A generalization of Filon-Clenshaw-Curtis quadrature for highly oscillatory integrals

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Abstract

The Filon–Clenshaw–Curtis method (FCC) for the computation of highly oscillatory integrals has been proposed by Domínguez, Graham and Smyshlayev and is known to attain surprisingly high precision. Yet, for large values of frequency ω it is not competitive with other versions of the Filon method, which use high derivatives at critical points and exhibit high asymptotic order. In this paper we propose to extend FCC to a new method, FCC+, which can attain an arbitrarily high asymptotic order while preserving the advantages of FCC. Numerical experiments are provided to illustrate that FCC+ shares the advantages of both familiar Filon methods and FCC, while avoiding their disadvantages.

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1 Introduction

The highly oscillatory integral

$$I_\omega[f] = \int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x, \qquad \omega \geq 0, \tag{1.1}$$

where $f, g \in \mathbf{C}^\infty[-1,1]$, occurs in a wide range of applications, e.g. the numerical solution of oscillatory differential and integral equations, acoustic and electromagnetic scattering and fluid mechanics. It is a difficult problem when approached by classical quadrature methods. However, once the mathematics of high oscillation is properly understood, the quadrature of (1.1) becomes fairly simple and affordable. Indeed, high oscillation is actually helpful in the design of a computational method, rather than a stumbling block. A number of innovative methods have been developed in the last two decades: an asymptotic expansion and Filon-type method (Iserles 2004, Iserles & Nørsett 2004, Iserles &

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Nørsett 2005), Levin's method (Levin 1996, Olver 2006) and numerical steepest descent (Huybrechs & Vandewalle 2006). These methods behave very well for $\omega \gg 1$.

The emphasis in the design of above methods has been on large ω, yet there is significant merit in methods which are uniformly good for all $\omega \geq 0$. This reflects much of recent research. Complex-valued Gaussian quadrature (Asheim & Huybrechs 2013, Deaño, Huybrechs & Iserles 2015) is constructed (and equally efficient) for all $\omega \geq 0$. The FCC method has been introduced in (Domínguez, Graham & Smyshlyaev 2011, Domínguez, Graham & Kim 2013, Domínguez 2014) with very low asymptotic order – its error for $\omega \gg 0$ (in the absence of stationary points) decays like $O(\omega^{-2})$, while the methods above can attain any $O(\omega^{-s})$ for $s \geq 2$. The current authors have recently analysed in (Gao & Iserles 2016) the error of the extended Filon method for the full range of ω.

The idea underlying all Filon-type methods is to replace the non-oscillatory function f in (1.1) by a polynomial p. Suppose for the time being that there is no stationary point, i.e. that $g' \neq 0$ in $[-1,1]$, and recall the asymptotic expansion

$$I_\omega[f] \sim -\sum_{k=0}^{s-1} \frac{1}{(-\mathrm{i}\omega)^{k+1}} \left[\frac{\sigma_k[f](1)}{g'(1)} \mathrm{e}^{\mathrm{i}\omega g(1)} - \frac{\sigma_k[f](-1)}{g'(-1)} \mathrm{e}^{\mathrm{i}\omega g(-1)} \right] + O\left(\omega^{-(s+1)}\right), \qquad (1.2)$$

where

$$\begin{aligned} \sigma_0[f](x) &= f(x), \\ \sigma_k[f](x) &= \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sigma_{k-1}[f](x)}{g'(x)} = \sum_{j=0}^{k} \sigma_{k,j}(x) f^{(j)}(x), \qquad \sigma_{k,k} = \frac{1}{{g'}^k(x)} \neq 0, \end{aligned}$$

(Iserles & Nørsett 2005). The functions $\sigma_{k,j}$ are independent of f, depending just on g' and its derivatives. Note that the error of Filon $I_\omega[f-p]$ is still a highly oscillatory integral. Replacing f by $f-p$ in (1.2), we obtain the error of a Filon-type method. To derive the asymptotic order $O(\omega^{-s-1})$, we let

$$p^{(j)}(1) = f^{(j)}(1), \quad p^{(j)}(-1) = f^{(j)}(-1), \qquad j = 0, 1, \cdots, s-1,$$

which determines a Filon method with a Hermite interpolation polynomial p of degree $2s-1$, referred here as the sth *plain Filon method*

$$\mathcal{Q}_\omega^{\mathbf{F},s}[f] = \int_{-1}^{1} p(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x.$$

The error of a plain Filon method can be reduced (without increasing its asymptotic order) by adding extra N interpolation points in the interval $(-1,1)$ and this leads to an *extended Filon method* (Gao & Iserles 2016). The *Filon–Clenshaw–Curtis (FCC)* procedure of (Domínguez et al. 2011, Domínguez et al. 2013) is a particular case of extended Filon where the interpolation points are chosen as $\cos(k\pi/N)$, $k = 0, \ldots, N$, and it enjoys a number of important advantages. Firstly, everything is explicit,

$$\begin{aligned} p_0 &= \frac{1}{N} \left[\frac{1}{2} f(1) + \sum_{\ell=1}^{N-1} f\left(\cos \frac{\ell\pi}{N}\right) + \frac{1}{2} f(-1) \right], \\ p_n &= \frac{2}{N} \left[\frac{1}{2} f(1) + \sum_{\ell=1}^{N-1} f\left(\cos \frac{\ell\pi}{N}\right) \cos \frac{\ell n \pi}{N} + \frac{(-1)^n}{2} f(-1) \right], \quad n = 1, \ldots, N-1, \end{aligned}$$

$$p_N = \frac{1}{N}\left[\frac{1}{2}f(1) + \sum_{\ell=1}^{N-1} f\left(\cos\frac{\ell\pi}{N}\right)(-1)^\ell + \frac{(-1)^N}{2}f(-1)\right]$$

and

$$\mathcal{Q}_\omega^{\mathbf{FCC},N,1}[f] = \int_{-1}^{1} p(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x = \sum_{n=0}^{N} p_n \int_{-1}^{1} \mathbf{T}_n(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x,$$

where $\mathbf{T}_n(x)$ is the Chebyshev polynomial of the first kind. Note that for large values of N we can compute the p_ns with Discrete Cosine Transform I (DCT-I) in $O(N \log N)$, rather than $O(N^2)$ operations.

The error of FCC (in absence of stationary points) can be computed, in a large measure because of the explicit form of the coefficients, and it is

$$\mathcal{Q}_\omega^{\mathbf{FCC},N,1}[f] - I_\omega[f] = O\left(\omega^{-2}N^{-r}\right),$$

where r is the regularity of f, which is consistent with asymptotic order 2. (Recall that ± 1 are interpolation points and this is fully compliant with the reasoning underlying extended Filon methods.) However, the low asymptotic order notwithstanding, FFC produces a fairly small error.

As an example, we display in Fig. 1.1 the errors (in logarithmic scale) committed by plain Filon with $s = 1, 2, 3, 4$ (from the top to bottom, in the right figure) and by FCC with $N = 10$ for

$$f(x) = \frac{1+x}{1+x^2}, \qquad g(x) = x. \tag{1.3}$$

Note that while $\mathcal{Q}_\omega^{\mathbf{FCC},N,1}[f]$ has asymptotic error $O(\omega^{-2})$, a plain Filon method $\mathcal{Q}_\omega^{\mathbf{F},s}[f]$ carries an asymptotic error of $O(\omega^{-s-1})$. For $s = 1$ the two asymptotic errors are the same and it can be seen from the figure on the left that FCC emerges as a decisive winner for the entire range of frequencies $\omega \geq 0$. However, the figure on the right confirms the clear fact that higher asymptotic order always wins for sufficiently large ω. Recalling that $\mathcal{Q}_\omega^{\mathbf{FCC},N,1}[f]$ requires $N+1$ function evaluations, while $\mathcal{Q}_\omega^{\mathbf{F},s}[f]$ 'costs' $2s$ function and derivative evaluations, it transpires that the better performance of Filon with $s \geq 2$ for $\omega \gg 1$ need not be accompanied by greater computational cost.

Fig. 1.2 redraws Fig. 1.1 (left) in a different form, scaling the absolute value of the error by ω^2. Since asymptotically the error behaves like $O(\omega^{-2})$, we expect both Filon (the top) and FCC (the bottom) to tend to a straight line (or at least being bounded away from zero and infinity) for $\omega \gg 1$, and this is confirmed in the figure – clearly, FCC is much more accurate!

Another way of looking at our methods is by examining the interpolation error. In Fig. 1.3, we sketch the error function $f - p$ for FCC with $N = 8$ (the left figure) and a plain Filon method for $s = 1, 2, 3, 4$ (plum, dark violet, indian red, olive drab, from the bottom to top in the right figure). As we might have expected, FCC, based on interpolation at Chebyshev points of the second kind, gives hugely better minimax approximation, while for plain Filon $p - f$ is larger in magnitude, but flat near the endpoints. It is precisely this flatness that explains superior performance for $\omega \gg 1$.

Let us set up the competing advantages of the two methods:

1. Plain Filon demonstrates much better accuracy for $\omega \gg 1$ which, after all, is the entire point of highly oscillatory quadrature.

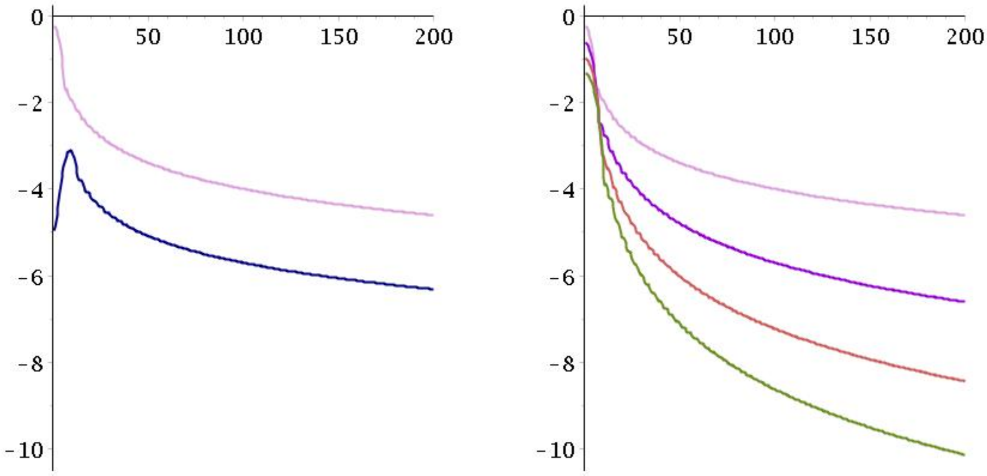

Figure 1.1: Logarithmic errors. On the left $\mathcal{Q}_\omega^{\mathbf{FCC},8,1}[f]$ (navy blue) and $\mathcal{Q}_\omega^{\mathbf{F},1}[f]$ (plum), while on the right $\mathcal{Q}_\omega^{\mathbf{F},s}[f]$, $s = 1, 2, 3, 4$ (The colors are plum, dark violet, indian red, olive drab, from the top to bottom).

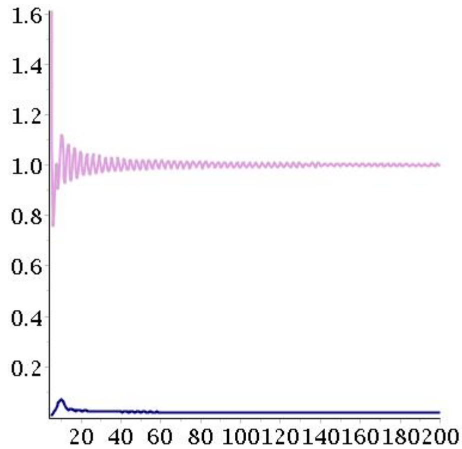

Figure 1.2: The absolute value of the error, scaled by ω^2: $\mathcal{Q}_\omega^{\mathbf{FCC},8,1}[f]$ in navy blue (the bottom) and $\mathcal{Q}_\omega^{\mathbf{F},1}[f]$ in plum (the top).

2. FCC behaves much better for small ω and has smaller uniform error for $\omega \geq 0$. "Better", rather than "best": even better behaviour can be obtained replacing Chebyshev by Jacobi points, at the price of a minor deterioration in asymptotic behaviour (Gao & Iserles 2016).

3. An issue often disregarded in papers on highly oscillatory quadrature is that FCC has a considerably simpler form: this is important once we wish to compute (1.1) for a large number of different values of ω. Specifically, we can represent any extended Filon method in the form
$$\sum_{\ell=0}^{N} \sum_{k=0}^{m_\ell - 1} b_{\ell,k}(\omega) f^{(k)}(c_\ell)$$
– here $c_0 = 1 < c_1 < \cdots < c_{N-1} < c_N = 1$ are the interpolation points with weights m_ℓ: $N = 1$, $m_0 = m_1 = s - 1$ for plain Filon and $c_\ell = \cos(\pi\ell/N)$, $m_\ell \equiv 1$

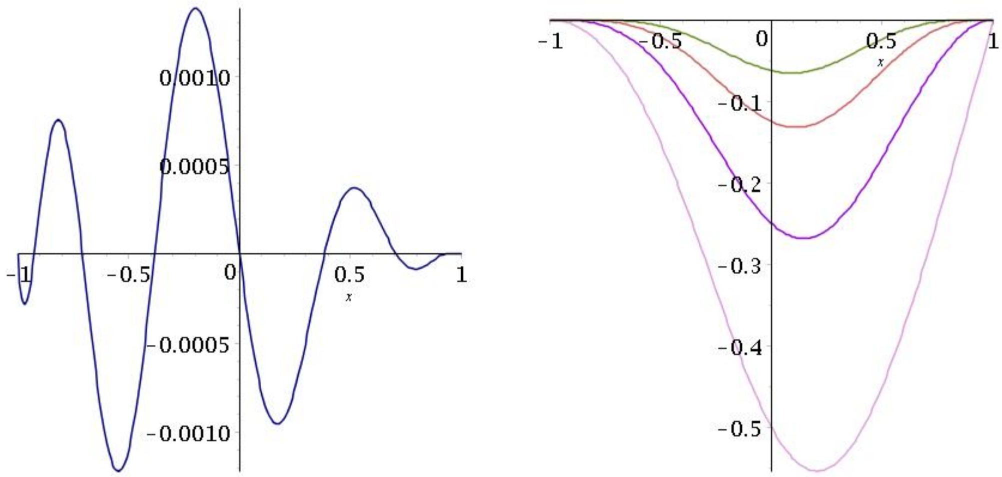

Figure 1.3: The functions $p - f$ for FCC with $N = 8$ (on the left) and for Filon with $s = 1, 2, 3, 4$ on the right. The corresponding colors are plum, dark violet, indian red, olive drab, from the bottom to top.

for FCC. Given an interpolating polynomial $p(x) = \sum_{\ell=0}^{N} \sum_{k=0}^{m_\ell - 1} p_{\ell,k}(x) f^{(k)}(c_\ell)$, we have $b_{\ell,k}(\omega) = \int_{-1}^{1} p_{\ell,k}(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x$. For plain Filon the generalised weights $b_{\ell,k}$ are fairly complicated, e.g. for $s = 2$ and $g(x) = x$ we have

$$
\begin{aligned}
b_{0,0}(\omega) &= \frac{\mathrm{e}^{-\mathrm{i}\omega}}{-\mathrm{i}\omega} - \frac{3\cos\omega}{(-\mathrm{i}\omega)^3} - \frac{3\mathrm{i}\sin\omega}{(-\mathrm{i}\omega)^4}, \quad b_{0,1}(\omega) = \frac{\mathrm{e}^{-\mathrm{i}\omega}}{(-\mathrm{i}\omega)^2} - \frac{\mathrm{e}^{\mathrm{i}\omega} + 2\mathrm{e}^{-\mathrm{i}\omega}}{(-\mathrm{i}\omega)^3} - \frac{3\mathrm{i}\sin\omega}{(-\mathrm{i}\omega)^4}, \\
b_{1,0}(\omega) &= -\frac{\mathrm{e}^{\mathrm{i}\omega}}{-\mathrm{i}\omega} + \frac{3\cos\omega}{(-\mathrm{i}\omega)^3} + \frac{3\mathrm{i}\sin\omega}{(-\mathrm{i}\omega)^4}, \quad b_{1,1}(\omega) = -\frac{\mathrm{e}^{\mathrm{i}\omega}}{(-\mathrm{i}\omega)^2} - \frac{2\mathrm{e}^{\mathrm{i}\omega} + \mathrm{e}^{-\mathrm{i}\omega}}{(-\mathrm{i}\omega)^3} - \frac{3\mathrm{i}\sin\omega}{(-\mathrm{i}\omega)^4}.
\end{aligned}
$$

Complexity grows rapidly for larger s. An alternative is to compute $\int_{-1}^{1} p(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x$ but this must be done (having formed p, e.g. by solving a linear system) separately for every ω. The formation of FCC, however, is considerably simpler! We first compute the fast cosine transform $\{\hat{p}_\ell\}_{\ell=0}^{N}$ of the sequence $f(c_\ell)$, $\ell = 0, \ldots, N$ using $O(N \log N)$ operations– the interpolating polynomial is then

$$
p(x) = \sum_{\ell=0}^{N} \hat{p}_\ell \mathbf{T}_\ell(x) \quad \text{hence} \quad \mathcal{Q}_\omega^{\mathbf{FCC},N,1}[f] = \sum_{\ell=0}^{N} \hat{p}_\ell \hat{b}_\ell(\omega),
$$

where $\hat{b}_\ell(\omega) = \int_{-1}^{1} \mathbf{T}_\ell(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x$ can be formed rapidly (Domínguez et al. 2013).

It is obvious how to reconcile 1 and 2: choose a polynomial p of degree $N+2s-2$, where $N, s \geq 1$, which interpolates f at $\cos(k\pi/N)$, $k = 1, \ldots, N-1$, and $f^{(i)}$, $i = 0, \ldots, s-1$, at ± 1, and compute $\int_{-1}^{1} p(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x$ – this is precisely the method we term "FCC+" and denote by $\mathcal{Q}_\omega^{\mathbf{FCC},N,s}[f]$. The method and its error have been already analysed in (Gao & Iserles 2016). The contention of the current paper is the method also ticks point 3: it can be derived computed to standard FCC and requires just $O(N \log N) + O(Ns) + O(s^3)$ operations.

In Section 2 we introduce FCC+ in an orderly manner and describe its basic properties. In Section 3 we present a construction of FCC+ which is consistent with the above point 3 and present some numerical experiments. We conclude with a brief Section 4, reviewing the results of this paper.

2 Basic properties of FCC+

Letting $s, N \geq 1$, we seek a polynomial p of degree $N+2s-2$ such that

$$\begin{aligned} &p^{(i)}(-1) = f^{(i)}(-1), \quad p^{(i)}(1) = f^{(i)}(1), \qquad i = 1, \ldots, s-1, \\ &p\left(\cos\frac{j\pi}{N}\right) = f\left(\cos\frac{j\pi}{N}\right), \qquad j = 0, \ldots, N. \end{aligned} \tag{2.1}$$

(Note that the case $i=0$ is automatically covered by $j=0$ and $j=N$.) We then let

$$\mathcal{Q}_\omega^{\mathbf{FCC},N,s}[f] = \int_{-1}^{1} p(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x.$$

This is our FCC+ method. We do not require $g' \neq 0$ in $[-1,1]$, yet note that, once this condition is satisfied, $\mathcal{Q}_\omega^{\mathbf{FCC},N,s}[f] - I_\omega[f] \sim O\left(\omega^{-s-1}\right)$, while for $\omega \to 0$ we have Birkhoff–Hermite quadrature with Clenshaw–Curtis nodes. It is convenient to represent

$$p(x) = \sum_{m=0}^{N+2s-2} p_m \mathbf{T}_m(x).$$

The next conceptual step is to calculate the coefficients p_m fast using DCT-I.

We need first to compute $\mathbf{T}_m^{(k)}(\pm 1)$ for relevant values of k and m. To this end we recall from DLMF (`http://dlmf.nist.gov`) that

$$\begin{aligned} &18.7.3: \qquad \mathbf{T}_m(x) = \frac{\mathbf{P}_m^{(-1/2,-1/2)}(x)}{\mathbf{P}_m^{(-1/2,-1/2)}(1)}, \\ &18.6.1: \qquad \mathbf{P}_m^{(\alpha,\alpha)}(1) = \frac{(\alpha+1)_m}{m!}, \qquad \mathbf{P}_m^{(\alpha,\alpha)}(-1) = (-1)^m \mathbf{P}_m^{(\alpha,\alpha)}(1), \\ &18.9.15: \qquad \frac{\mathrm{d}\mathbf{P}_m^{(\alpha,\alpha)}(x)}{\mathrm{d}x} = \frac{1}{2}(m+2\alpha+1)\mathbf{P}_{m-1}^{(\alpha+1,\alpha+1)}(x). \end{aligned}$$

Iterating the last expression, we have

$$\frac{\mathrm{d}^k \mathbf{P}_m^{(\alpha,\alpha)}(x)}{\mathrm{d}x^k} = \frac{1}{2^k}(m+2\alpha+1)_k \mathbf{P}_{m-k}^{(\alpha+k,\alpha+k)}(x),$$

in particular

$$\frac{\mathrm{d}^k \mathbf{P}_m^{(-1/2,-1/2)}(x)}{\mathrm{d}x^k} = \frac{(m)_k}{2^k}\mathbf{P}_{m-k}^{(k-\frac{1}{2},k-\frac{1}{2})}(x).$$

Consequently,

$$\mathbf{T}_m^{(k)}(x) = \frac{(m)_k}{2^k}\frac{\mathbf{P}_{m-k}^{(k-\frac{1}{2},k-\frac{1}{2})}(x)}{\mathbf{P}_m^{(-1/2,-1/2)}(1)}.$$

We deduce that

$$\mathbf{T}_m^{(k)}(1) = \frac{(m)_k}{2^k}\frac{\mathbf{P}_{m-k}^{(k-\frac{1}{2},k-\frac{1}{2})}(1)}{\mathbf{P}_m^{(-1/2,-1/2)}(1)} = \frac{(m)_k}{2^k}\frac{(k+\frac{1}{2})_{m-k}}{(m-k)!}\frac{m!}{(\frac{1}{2})_m},$$

where

$$(m)_k = \frac{(m+k-1)!}{(m-1)!}, \qquad (k+\tfrac{1}{2})_{m-k} = \frac{(2m)!k!}{4^{m-k}m!(2k)!}.$$

Therefore

$$\mathbf{T}_m^{(k)}(1) \quad = \quad \frac{2^k k! m(m+k-1)!}{(2k)!(m-k)!}, \qquad m,k \geq 0, \quad m+k \geq 1, \tag{2.2}$$

$$\mathbf{T}_m^{(k)}(-1) \;=\; (-1)^{m-k}\mathbf{T}_m^{(k)}(1).$$

Over to (2.1). Incorporating (2.2) and the definition of Chebyshev polynomials, we obtain the linear system

$$\begin{aligned}
\sum_{m=1}^{N+2s-2} \frac{m(m+i-1)!}{(m-i)!} p_m &= \frac{(2i)!}{2^i i!} f^{(i)}(1), \\
\sum_{m=1}^{N+2s-2} (-1)^{m-i}\frac{m(m+i-1)!}{(m-i)!} p_m &= \frac{(2i)!}{2^i i!} f^{(i)}(-1), \qquad i=1,\ldots,s-1, \\
\sum_{m=0}^{N+2s-2} \cos\!\left(\frac{jm\pi}{N}\right) p_m &= f\!\left(\cos\frac{j\pi}{N}\right), \qquad j=0,\ldots,N.
\end{aligned} \tag{2.3}$$

Let

$$\begin{aligned}
&\hat{p}_0 = 2p_0, \qquad \hat{p}_k = p_k, \quad k=1,\ldots,N-1, \qquad \hat{p}_N = 2p_N, \\
&h_j = f\!\left(\cos\frac{j\pi}{N}\right) - \sum_{m=N+1}^{N+2s-2} \cos\!\left(\frac{jm\pi}{N}\right) p_m, \\
&\quad = f_j - \sum_{m=N+1}^{N+2s-2} \cos\!\left(\frac{jm\pi}{N}\right) p_m, \quad f_j = f\!\left(\cos\frac{j\pi}{N}\right), \qquad j=0,\ldots,N.
\end{aligned}$$

The bottom line in (2.3) is equivalent to

$$\frac{1}{2}\hat{p}_0 + \sum_{m=1}^{N-1}\cos\!\left(\frac{jm\pi}{N}\right)\hat{p}_m + \frac{(-1)^j}{2}\hat{p}_N = h_j, \qquad j=0,\ldots,N.$$

This is DCT-I, $\mathcal{C}_N\hat{\mathbf{p}} = \mathbf{h}$, and its inverse is $\mathcal{C}_N^{-1} = (2/N)\mathcal{C}_N$. We deduce that

$$\hat{p}_m = \frac{2}{N}\left[\frac{1}{2}h_0 + \sum_{j=1}^{N-1}\cos\!\left(\frac{jm\pi}{N}\right)h_j + \frac{(-1)^m}{2}h_N\right], \qquad m=0,\ldots,N. \tag{2.4}$$

Consequently, the coefficients p_m, $m=0,\ldots,N$, can be calculated in $O(N\log N)$ operations with DCT-I from the *unknown* coefficients p_m for $m = N+1,\ldots,N+2s-2$. Specifically, we need first to solve a *linear* system of $2s-2$ unknowns $p_{N+1},\ldots,p_{N+2s-2}$, and subsequently recover $p_0,\ldots,p_N$ as above.

3 The construction of FCC+

In this section, we complete the construction of FCC+ for a general $s \ge 2$ by identifying explicitly the linear system for $p_{N+1},\ldots,p_{N+2s-2}$. We commence with the simplest case, $s=2$, subsequently generalising to all $s \ge 2$.

3.1 $s=2$

We have

$$h_j = f_j - \cos\!\left(\frac{j(N+1)\pi}{N}\right)p_{N+1} - \cos\!\left(\frac{j(N+2)\pi}{N}\right)p_{N+2}$$

$$= f_j - (-1)^j \cos\frac{j\pi}{N} p_{N+1} - (-1)^j \cos\frac{2j\pi}{N} p_{N+2}, \qquad j = 0, \ldots, N,$$

and it follows from (2.4) that

$$\begin{aligned}
\hat{p}_m = \frac{2}{N} \Bigg\{ & \frac{1}{2} f_0 - \frac{1}{2} p_{N+1} - \frac{1}{2} p_{N+2} \\
& + \sum_{j=1}^{N-1} \cos\left(\frac{jm\pi}{N}\right) \left[f_j - (-1)^j \cos\frac{j\pi}{N} p_{N+1} - (-1)^j \cos\frac{2j\pi}{N} p_{N+2} \right] \\
& \quad + \frac{(-1)^m}{2} f_N + \frac{(-1)^{m+N}}{2} p_{N+1} - \frac{(-1)^{m+N}}{2} p_{N+2} \Bigg\} \\
= \frac{2}{N} & \left[\frac{1}{2} f_0 + \sum_{j=1}^{N-1} \cos\left(\frac{jm\pi}{N}\right) f_j + \frac{(-1)^m}{2} f_N \right] \\
& - \frac{2}{N} p_{N+1} \left[\frac{1}{2} + \sum_{j=1}^{N-1} (-1)^j \cos\left(\frac{jm\pi}{N}\right) \cos\frac{j\pi}{N} - \frac{(-1)^{m+N}}{2} \right] \\
& - \frac{2}{N} p_{N+2} \left[\frac{1}{2} + \sum_{j=1}^{N-1} (-1)^j \cos\left(\frac{jm\pi}{N}\right) \cos\frac{2j\pi}{N} + \frac{(-1)^{m+N}}{2} \right].
\end{aligned}$$

Since

$$\sum_{j=0}^{N-1} (-1)^j \cos\frac{jM\pi}{N} = \begin{cases} \dfrac{1-(-1)^{N+M}}{2}, & M \neq N, \\ N, & M = N, \end{cases}$$

for $m = 0, 1, \ldots, N-2$ and $m = N$, we have

$$\begin{aligned}
& \frac{1}{2} + \sum_{j=1}^{N-1} (-1)^j \cos\left(\frac{jm\pi}{N}\right) \cos\frac{j\pi}{N} - \frac{(-1)^{m+N}}{2} \\
&= \frac{1}{2} + \frac{1}{2} \sum_{j=1}^{N-1} (-1)^j \cos\frac{j(m+1)\pi}{N} + \frac{1}{2} \sum_{j=1}^{N-1} (-1)^j \cos\frac{j(m-1)\pi}{N} - \frac{(-1)^{m+N}}{2} \\
&= \frac{1}{2} \sum_{j=0}^{N-1} (-1)^j \cos\frac{j(m+1)\pi}{N} + \frac{1}{2} \sum_{j=0}^{N-1} (-1)^j \cos\frac{j(m-1)\pi}{N} - \frac{1+(-1)^{m+N}}{2} = 0,
\end{aligned}$$

while for $m = N-1$ the sum is $N/2$.

In exactly the same fashion

$$\begin{aligned}
& \frac{1}{2} + \sum_{j=1}^{N-1} (-1)^j \cos\left(\frac{jm\pi}{N}\right) \cos\frac{2j\pi}{N} + \frac{(-1)^{m+N}}{2} \\
&= \frac{1}{2} \sum_{j=0}^{N-1} (-1)^j \cos\frac{j(m+2)\pi}{N} + \frac{1}{2} \sum_{j=0}^{N-1} (-1)^j \cos\frac{j(m-2)\pi}{N} - \frac{1-(-1)^{m+N}}{2}
\end{aligned}$$

and we deduce that the sum equals zero for $m \neq N-2$ and $N/2$ for $m = N-2$.

Let

$$\check{p}_m = \frac{2}{N} \left[\frac{1}{2} f_0 + \sum_{j=1}^{N-1} \cos\left(\frac{jm\pi}{N}\right) f_j + \frac{(-1)^m}{2} f_N \right], \qquad m = 0, \ldots, N \tag{3.1}$$

be the coefficients that feature in the original FCC. We thus deduce that

$$\hat{p}_m = \check{p}_m, \qquad m \neq N-2, N-1,$$
$$\hat{p}_{N-2} = \check{p}_{N-2} - p_{N+2}, \qquad \hat{p}_{N-1} = \check{p}_{N-1} - p_{N+1},$$

– except for $m = N-2$ and $m = N-1$, exactly like in standard FCC!

We have the two remaining equations from (2.3), namely

$$\sum_{m=1}^{N+2} m^2 p_m = f'(1), \qquad \sum_{m=1}^{N+2} (-1)^{m-1} m^2 p_m = f'(-1).$$

Therefore

$$4Np_{N+1} + 8Np_{N+2} = f'(1) - \sum_{m=1}^{N-1} m^2 \check{p}_m - \frac{N^2}{2}\check{p}_N,$$
$$4Np_{N+1} - 8Np_{N+2} = (-1)^N f'(-1) + \sum_{m=1}^{N-1} (-1)^{N-m} m^2 \check{p}_m + \frac{N^2}{2}\check{p}_N.$$

All this results in two linear equations,

$$p_{N+1} = \frac{1}{8N}\left\{ f'(1) + (-1)^N f'(-1) - \sum_{m=1}^{N-1} [1 - (-1)^{N-m}] m^2 \check{p}_m \right\},$$
$$p_{N+2} = \frac{1}{16N}\left\{ f'(1) - (-1)^N f'(-1) - \sum_{m=1}^{N-1} [1 + (-1)^{N-m}] m^2 \check{p}_m - N^2 \check{p}_N \right\},$$

for the unknowns p_{N+1} and p_{N+2}.

3.2 A general $s \geq 2$

Given a general $s \geq 2$, we have

$$h_j = f_j - \sum_{n=N+1}^{N+2s-2} \cos\left(\frac{jn\pi}{N}\right) p_n = f_j - \sum_{n=1}^{2s-2} (-1)^j \cos\left(\frac{jn\pi}{N}\right) p_{N+n}, \quad j = 0, \ldots, N.$$

Subject to the definition (3.1), for $m = 0, \ldots, N$, the formula (2.4) can be written as

$$\begin{aligned}
\hat{p}_m &= \frac{2}{N}\left[\frac{1}{2}h_0 + \sum_{j=1}^{N-1} \cos\left(\frac{jm\pi}{N}\right) h_j + \frac{(-1)^m}{2} h_N\right] \\
&= \check{p}_m - \frac{2}{N}\sum_{n=1}^{2s-2} p_{N+n}\left[\frac{1}{2} + \sum_{j=1}^{N-1} (-1)^j \cos\left(\frac{jm\pi}{N}\right)\cos\left(\frac{jn\pi}{N}\right) + \frac{(-1)^{m+n+N}}{2}\right] \\
&= \check{p}_m - \frac{1}{N}\sum_{n=1}^{2s-2} p_{N+n}\left[\sum_{j=0}^{N-1} (-1)^j \cos\left(\frac{j(m+n)\pi}{N}\right) + \sum_{j=0}^{N-1} (-1)^j \cos\left(\frac{j(m-n)\pi}{N}\right)\right. \\
&\qquad \left. + (-1)^{m+n+N} - 1\right].
\end{aligned}$$

Since

$$\sum_{j=0}^{N-1} (-1)^j \cos\left(\frac{j(m-n)\pi}{N}\right) = \frac{1 - (-1)^{m-n+N}}{2}, \quad m = 0, \ldots, N, \;\; n = 1, \ldots, 2s-2,$$

$$\sum_{j=0}^{N-1}(-1)^j\cos\Bigl(\frac{j(m+n)\pi}{N}\Bigr)=\frac{1-(-1)^{m+n+N}}{2},\quad m+n\neq N,$$
$$\sum_{j=0}^{N-1}(-1)^j\cos\Bigl(\frac{j(m+n)\pi}{N}\Bigr)=N,\quad m=N-2s+2,\dots,N-1,\ \ n=N-m,$$

we obtain

$$\begin{aligned}
&\hat p_m=\check p_m, && m=0,\dots,N-2s+1 \ \text{ or } \ m=N,\\
&\hat p_m=\check p_m-p_{2N-m}, && m=N-2s+2,\dots,N-1.
\end{aligned}\tag{3.2}$$

Over to the remaining conditions in (2.3). Substituting (3.2), we have

$$\begin{aligned}
&\sum_{m=1}^{2s-2}\left[\frac{(N+m)(N+m+i-1)!}{(N+m-i)!}-\frac{(N-m)(N-m+i-1)!}{(N-m-i)!}\right]p_{N+m}\\
&=\frac{(2i)!}{2^i i!}f^{(i)}(1)-\sum_{m=1}^{N-1}\frac{m(m+i-1)!}{(m-i)!}\check p_m-\frac{N(N+i-1)!}{2(N-i)!}\check p_N,\\
&\sum_{m=1}^{2s-2}(-1)^m\left[\frac{(N+m)(N+m+i-1)!}{(N+m-i)!}-\frac{(N-m)(N-m+i-1)!}{(N-m-i)!}\right]p_{N+m}\\
&=(-1)^{N+i}\frac{(2i)!}{2^i i!}f^{(i)}(-1)-(-1)^N\sum_{m=1}^{N-1}(-1)^m\frac{m(m+i-1)!}{(m-i)!}\check p_m-\frac{N(N+i-1)!}{2(N-i)!}\check p_N
\end{aligned}$$

for $i=1,\dots,s-1$. This can be separated into two smaller systems adding and subtracting the equations, but just now we will not pursue this route. Solving the equations directly, it can be derived that

$s=2$:

$$\begin{aligned}
p_{N+1}&=\frac{f'(1)+(-1)^N f'(-1)}{8N}-\frac{1}{8N}\sum_{m=1}^{N-1}[1-(-1)^{N-m}]m^2\check p_m,\\
p_{N+2}&=\frac{f'(1)-(-1)^N f'(-1)}{16N}-\frac{1}{16N}\sum_{m=1}^{N-1}[1+(-1)^{N-m}]m^2\check p_m-\frac{N}{16}\check p_N.
\end{aligned}$$

$s=3$:

$$\begin{aligned}
p_{N+1}&=\frac{2N^2+17}{128N}[f'(1)+(-1)^N f'(-1)]-\frac{3}{128N}[f''(1)-(-1)^N f''(-1)]\\
&\quad-\frac{1}{128N}\sum_{m=1}^{N-1}m^2(2N^2-m^2+18)[1-(-1)^{N-m}]\check p_m,\\
p_{N+2}&=\frac{2N^2+31}{384N}[f'(1)-(-1)^N f'(-1)]-\frac{1}{128N}[f''(1)+(-1)^N f''(-1)]\\
&\quad-\frac{1}{384N}\sum_{m=1}^{N-1}m^2(2N^2-m^2+32)[1+(-1)^{N-m}]\check p_m-\frac{N(N^2+32)}{384}\check p_N,\\
p_{N+3}&=-\frac{2N^2+1}{384N}[f'(1)+(-1)^N f'(-1)]+\frac{1}{128N}[f''(1)-(-1)^N f''(-1)]
\end{aligned}$$

$$
+ \frac{1}{384N} \sum_{m=1}^{N-1} m^2(2N^2 - m^2 + 2)[1 - (-1)^{N-m}]\check{p}_m,
$$

$$
p_{N+4} = -\frac{2N^2+7}{768N}[f'(1) - (-1)^N f'(1)] + \frac{1}{256N}[f''(1) + (-1)^N f''(-1)]
$$

$$
+ \frac{1}{768N} \sum_{m=1}^{N-1} m^2(N^2 - m^2 + 8)[1 + (-1)^{N-m}]\check{p}_m + \frac{N(N^2+8)}{768}\check{p}_N.
$$

For $s \geq 4$ is probably makes more sense to solve the equations directly than to write down a general solution like above.

3.3 A numerical example

We have demonstrated that FCC+ can attain any asymptotic order $s \geq 1$ at a cost similar to standard FCC: a single DCT-I computation and a solution of an $(2s-2) \times (2s-2)$ linear system. (Since s is likely to be small, the extra expense of solving a linear system is marginal.) To illustrate the gain in accuracy, we revisit the problem (1.3). Fig. 3.1 displays the magnitude of the error, $\log \left| \mathcal{Q}_\omega^{\mathbf{FCC},8,s}[f] - I[f] \right|$, for $s = 1$ (i.e., plain FCC), 2, 3, 4 and $N = 8$, from the top to bottom. We observe that the curves have for $\omega \gg 1$ the same *slope* as plain Filon methods in Fig. 1.1 (right), but the curves lie much lower: the error is considerably smaller! Moreover, for small $\omega \geq 0$ the performance is starkly better than of plain Filon and marginally improves with greater $s \geq 1$.

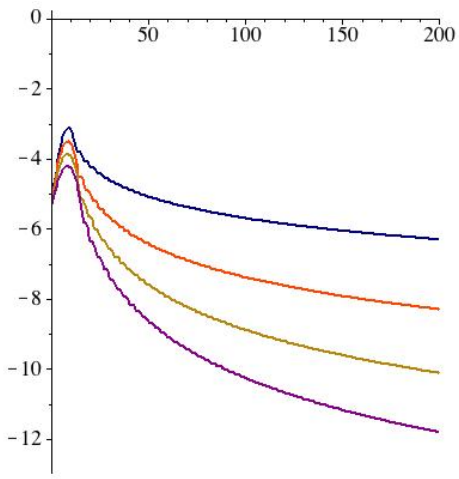

Figure 3.1: The error (in logarithmic scale) of $\mathcal{Q}_\omega^{\mathbf{FCC},8,1}[f]$ (navy blue), $\mathcal{Q}_\omega^{\mathbf{FCC},8,2}[f]$ (orange red), $\mathcal{Q}_\omega^{\mathbf{FCC},8,3}[f]$ (dark goldenrod) and $\mathcal{Q}_\omega^{\mathbf{FCC},8,4}[f]$ (dark magenta). The corresponding order is from the top to bottom.

4 Conclusions

Several effective algorithms to compute highly oscillatory integrals have emerged in the last two decades. Among these methods, an extended Filon method enjoys the advantage of simplicity and flexibility: once we can compute the *moments* $\int_{-1}^{1} x^k e^{i\omega g(x)} dx$, $k \in \mathbb{Z}_+$, we can construct an extended Filon method with great ease.

Choosing an extended Filon method, we need to make three choices: how many derivatives to compute at critical points, how many extra interpolation points to add and how to choose these interpolation points. We are guided by three goals: good performance for large ω, good performance for small $\omega \geq 0$ (and hence good uniform performance) and simplicity of the underlying expressions and ease of their computation. Plain Filon method exhibits excellent behaviour for $\omega \gg 1$, while FCC is superior for small $\omega \geq 0$, can be derived cheaply. and has a pleasingly simple form. In this paper we have introduced an approach that shares the advantages of both.

In this paper we have focused on the case $g'(x) \neq 0$, $x \in [-1, 1]$, but, like FCC in (Domínguez et al. 2013), our approach can be easily generalised to the presence of stationary points, where g' vanishes.

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