1}^{L_k}\frac{1}{(-\mathrm{i}\omega)^\ell}\left[\frac{\eta_{k-1,\ell-1}[f](x) - \sum\limits_{n=0}^{r-1}\frac{\eta_{k-1,\ell-1}^{(n)}[f](0)}{n!}x^n}{g'(x)}\mathrm{e}^{\mathrm{i}\omega g(x)}\right]\Bigg|_0^b\\
&+\frac{1}{(-\mathrm{i}\omega)^{L_k}}\int_0^b \eta_{k-1,L_k}[f](x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x. \qquad (2.16)
\end{aligned}
$$

Based on the result in [13], the error order of the integral on the left side of (2.16) is $O(\omega^{-L_k-\frac{1}{r+1}})$, given that $\mu_0 \sim O(\omega^{-\frac{1}{r+1}})$.

Considering the expansions (2.13) and (2.16), we get the asymptotic expansion for the integral (1.2) with an order-r stationary point at $x=0$,

$$
\begin{aligned}
I[f] \sim &\sum_{k=0}^{s-1}\frac{1}{(-\mathrm{i}\omega)^k}\sum_{j=0}^{r}\frac{\nu_j}{j!}\rho_k^{(j)}[f](0) - \sum_{k=1}^{s}\frac{\mathrm{e}^{\mathrm{i}\omega g(b)}}{(-\mathrm{i}\omega)^k}\frac{\log b}{g'(b)}\left[\rho_{k-1}[f](b) - \sum_{j=0}^{r}\frac{\rho_{k-1}^{(j)}[f](0)}{j!}b^j\right]\\
&+\sum_{k=1}^{s}\frac{1}{(-\mathrm{i}\omega)^k}\sum_{\ell=0}^{L_k-1}\frac{1}{(-\mathrm{i}\omega)^\ell}\sum_{n=0}^{r-1}\frac{\mu_n(0,\omega)}{n!}\eta_{k-1,\ell}^{(n)}[f](0)\\
&-\sum_{k=1}^{s}\frac{1}{(-\mathrm{i}\omega)^k}\sum_{\ell=1}^{L_k}\frac{1}{(-\mathrm{i}\omega)^\ell}\left[\frac{\eta_{k-1,\ell-1}[f](x) - \sum\limits_{n=0}^{r-1}\frac{\eta_{k-1,\ell-1}^{(n)}[f](0)}{n!}x^n}{g'(x)}\mathrm{e}^{\mathrm{i}\omega g(x)}\right]\Bigg|_0^b\\
&+\sum_{k=1}^{s}\frac{1}{(-\mathrm{i}\omega)^k}O\left(\omega^{-L_k-\frac{1}{r+1}}\right) + O\Big(\omega^{-s-\frac{1}{r+1}}\log\omega\Big). \qquad (2.17)
\end{aligned}
$$

To maintain the asymptotic order of $O\Big(\omega^{-s-\frac{1}{r+1}}\log\omega\Big)$, the index L_k should be chosen so that $L_k \geq s-k$.

As an example of the asymptotic expansion method for the logarithmic highly oscillatory integral, consider the integral

$$\int_0^1 \log x f(x) \mathrm{e}^{\mathrm{i}\omega x} \mathrm{d}x, \tag{2.18}$$

where $f(x) = (1+x)^{-1}$. The expansion (2.11) is truncated to $s = 1, 2, 3$ with the associated asymptotic error $O\left(\omega^{-s-1} \log \omega\right)$. The expansions are

$$\begin{aligned}
Q^{A,1}[f](\omega) &= \nu_0(\omega), \\
Q^{A,2}[f](\omega) &= Q^{A,1}[f](\omega) - \frac{\mathrm{i}}{\omega}\nu_0(\omega) + \frac{-\frac{\mathrm{e}^{\mathrm{i}\omega}}{2} + 1}{\omega^2}, \\
Q^{A,3}[f](\omega) &= Q^{A,2}[f](\omega) + \frac{-2}{\omega^2}\nu_0(\omega) + \frac{\mathrm{i}\left(\frac{\mathrm{e}^{\mathrm{i}\omega}}{4} - 1\right)}{\omega^3} + \frac{\mathrm{i}\left(\frac{3}{4}\mathrm{e}^{\mathrm{i}\omega} - 2\right)}{\omega^3}.
\end{aligned}$$

Fig. 2.3 displays the error for the asymptotic methods $Q^{A,s}$, $s = 1, 2, 3$ with increasing ω. Again, inclusion of more terms results in a reduced error.

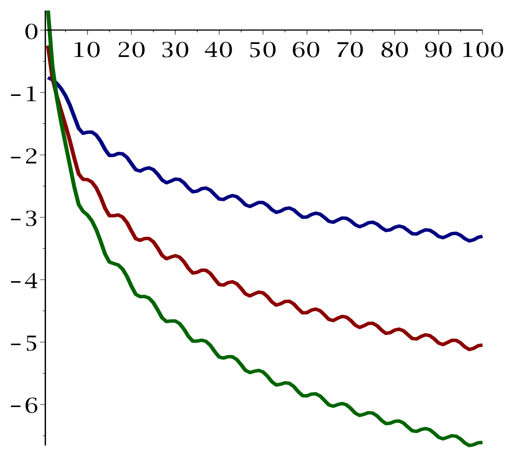

Figure 2.3: The error, $\log\left|Q^{A,s}[f] - I[f]\right|$, as a function of ω with $g(x) = x$. The colours are navyblue (the top), darkred (the middle) and darkgreen (the bottom) for $s = 1, 2, 3$.

For the stationary-point case, we consider the same integral but with $g(x) = x^2$,

$$\int_0^1 \log x f(x) \mathrm{e}^{\mathrm{i}\omega x^2} \mathrm{d}x. \tag{2.19}$$

The asymptotic expansion terms are calculated based on formula (2.17) for $s = 1, 2, 3$,

$$\begin{aligned}
Q^{A,1}[f](\omega) &= \nu_0(\omega) - \nu_1(\omega), \\
Q^{A,2}[f](\omega) &= Q^{A,1}[f](\omega) + \frac{\mathrm{i}}{2\omega}\nu_0(\omega) - \frac{\mathrm{i}}{\omega}\nu_1(\omega) + \frac{\mathrm{i}}{2\omega}\mu_0(0,\omega) + \frac{-\frac{\mathrm{e}^{\mathrm{i}\omega}}{8} + \frac{1}{4}}{\omega^2}, \\
Q^{A,3}[f](\omega) &= Q^{A,2}[f](\omega) - \frac{3}{4\omega^2}\nu_0(\omega) + \frac{2}{\omega^2}\nu_1(\omega) - \frac{1}{\omega^2}\mu_0(0,\omega) \\
&\quad + \frac{\mathrm{i}\left(-\frac{3\mathrm{e}^{\mathrm{i}\omega}}{32} + \frac{1}{4}\right)}{\omega^3} + \frac{\mathrm{i}\left(-\frac{7\mathrm{e}^{\mathrm{i}\omega}}{32} + \frac{1}{2}\right)}{\omega^3}.
\end{aligned}$$

The order of the error is $O\left(\omega^{-s-\frac{1}{2}}\log\omega\right)$, $s=1,2,3$. As evident from Fig. 2.4, the appropriate asymptotic expansion method for the logarithmic oscillatory integral with stationary points results in an error that reduces with increasing ω. However, it blows up when ω approaches to 0. Again, inclusion of more terms increases accuracy.

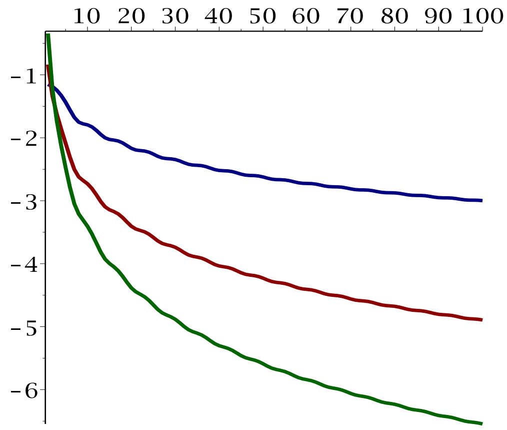

Figure 2.4: The error, $\log\left|Q^{A,s}[f]-I[f]\right|$, as a function of ω with $g(x)=x^2$. The navy blue (the top), dark red (the middle) and dark green (the bottom) corresponds to $s=1,2,3$, respectively.

3 The Filon method for singular highly oscillatory integrals

3.1 The Filon method for power-law singularity

A popular alternative to the asymptotic expansion is a Filon-type method. The essence of the technique is to interpolate the non-oscillatory function $f(x)$ subject to interpolation conditions that are determined by the asymptotic expansion. Given interpolation nodes $c_1=0<c_2<\cdots<c_\nu=b$ with multiplicities $m_1,m_2,\cdots,m_\nu\in\mathbb{N}$, respectively, we approximate $f(x)$ by a Hermite interpolatory polynomial $p_n(x)=\sum_{m=0}^{n}d_m x^m$, $n=\sum_{\ell=1}^{\nu}m_\ell-1$. We determine the coefficients d_m by solving the system of equations

$$p_n^{(j)}(c_\ell)=f^{(j)}(c_\ell),\quad \ell=1,\cdots,\nu;\quad j=0,1,\cdots,m_\ell-1.$$

The Filon method is defined as

$$Q^F[f]=\int_0^b x^{-\alpha}p_n(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x=\sum_{m=0}^{n}d_m I[x^m],$$

where $I[x^m]$ is the moment function.

Theorem 7. *Let $\nu\geq 2$, $c_0=0, c_\nu=b$, $\min\{m_1,m_\nu\}\geq s$, then*

$$I[f]-Q^F[f]\sim O(\omega^{-s-(1-\alpha)}).$$

Proof. Let $r(x)=f(x)-p_n(x)$.

$$I[r(x)]\sim\mu_0(\alpha;\omega)\sum_{k=0}^{s-1}\frac{\sigma_k[r](0)}{(-\mathrm{i}\omega)^k}-\sum_{k=1}^{s}\frac{1}{(-\mathrm{i}\omega)^k}\frac{\sigma_{k-1}[r](b)-\sigma_{k-1}[r](0)}{b^\alpha g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)}$$

$$+\frac{1}{(-\mathrm{i}\omega)^s}\int_0^b x^{-\alpha}\sigma_s[r](x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x$$

in which the functional $\sigma_k[r]$, whose form we recall from the proof of Theorem 2,

$$\sigma_0[f](x)=f(x),$$
$$\sigma_{k+1}[f](x)=x^{\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sigma_k[f](x)-\sigma_k[f](0)}{x^{\alpha}g'(x)},\qquad k\ge 0,$$

is determined from the error function $r(x)$. If

$$\sigma_k[r](0)=0,\quad \sigma_k[r](b)=0,\quad k=0,1,\cdots,s-1, \tag{3.1}$$

then the error is $O\left(\omega^{-s-(1-\alpha)}\right)$.

Next we determine what are the requirements to meet these conditions. Following from its definition, $\sigma_1[f](0)$ depends on $f'(0)$.
Let

$$f(x)=\sum_{j=0}^{\infty}\frac{f^{(j)}(0)}{j!}x^j,\qquad g(x)=\sum_{j=0}^{\infty}\frac{g^{(j)}(0)}{j!}x^j,\qquad \sigma_k[f](x)=\sum_{j=0}^{\infty}\frac{\sigma_k^{(j)}[f](0)}{j!}x^j$$

Since $\sigma_k[f](x)$ is an analytic function,

$$\sigma_2[f](x)=\frac{(1-\alpha)\sigma_1'[f](0)}{g'(0)}+O(x)$$

which indicates that $\sigma_2[f](0)$ is determined by $f'(0)$, $f''(0)$. Similarly, $\sigma_k[f](0)$ is a linear combination of $f'(0)$, $f''(0)$, $\cdots$, $f^{(k)}(0)$. In addition, for $x\ne 0$, it may be shown easily that each $\sigma_k[f](x)$, is a linear combination of $f(x)$, $f(0)$, $f'(x)$, $f'(0)$, $\cdots$, $f^{(k-1)}(x)$, $f^{(k-1)}(0)$, $f^{(k)}(x)$. Therefore, the conditions in (3.1) are met if

$$r(0)=r'(0)=\cdots=r^{(s-1)}(0)=0,$$
$$r(b)=r'(b)=\cdots=r^{(s-1)}(b)=0.$$

Hence, by setting

$$p_n^{(j)}(c_\ell)=f^{(j)}(c_\ell),\quad \ell=1,\cdots,\nu;\quad j=0,1,\cdots,m_\ell-1.$$

the conditions in 3.1 follow and the error is therefore $O\left(\omega^{-s-(1-\alpha)}\right)$. □

Once $x=0$ is an order-r stationary point for the integral, the Filon method is constructed based on the expansion in (2.6).

Theorem 8. *Given $\nu\ge 2$ and $c_0=0$, $c_\nu=b$. Let $m_1\ge s(r+1)$ and $m_\nu\ge s$, then*

$$I[f]-Q^F[f]\sim O\Big(\omega^{-s-\frac{1-\alpha}{r+1}}\Big).$$

Proof. Substituting $r(x)=f(x)-p_n(x)$ into (2.6), it is observed that if $\rho_k^{(j)}[r](0)=0$, $k=0,\cdots,s-1$, $j=0,\cdots,r$ and $\rho_k[r](b)=0$, $k=0,1,\cdots,s-1$, then the error is $O\left(\omega^{-s-\frac{1-\alpha}{r+1}}\right)$. We examine in detail the calculation of $\rho_k[f](x)$ in (2.6) and show that it depends on $f(x)$, $f'(x)$, $\cdots$, $f^{(k)}(x)$, $x\ne 0$. Let

$$f(x)=\sum_{j=0}^{\infty}\frac{f^{(j)}(0)}{j!}x^j,\qquad g(x)=\sum_{j=r+1}^{\infty}\frac{g^{(j)}(0)}{j!}x^j,$$

then

$$
\begin{aligned}
\rho_1[f](x) &= x^{\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)-\sum\limits_{j=0}^{r}\frac{f^{(j)}(0)}{j!}x^j}{x^{\alpha}g'(x)}\\
&= \frac{\sum\limits_{j=r}^{\infty}\frac{f^{(j+1)}(0)}{j!}x^j}{\sum\limits_{j=r}^{\infty}\frac{g^{(j+1)}(0)}{j!}x^j} - \alpha\frac{\sum\limits_{j=r+1}^{\infty}\frac{f^{(j)}(0)}{j!}x^j}{\sum\limits_{j=r+1}^{\infty}\frac{g^{j}(0)}{(j-1)!}x^j} - \frac{\left[\sum\limits_{j=r+1}^{\infty}\frac{f^{(j)}(0)}{j!}x^j\right]\left[\sum\limits_{j=r-1}^{\infty}\frac{g^{(j+2)}(0)}{j!}x^j\right]}{\left[\sum\limits_{j=r}^{\infty}\frac{g^{(j+1)}(0)}{j!}x^j\right]\left[\sum\limits_{j=r}^{\infty}\frac{g^{(j+1)}(0)}{j!}x^j\right]}.
\end{aligned}
$$

It follows that

$$
\rho_1[f](0) = \left(1-\frac{r+\alpha}{r+1}\right)\frac{f^{(r+1)}(0)}{g^{(r+1)}(0)}.
$$

Moreover,

$$
\rho_1^{(1)}[f](0) = \lim_{x\to 0}\frac{\rho_1[f](x)-\rho_1[f](0)}{x}
$$

is a linear combination of $f^{(r+1)}(0)$ and $f^{(r+2)}(0)$. We thus deduce that $\rho_1^{(j)}[f](0)$ depends on $f^{(\ell)}$, $\ell = r+1, r+2, \cdots, r+1+j$. Similarly, assuming $\rho_1[f](x) = \sum\limits_{j=0}^{\infty}\frac{\rho_1^{(j)}[f](0)}{j!}x^j$, $\rho_2[f](0)$ is determined by $\rho_1^{(r+1)}[f](0)$, that is, the derivatives $f^{(\ell)}$, $\ell = r+1, r+2, \cdots, 2(r+1)$ determine the value of $\rho_2[f](0)$. Also the r-th order derivative $\rho_2^{(r)}[f](0)$ is related to $f^{(\ell)}$, $\ell = r+1, r+2, \cdots, 2(r+1)+r$. Thus, we conclude that $\rho_k[f](0)$ involves $f^{(\ell)}$, $\ell = r+1, r+2, \cdots, k(r+1)$ and $\rho_k^{(j)}[f](0)$ includes $f^{(\ell)}$, $\ell = r+1, r+2, \cdots, k(r+1)+j$. Hence, setting

$$
\begin{aligned}
r^{(\ell)}(0) &= 0, \quad \ell = 0, 1, \cdots, (s-1)(r+1)+r,\\
r^{(\ell)}(b) &= 0, \quad \ell = 0, 1, \cdots, s-1,
\end{aligned}
$$

which is equivalent to setting

$$
\rho_k^{(j)}[r](0) = 0, k = 0, \cdots, s-1, j = 0, \cdots, r \;\; \rho_k[r](b) = 0, k = 0, 1, \cdots, s-1
$$

and the theorem follows. □

Since the error in approximating the analytic function $f(x)$ by the polynomial $p_n(x)$ is small when $\omega = 0$, a Filon-type method can approximate the integral for all $\omega \geq 0$, unlike the asymptotic expansion method. In addition, the Filon-type method can attain the same asymptotic error order as the asymptotic expansion for large ω so the Filon method is deemed superior.

To illustrate the theoretical analysis, we consider again the example (2.7) and construct the Filon-type method as follows

$$
\begin{aligned}
Q^{F,1}[f](\omega) &= -\frac{\mu_1(\alpha,\omega)}{2} + \mu_0(\alpha,\omega),\\
&\quad \text{with} \quad f(0) = p_n(0), \quad f(1) = p_n(1);\\
Q^{F,2}[f](\omega) &= -\frac{\mu_3(\alpha,\omega)}{4} + \frac{3\mu_2(\alpha,\omega)}{4} - \mu_1(\alpha,\omega) + \mu_0(\alpha,\omega),\\
&\quad \text{with} \quad f^{(j)}(0) = p_n^{(j)}(0), \quad j = 0,1, \quad f^{(j)}(1) = p_n^{(j)}(1), \quad j = 0,1;
\end{aligned}
$$

$$
\begin{aligned}
Q^{F,3}[f](\omega) = &-\frac{\mu_5(\alpha,\omega)}{8} + \frac{\mu_4(\alpha,\omega)}{2} - \frac{7\mu_3(\alpha,\omega)}{8} + \mu_2(\alpha,\omega) \\
&- \mu_1(\alpha,\omega) + \mu_0(\alpha,\omega), \\
&\text{with} \quad f^{(j)}(0) = p_n^{(j)}(0), \quad j = 0,1,2, \quad f^{(j)}(1) = p_n^{(j)}(1), \quad j = 0,1,2.
\end{aligned}
$$

We display the error $\log_{10}\left|Q^{F,s} - I\right|$ in Fig. 3.1, for $s = 1,2,3$. As we can see, unlike in Fig. 2.1, the error does not blow up when ω is near 0 while when $\omega \to \infty$, the error of the Filon method behaves better than that of the asymptotic method with the same s.

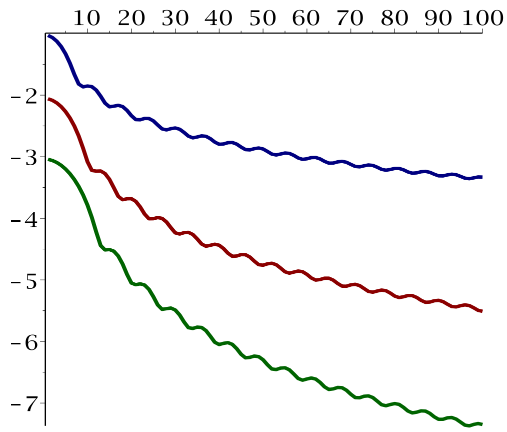

Figure 3.1: The error, $\log\left|Q^{F,s}[f] - I[f]\right|$, as a function of ω with $g(x) = x$, for $s = 1$ (navy blue, the top), 2 (dark red, the middle) and 3 (dark green, the bottom).

Next we consider the same integral with $g(x) = x^2$. Again, three Filon methods are presented as

$$
\begin{aligned}
Q^{F,1}[f](\omega) = &\frac{\mu_2(\alpha,\omega)}{2} - \mu_1(\alpha,\omega) + \mu_0(\alpha,\omega), \\
&\text{with} \quad f^{(j)}(0) = p_n^{(j)}(0), \quad j = 0,\cdots,1, \quad f(1) = p_n(1);
\end{aligned}
$$

$$
\begin{aligned}
Q^{F,2}[f](\omega) = &\frac{\mu_5(\alpha,\omega)}{-4} + \frac{3\mu_4(\alpha,\omega)}{4} - \mu_3(\alpha,\omega) + \mu_2(\alpha,\omega) - \mu_1(\alpha,\omega) + \mu_0(\alpha,\omega), \\
&\text{with} \quad f^{(j)}(0) = p_n^{(j)}(0), \quad j = 0,\cdots,3, \quad f^{(j)}(1) = p_n^{(j)}(1), \quad j = 0,1;
\end{aligned}
$$

$$
\begin{aligned}
Q^{F,3}[f](\omega) = &\frac{\mu_8(\alpha,\omega)}{8} - \frac{\mu_7(\alpha,\omega)}{2} + \frac{7\mu_6(\alpha,\omega)}{8} - \mu_5(\alpha,\omega) + \mu_4(\alpha,\omega) - \mu_3(\alpha,\omega) \\
&+ \mu_2(\alpha,\omega) - \mu_1(\alpha,\omega) + \mu_0(\alpha,\omega), \\
&\text{with} \quad f^{(j)}(0) = p_n^{(j)}(0), \quad j = 0,\cdots,5, \quad f^{(j)}(1) = p_n^{(j)}(1), \quad j = 0,1,2.
\end{aligned}
$$

The error $\log_{10}\left|Q^{F,s} - I\right|$ is plotted in Fig. 3.2, for $s = 1,2,3$. Note that the error of the Filon method is substantially smaller than the asymptotic method for both small and large ω.

3.2 The Filon method for logarithmic singularity

Consider the logarithmic integral without a stationary point.

$$
Q^F[f] = \int_0^b \log x \, p_n(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x = \sum_{m=0}^{n} d_m I[x^m],
$$

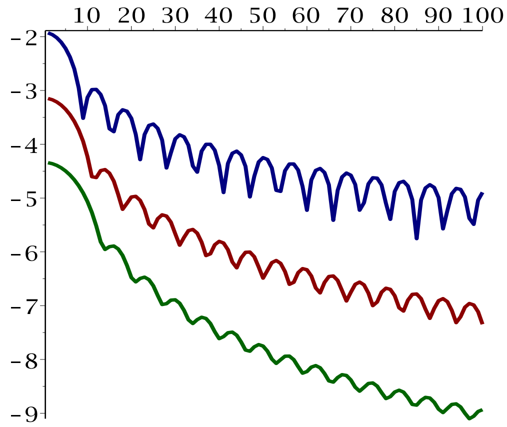

Figure 3.2: The error, $\log\left|Q^{F,s}[f]-I[f]\right|$, as a function of ω with $g(x)=x^2$. The colours are navy blue (the top), dark red (the middle) and dark green (the bottom) for $s=1,2,3$.

Theorem 9. *Suppose that $g'(x)\neq 0$ within $[0,b]$. Then*

$$I[f]-Q^F[f]\sim O\left(\omega^{-s-1}\log\omega\right),$$

for $q\geq 2$, where $c_1=0$, $c_q=b$, $m_1\geq s$, $m_q\geq s$.

Proof. In the asymptotic expansion (2.11), for every $k\geq 0$, $p=0,\cdots,k$,

$$\sigma_k[f](x)=\sum_{p=0}^{k}\sigma_{k,p}(x)f^{(p)}(x)+\sigma_{0,p}(x)f(0),$$

where $\sigma_{k,k}(x)\neq 0$ and $\sigma_{k,p}$ is a combination of derivatives of $g(x)$. We compute

$$\gamma_{k-1,0}[f](x)=\sum_{p=0}^{k-1}\frac{\sigma_{k-1,p}(x)f^{(p)}(x)-\sigma_{k-1,p}(0)f^{(p)}(0)}{xg'(x)}+[\sigma_{0,p}(x)-\sigma_{0,p}(0)]f(0),$$
$$\gamma_{k-1,1}[f](x)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\gamma_{k-1,0}[f](x)}{g'(x)}\right).$$

Hence, we deduce that $\sigma_{s-1}[f](b)$ and $\gamma_{s-1,\ell}[f](b)$ are a linear combination of $f^{(j)}(b)$, $j=0,\cdots,s-1$. Also $\sigma_{s-1}[f](0)$ and $\gamma_{s-1,\ell}[f](0)$ can be expressed in terms of the $f^{(j)}(0)$, $j=0,1,\cdots,s-1$. Hence, with nodes c_j and multiplicities m_j, $j=1,2,\cdots,q$, if we substitute the error function $r(x)=f(x)-p_n(x)$ into the asymptotic expansion, the result is an error of $O\left(\omega^{-(s+1)}\log\omega\right)$. ∎

We consider the same example as in (2.18). The interpolation function $p_n(x)$ is formed in a similar manner to that in Section 3.1 except that $\mu_m(\alpha,\omega)$ is replaced by $\nu_m(\omega)$, defined in (**??**). In the plot of Fig. 3.3, we depict the error function $\log|Q^{F,s}-I|$, $s=1$ (navy blue), 2(dark red), 3(dark green). The plots of the error function are in accordance with the error order $O\left(\omega^{-s-1}\log\omega\right)$.

Now we consider the case where stationary points are present.

Theorem 10. *Given $q\geq 2$. Let $c_1=0$, $c_q=b$, $m_1\geq s(r+1)$ and $m_q\geq s$, then*

$$I[f]-Q^F[f]\sim O\left(\omega^{-\left(s+\frac{1}{r+1}\right)}\log\omega\right),$$

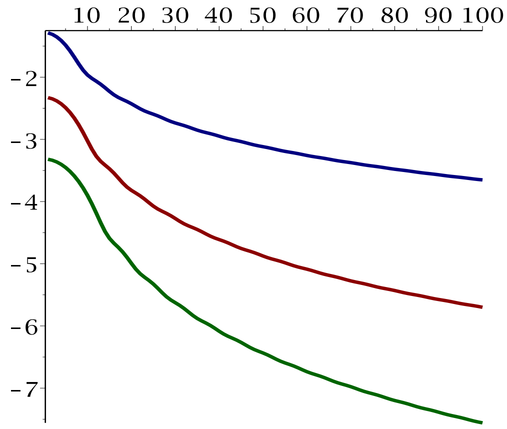

Figure 3.3: The error, $\log\left|Q^{F,s}[f]-I[f]\right|$, as a function of ω with $g(x)=x$ and $\omega\in[1,100]$. Navy blue (the top), dark red (the middle) and dark green (the bottom) correspond to $s=1,2,3$ respectively.

where

$$Q^F[f]=\sum_{m=0}^{n} d_m I[x^m].$$

Proof. Consider a first-order stationary point at $x=0$. We need to analyse the relationships among $\rho_{k-1}[f](x)$, $\eta_{k-1,\ell-1}[f](x)$, f and its derivatives in (2.17). Based on the definitions, it can be deduced that $\rho_{k-1}[f](b)$ is a combination of $f^{(j)}(b)$, $j=0,1,\cdots,k-1$. Similarly, $\eta_{k-1,\ell-1}[f](b)$ is also determined from $f^{(j)}(b)$, $j=0,1,\cdots,k+\ell-2$ for $k=1,\cdots,s$ and $\ell=1,\cdots,L_k$, $L_k=s-k+1$. Thus, the values of $\rho_{k-1}[f](b)$ and $\eta_{k-1,\ell-1}[f](b)$ in the asymptotic expansion depend upon $f^{(j)}(b)$, $j=0,1,\cdots,s-1$.

Now consider the endpoint $x=0$. Let

$$f(x)=\sum_{j=0}^{\infty}\frac{f^{(j)}(0)}{j!}x^j,\quad g(x)=\sum_{j=r+1}^{\infty}\frac{g^{(j)}(0)}{j!}x^j.$$

Incorporating this into the definition of ρ_k yields

$$\rho_1[f](0)=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sum_{j=r+1}^{\infty}\frac{f^{(j)}(0)}{j!}x^j}{\sum_{j=r}^{\infty}\frac{g^{(j+1)}(0)}{j!}x^j}\right]\Bigg|_0=\frac{f^{(r+1)}(0)}{(r+1)g^{(r+1)}(0)},$$

which implies that $\rho_1^{(k)}[f](0)$ depends on $f^{(j)}(0)$, $j=r+1,r+2,\cdots,r+1+k$. Since

$$\rho_2[f](0)=\frac{\rho_1^{(r+1)}[f](0)}{(r+1)g^{(r+1)}(0)},$$

it follows that $\rho_2[f](0)$ is a linear combination of $f^{(j)}(0)$, $j=r+1,r+2,\cdots,2(r+1)$. An immediate consequence is that $\rho_{k-1}[f](0)$ is a combination of $f^{(j)}(0)$, $j=r+1,r+2,\cdots,(k-1)(r+1)$. In addition,

$$\eta_{k-1,0}[f](0)=\frac{\sum_{j=r+1}^{\infty}\frac{\rho_{k-1}^{(j)}[f](0)}{j!}x^j}{\sum_{j=r+1}^{\infty}\frac{g^{(j)}(0)}{(j-1)!}x^j}\Bigg|_0=\frac{\rho_{k-1}^{(r+1)}[f](0)}{(r+1)g^{(r+1)}(0)},$$

in which $\rho_{k-1}^{(r+1)}[f](0)$ depends on $f^{(j)}(0)$, $j = r+1, r+2, \cdots, k(r+1)$. Applying the same technique to $\eta_{k-1,1}[f]$, we observe that $\eta_{k-1,1}[f](0)$ depends linearly on $\eta_{k-1,0}^{(r)}[f](0)$ and $\eta_{k-1,0}^{(r+1)}[f](0)$ which involves a linear combination of $f^{(j)}(0)$, $j = r+1, r+2, \cdots, (k+1)(r+1)$. More generally, it is deduced that $\eta_{k-1,\ell}[f](0)$ is related to $f^{(j)}(0)$, $j = r+1, r+2, \cdots, (k+\ell)(r+1)$ and $\eta_{k-1,\ell}^{(n)}[f](0)$ is related to $f^{(j)}(0)$, $j = r+1, r+2, \cdots, (k+\ell)(r+1)+n$.

Hence, once we substitute the error function $r(x) = f(x) - p_n(x)$ into the asymptotic expansion (2.17), the theorem follows. □

Our example is (2.19). We form the interpolation polynomial p_n with $\nu_m(\omega)$ instead of $\mu_m(\alpha, \omega)$. The plot in Fig. 3.4 displays the logarithmic error of Filon methods $Q^{F,1}$, $Q^{F,2}$ and $Q^{F,3}$. As indicated in Theorem 10, the asymptotic order $O\left(\omega^{-(s+\frac{1}{2})}\log\omega\right)$ and the numerical results are in agreement with this.

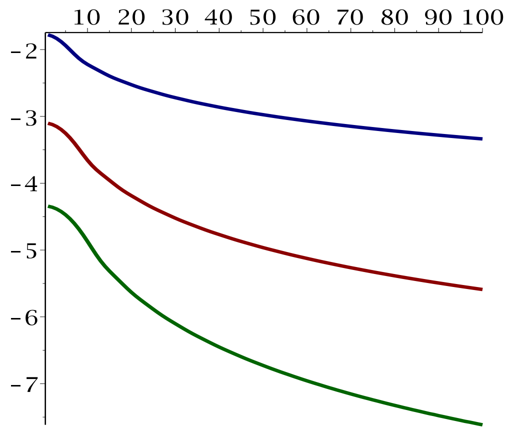

Figure 3.4: The error, $\log\left|Q^{F,s}[f] - I[f]\right|$, as a function of ω for the highly oscillatory integral with $g(x) = x^2$ and $\omega \in [1, 100]$. The colours navy blue, dark red, dark green correspond to $s = 1$ (the top), 2 (the middle), 3 (the bottom) respectively.

4 Complex-valued Gaussian quadrature

A powerful alternative to standard methods of quadrature for highly oscillatory functions is complex-valued Gaussian quadrature, whereby $\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x$ is the underlying measure. This approach has been analysed in great detail in [6] for the regular integral $\int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}x$. In this section, we are concerned with complex-valued Gaussian quadrature for singular highly oscillatory integrals of the form

$$I[f] = \int_0^b f(x)h(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x, \quad \omega \gg 1,$$

where $h(x) = x^{-\alpha}$ or $h(x) = \log x$ is weakly singular at $x = 0$ and $f(x)$ is an analytic function. We seek an n-point Gaussian quadrature formula

$$I[f] \sim \sum_{j=1}^{n} w_j f(x_j) = Q^{G,n}[f](\omega)$$

where x_j, $j = 1, \cdots, n$ are the zeros of a monic orthogonal polynomial $P_n^\omega(x)$ of degree n on $[0, b]$ with the *complex* weight function $h(x)\mathrm{e}^{\mathrm{i}\omega g(x)}$,

$$\int_0^b P_n^\omega(x) x^j h(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x = 0, \quad j = 0, \cdots, n-1, \quad n \in \mathbb{Z}_+,$$

and w_j are the corresponding weights.
We first construct $P_n^\omega(x)$. To do this, consider the Hankel matrix formed with μ_j for the power singularity and ν_j for the logarithmic singularity.

$$H_n = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix},$$

$$h_n = \det H_n, \quad n \in \mathbb{N}.$$

Then the orthogonal polynomial $P_n^\omega(x)$ is formed from [14]

$$P_n^\omega(x) = \frac{1}{h_{n-1}} \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_n & x \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} & x^n \end{bmatrix}.$$

The properties of polynomials orthogonal with respect to the complex weight $\mathrm{e}^{\mathrm{i}\omega x}$ are discussed in detail in [2, 6]. In this paper, we are concerned with the accuracy of complex Gaussian quadrature when it is applied to the singular highly oscillatory integrals and we compare it to the Filon method.

$$\begin{aligned} P_n^\omega(x_j) &= 0, \qquad j = 1, \ldots, n \\ \sum_{j=1}^n w_j f(x_j) &= Q^{G,n}[f](\omega), \end{aligned} \tag{4.1}$$

where the wights can be found by solving the linear system

$$\sum_{j=1}^n w_j x_j^k = I[x^k], \qquad k = 0, \ldots, n-1.$$

We do not analyse complex-valued Gaussian quadrature but note in passing that even the simplest non-singular case, $g(x) = x$, requires extensive mathematical machinery, since we are outside the conditions of the classical theory of Gaussian quadrature [6], while there is no theory for the general case without singularities.

To compare complex-valued quadrature to the Filon methods, we set n equal to the number of the interpolation conditions of the Filon method and consequently, both methods will involve the same number of evaluations of the function $f(x)$.

4.1 Numerical experiments for the power-law singularity

We apply the complex-valued Gaussian quadrature to the integrals in (2.7) and (2.8) and depict the error $\log_{10} \left| Q^{G,n}[f] - I[f] \right|$ in Fig. 4.1. The case without stationary points is

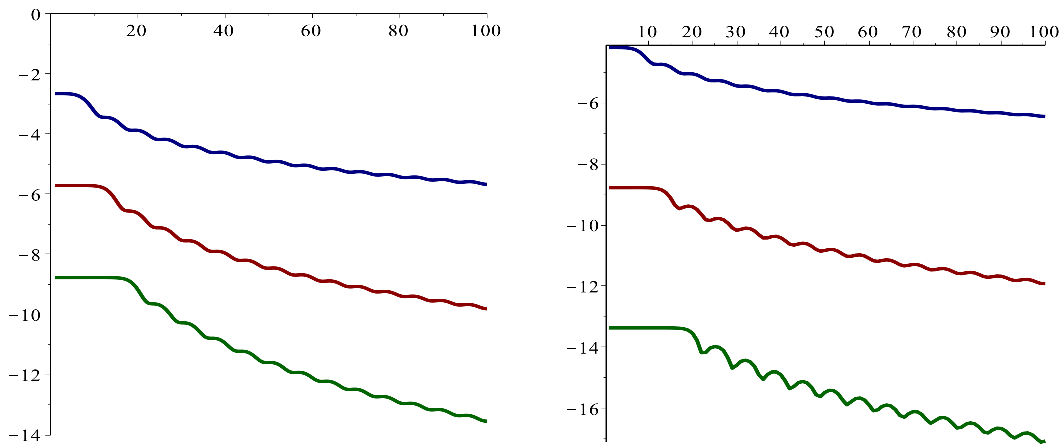

Figure 4.1: $\log_{10}\left|Q^{G,n}[f](\omega) - I[f](\omega)\right|$ as a function of ω for the integral with a power-law singularity. The left: $g(x) = x$. The colours are navy blue (the top), dark red and dark green (the bottom) for $n = 2, 4, 6$; The right: $g(x) = x^2$ and the same colour scheme for $n = 3, 6, 9$.

on the left of Fig. 4.1 while the case of 1 stationary point is on the right. We set $n = 2, 4$ on the left and $n = 3, 6$ on the right of Fig. 4.1. When $n = 2$, we obtain the two zeros of $P_2(x) = 0$. Thus the quadrature in (4.1) involves two function evaluations. Counting function evaluations, this is equivalent to $Q^{F,1}$, which involves two function evaluations at $x = 0$ and $x = 1$. However, when the results in Fig. 4.1 are compared to those in Fig. 3.1, it can be seen that the error of $Q^{G,2}$ is considerably smaller than the error of $Q^{F,2}$. Thus, the method is significantly more accurate than a Filon method that uses the same number of function evaluations. Similarly, although $Q^{G,4}$ involves the same number of function evaluations as $Q^{F,2}$, its accuracy is greater than $Q^{F,3}$. Now consider the case with a stationary point. We examine the result for $n = 3$. The number of function evaluations for $Q^{G,3}$ is equivalent to $Q^{F,1}$ as both involves evaluating the function at three points. However, when the result in Fig. 4.1 is compared to that in Fig. 3.2, the size of the error is similar to that of $Q^{F,3}$ clearly indicating its superior accuracy for the same number of function evaluations. Similar conclusions can be drawn for $n = 6$.

4.2 Numerical experiments for a logarithmic singularity

We calculate the integrals in (2.18) and (2.19). The logarithmic errors $\log_{10}\left|Q^{G,n} - I[f]\right|$ are displayed in Fig. 4.2. The cases $n = 2, 4$ for $g(x) = x$ are on the left and the cases of $n = 3, 6$ for $g(x) = x^2$ on the right. The difference between the precision of the complex-valued Gaussian quadrature and Filon method is again striking. Complex-valued Gaussian quadrature delivers substantially higher accuracy for the same number of function evaluations. However, it should be noted that in complex quadrature there is the additional cost of the computation of the orthogonal polynomial and its zeros. In other words, the simple comparison of function evaluations is incomplete, because it disregards the considerably higher price tag of linear algebra for complex-valued Gaussian quadrature.

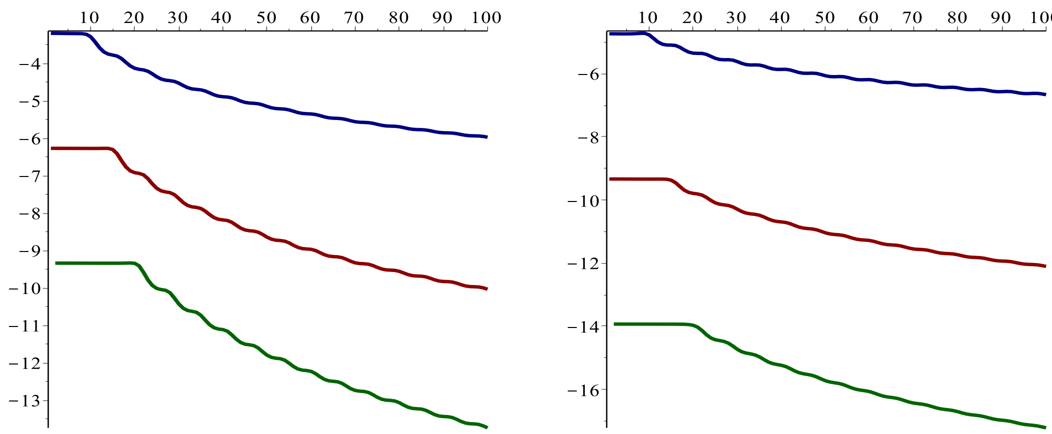

Figure 4.2: $\log_{10} \left| Q^{G,n}[f](\omega) - I[f](\omega) \right|$ as a function of ω for the integral with a logarithmic singularity. The left plot: $g(x) = x$. The colours are navy blue (the top), dark red and dark green (the bottom) for $n = 2, 4, 6$; The right: $g(x) = x^2$ and the same colour scheme for $n = 3, 6, 9$.

5 Conclusions

In this paper we have presented three methods for the computation of oscillatory integrals with logarithmic and power-law singularities. The first is a truncated asymptotic expansion. While this method is accurate for high frequencies, it fails for low frequencies. The second is a Filon-type method. It overcomes the problems at low frequencies and numerical results indicate its superiority *vis-á-vis* the asymptotic expansion method. The third method is complex-valued Gaussian quadrature. While there is no extant theory for this approach, numerical results indicate that it achieves the greatest asymptotic order and the best behaviour for small $\omega \geq 0$.

The main thrust of this paper is in demonstrating that existing theory of highly oscillatory quadrature can be extended to the singular scenario, although this requires great care and attention to detail. Once correct quadrature methods are employed, singular highly oscillatory integrals can be approximated by affordable and precise computations.

Acknowledgements

The first author wishes to thank Professor Jon Trevelyan (Durham University) for the very useful discussions about highly oscillatory logarithmic integral.

References

[1] S. Amari & J. Bornemann (1995). Efficient numerical computation of singular integrals with applications to electromagnetics, *IEEE Trans. Antennas & Propagation* **43**, 1343–1348.

[2] A. Asheim, A. Deaño, D. Huybrechs & H. Wang (2014). A Gaussian quadrature rule for oscillatory integrals on a bounded interval, *Discret. Contin. Dyn. Syst. A* **34**, 883–901.

[3] C.M. Bender & S.A. Orszag (1978). *Advanced Mathematical Methods for Scientists and Engineers*, Auckland: McGraw-Hill.

[4] S.N. Chandler-Wilde, I.G. Graham, S. Langdon & E.A. Spence (2012). Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering, *Acta Numer.* **21**, 89–305.

[5] P.J. Davis & P. Rabinowitz (1984). *Methods of Numerical Integration* (2nd ed.), New York: Academic Press.

[6] A. Deaño, D. Huybrechs & A. Iserles (2015). The kissing polynomials and their Hankel derminants. http://www.damtp.cam.ac.uk/user/na/NA_papers/NA2015_01.pdf

[7] V. Dominguez (2014). Filon–Clenshaw–Curtis rules for a class of highly-oscillatory integrals with logarithmic singularities. *J. Comp. Appld Maths*, **261** (2014), pp 299–319.

[8] G.A. Evans & K.C. Chung (2003). Some theoretical aspects of generalised quadrature methods, *J. Complexity* **19**, 272–285.

[9] L. O. Fichte, S. Lange & M. Clemens (2006). Numerical quadrature for the approximation of singular oscillating integrals, *Adv. Radio Sci.*, **4**, 11–15,

[10] M.E. Honnor, J. Trevelyan & D. Huybrechs (2010). Numerical evaluation of 2D partition of unity boundary integrals for Helmholtz problems, *J. Comp Appl Math* **234**, 1656–1662.

[11] D. Huybrechs & S. Vandewalle (2006). On the evaluation of highly oscillatory integrals by analytic continuation, *SIAM J. Numer. Anal.* **44**, 1026–1048.

[12] A. Iserles & S.P. Nørsett (2004). On quadrature methods for highly oscillatory integrals and their implementation, *BIT Numer. Math.* **44**, 755–772.

[13] A. Iserles & S.P. Nørsett (2005). Efficient quadrature of highly oscillatory integrals using derivatives, *Proc. Royal Soc. A.* **461**, 1383–1399.

[14] M.E.H. Ismail (2005). *Classical and quantum orthogonal polynomials in one variable*, Cambridge: Cambridge University Press.

[15] H. Kang, S. Xiang & G. He (2013). Computation of integrals with oscillatory and singular integrands using Chebyshev expansions, *J. Comp. Appld Maths* **242**, 141–156.

[16] D. Levin (1982). Procedures for computing one and two dimensional integrals of functions with rapid irregular oscillations, *Math. Comput.* **38**, 531–538.

[17] A. Maher & N.B. Pleshchinskii (2000). Plane electromagnetic wave scattering and diffraction in a stratified medium, *Int. Conf. on Mathematical Methods in Electromagnetic Theory (MMET 2000),* VOL. II, 426–428.

[18] K. Nesvit (2014). Scattering and propagation of the TE/TM waves on pre-fractal impedance grating in numerical results, *8th European Conference on Antennas and Propagation (EuCAP 2014)*, 2773–2777.

[19] J. Niegemann (2014). Efficient cubature rules for the numerical integration of logarithmic singularities, *2014 Int. Conf. on Electromagnetics in Advanced Applications (ICEAA)* 601–604.

[20] S. Olver (2006). Moment-free numerical integration of highly oscillatory functions, *IMA J. Numer. Anal.* **26**, 213–227.

[21] E. Perrey-Debain, J. Trevelyan & P. Bettess (2004). Wave boundary elements: a theoretical overview presenting applications in scattering of short waves, *Eng. Anal. Bound. Elem.* **28**, 131–141.

[22] E.M. Stein (1993). *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton, NJ: Princeton University Press.