Spectral computation of highly oscillatory integral equations in laser theory

Jing GAO *
School of Mathematics and Statistics, Xi'an Jiaotong University
E-mail: jgao@xjtu.edu.cn

Marissa CONDON
School of Electronic Engineering, Dublin City University
E-mail: marissa.condon@dcu.ie

Arieh ISERLES
DAMTP, Centre for Mathematical Sciences, University of Cambridge
E-mail: ai10@cam.ac.uk

Abstract

We are concerned in this paper with the numerical computation of the spectra of highly oscillatory integrals that arise in laser simulations. Discretised using the modified Fourier basis, the spectral problem for the integral equation is converted into two independent infinite systems of linear equations whose unknowns are the coefficients of the modified Fourier functions, namely the cosine and shifted sine functions, respectively. Each (m, n) entry of the resulting coefficient matrices can be represented exactly by expressions involving the error function with an argument that involves the oscillatory parameter ω and the numbers m and n. Moreover, considering the behaviour of the error function for a large argument, the asymptotics for each entry are analysed for large ω or for large m and n and this enables efficient truncation of the infinite systems. Numerical experiments are provided to illustrate the effectiveness of this method.

AMS subject classifications: 65R20, 65T40, 78A60.

Keywords: Fox–Li integral equation of the first kind, hyperbolic cross approximation, modified Fourier basis, error function, asymptotic expansion.

1 Introduction

Numerical simulation of lasers is an important topic in laser design and analysis. For example, it is important for exploration of the effect of aberrations on laser mirrors and in mode shaping and mode control. However, numerical simulation is typically a computationally intensive task for large Fresnel numbers (Yoo, Jeong, Lee, Rhee & Cho 2004). Much research has been done in this area and the iterative technique of Fox and Li (Fox & Li 1961) has been the basis for many algorithms. Commercial packages such as Light

*The work is supported by the Projects of International Cooperation and Exchanges NSFC-RS (Grant No. 11511130052), the Key Science and Technology Program of Shaanxi Province of China (Grant No. 2016GY-080), and the Fundamental Research Funds for the Central Universities.

Trans (`http://www.lighttrans.com/983.html`) employ this approach. Asymptotic theories have also played a vital role in the laser numerical simulation in (Horwitz 1973) and (Berry, Storm & Saarloos 2001). Knowledge of the eigenvalues and eigenfunctions of laser resonators plays an important part in the design and analysis of laser resonators. The corresponding mathematical model is (Berry 2001, Berry et al. 2001).

$$\int_{-1}^{1} f(y)\mathrm{e}^{\mathrm{i}\omega(x-\gamma y)^2}\mathrm{d}y = \lambda f(x), \quad x \in [-1,1], \tag{1.1}$$

where $\gamma \leq 1$ is related to the magnification constant, M, namely $\gamma = M^{-1}$ (Siegman 1974). The case $\gamma > 1$ corresponds to stable laser resonators such as the displacement measuring interferometer systems and range finders (Muntean & Svich 2000); For $\gamma < 1$, the model represents unstable resonators employed in high power lasers (Muntean & Svich 2000).

Asymptotic properties of the eigenfunctions for (1.1) have been analysed in (Horwitz 1973). Furthermore, compared to the case where $\gamma = 1$, the pseudospectra of (1.1) are observed to be smaller and exhibit fractal behaviour (Karman, McDonald, New & Woerdman 1999). For the case of (1.1) with $\gamma = 1$, known as the *Fox–Li integral equation*, Brunner, Iserles & Nørsett (2011) applied the finite section method to reduce (1.1) to an algebraic eigenvalue problem. In the case of the more general laser resonator (1.1) we present in this paper a new expression involving the error function for each entry of the coefficient matrix in the finite section method. We examine the asymptotics with a view to obtaining an efficient numerical implementation of this method that meets a specific accuracy requirement.

The finite section can be described as follows: A set of orthonormal basis functions $\mathcal{H} = \{\phi_n(x)\}$, $n \geq 0$, with $x \in [-1,1]$, is taken as the trial basis to approximate the unknown function $f(x) = \sum_{n=0}^{\infty} \hat{f}_n \phi_n(x)$. By taking a second set of othonormal test functions $\{\psi_n(x)\}$, the integral equation (1.1) is converted to an infinite algebraic linear system

$$\mathcal{Q}\hat{\mathbf{f}} = \lambda\hat{\mathbf{f}}, \tag{1.2}$$

where

$$\mathcal{Q} := (q_{m,n})_{m,n\geq 0}, \quad q_{m,n} = \int_{-1}^{1}\int_{-1}^{1} \psi_m(x)\phi_n(y)\mathrm{e}^{\mathrm{i}\omega(x-\gamma y)^2}\mathrm{d}x\,\mathrm{d}y$$

and the dimensions of $\mathcal{Q}$ and the vector $\hat{\mathbf{f}}$ are infinite. To implement the computation of the eigenvalue λ, the linear equation (1.2) is truncated, being guided by the decay rate of the entries in $\mathcal{Q}$. In this sense, the computation of the double oscillatory integral representing the entry is particularly important. It can be written in the general form

$$\begin{aligned}
\mathcal{I}_\omega[\phi_n, \psi_m] &= \int_{-1}^{1}\int_{-1}^{1} \psi_m(x)\phi_n(y)\mathrm{e}^{\mathrm{i}\omega(x-\gamma y)^2}\mathrm{d}x\mathrm{d}y \\
&= \int_{-1}^{1}\int_{-1-\gamma y}^{1-\gamma y} \psi_m(t+\gamma y)\phi_n(y)\mathrm{e}^{\mathrm{i}\omega t^2}\mathrm{d}t\mathrm{d}y \\
&= \int_{-1-\gamma}^{1+\gamma}\int_{\max\{-1,-(1+t)/\gamma\}}^{\min\{1,(1-t)/\gamma\}} \psi_m(t+\gamma y)\phi_n(y)\mathrm{d}y\mathrm{e}^{\mathrm{i}\omega t^2}\mathrm{d}t.
\end{aligned}$$

The integration intervals can be simplified noting that

$$\int_{-1-\gamma}^{1+\gamma}\int_{\max\{-1,-(1+t)/\gamma\}}^{\min\{1,(1-t)/\gamma\}} = \int_{-1+\gamma}^{1-\gamma}\int_{-1}^{1} + \int_{1-\gamma}^{1+\gamma}\int_{-1}^{(1-t)/\gamma} + \int_{-1-\gamma}^{-1+\gamma}\int_{-(1+t)/\gamma}^{1},$$

for $\gamma \in (0,1]$. Likewise, for $\gamma \geq 1$,

$$\int_{-1-\gamma}^{1+\gamma} \int_{\max\{-1,-(1+t)/\gamma\}}^{\min\{1,(1-t)/\gamma\}} = \int_{-1+\gamma}^{1+\gamma} \int_{-1}^{(1-t)/\gamma} + \int_{-1-\gamma}^{1-\gamma} \int_{-(1+t)/\gamma}^{1} + \int_{1-\gamma}^{-1+\gamma} \int_{-(1+t)/\gamma}^{(1-t)/\gamma}.$$

(Although we are concerned here with the parameter $0 < \gamma \leq 1$, the analysis in this paper can be easily extended to the case $\gamma > 1$).

The first conclusion is that $\mathcal{I}_\omega[\phi_n, \psi_m] \sim O\left(\omega^{-1/2}\right)$ for $\omega \gg 1$. The reason for this is that

$$\mathcal{I}_\omega[\phi_n, \psi_m] = \int_{-1-\gamma}^{1+\gamma} F(t) \mathrm{e}^{\mathrm{i}\omega t^2} \mathrm{d}t,$$

where

$$F(t) = \int_{\max\{-1,-(1+t)/\gamma\}}^{\min\{1,(1-t)/\gamma\}} \psi_m(t+\gamma y)\phi_n(y)\mathrm{d}y$$

is non-oscillatory, therefore the asymptotic decay is govern by a univariate oscillatory integral with a first-order stationary point (Deaño, Huybrechs & Iserles 2018).

We use the modified Fourier basis for application of the finite section method to (1.1). Thus let

$$\mathcal{H} = \{\phi_n(x) = \cos n\pi x \,:\, n \geq 0\} \cup \left\{\varphi_n(x) := \sin\left(n + \frac{1}{2}\right)\pi x \,:\, n \geq 0\right\},$$

be the orthonormal basis in $\mathbb{L}_2[-1,1]$ spanned by the eigenfunctions of the Laplace–Neumann problem (Iserles & Nørsett 2008). Note that the ϕ_ns are even and the φ_ns odd.

The eigenfunction f in (1.1) is expanded in the basis $\{\phi_n, \varphi_n\}$,

$$f(x) = \sum_{n=0}^{\infty} \hat{f}_n^C \phi_n(x) + \sum_{n=0}^{\infty} \hat{f}_n^S \varphi_n(x).$$

Taking the same test basis and inserting it in (1.1) yields a system of linear equations,

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{f}}^C \\ \hat{\mathbf{f}}^S \end{pmatrix} = \lambda \begin{pmatrix} \hat{\mathbf{f}}^C \\ \hat{\mathbf{f}}^S \end{pmatrix},$$

where the elements of the matrices are

$$\mathcal{A} := (a_{m,n})_{m,n\geq 0}, \qquad a_{m,n} = \int_{-1}^{1}\int_{-1}^{1} \phi_m(x)\phi_n(y)\mathrm{e}^{\mathrm{i}\omega(x-\gamma y)^2}\mathrm{d}y\mathrm{d}x, \tag{1.3}$$

$$\mathcal{B} := (b_{m,n})_{m,n\geq 0}, \qquad b_{m,n} = \int_{-1}^{1}\int_{-1}^{1} \phi_m(x)\varphi_n(y)\mathrm{e}^{\mathrm{i}\omega(x-\gamma y)^2}\mathrm{d}y\mathrm{d}x,$$

$$\mathcal{C} := (c_{m,n})_{m,n\geq 0}, \qquad c_{m,n} = \int_{-1}^{1}\int_{-1}^{1} \varphi_m(x)\phi_n(y)\mathrm{e}^{\mathrm{i}\omega(x-\gamma y)^2}\mathrm{d}y\mathrm{d}x,$$

$$\mathcal{D} := (d_{m,n})_{m,n\geq 0}, \qquad d_{m,n} = \int_{-1}^{1}\int_{-1}^{1} \varphi_m(x)\varphi_n(y)\mathrm{e}^{\mathrm{i}\omega(x-\gamma y)^2}\mathrm{d}y\mathrm{d}x, \tag{1.4}$$

and $\hat{\mathbf{f}}^C = \left(\hat{f}_0^C, \hat{f}_1^C, \cdots\right)^\top$, $\hat{\mathbf{f}}^S = \left(\hat{f}_0^S, \hat{f}_1^S, \cdots\right)^\top$. Therefore, the problem of the determination of the spectra of the operator has been converted to infinite-dimensional algebraic eigenvalue problem. Since the integrands of $b_{m,n}$ and $c_{m,n}$ on the square domain $[-1,1] \times [-1,1]$ satisfy

$$h(x,y) = -h(-x,-y),$$

it follows that

$$b_{m,n} = c_{m,n} = 0.$$

Thus, the system (1.2) reduces into two uncoupled sub-systems

$$\mathcal{A}\hat{\mathbf{f}}^C = \lambda \hat{\mathbf{f}}^C, \tag{1.5}$$

$$\mathcal{D}\hat{\mathbf{f}}^S = \lambda \hat{\mathbf{f}}^S. \tag{1.6}$$

The infinite matrices need be truncated for the practical computation of the eigenvalues. In the forthcoming subsections, we shall explore the asymptotics of $a_{m,n}$ and $d_{m,n}$ with large arguments m and n or ω to allow for a judicious truncation of the matrices. Firstly, we are interested in presenting explicit expressions for $a_{m,n}$ and $d_{m,n}$. We show that the entries of $\mathcal{Q}$ can be explicitly given by an expression involving the error function. Employing the asymptotic property of the error function with a large argument, the asymptotics of each entry of the matrix are analysed. Section 2 is devoted to the determination of explicit expressions for the elements of the coefficient matrices. The corresponding asymptotic expansions are formulated in Section 3. Numerical experiments are presented to confirm the validity of our asymptotic expressions. Section 4 presents numerical results illustrating practical eigenvalue computation. The paper is concluded in Section 5.

2 Explicit expressions for $a_{m,n}$ and $d_{m,n}$

In this section, we derive explicit expressions for $a_{m,n}$ and $d_{m,n}$.

Theorem 1. *Using the modified Fourier basis, the elements of the matrices $\mathcal{A}$ and $\mathcal{D}$ (cf. (1.3)) take the following form. For $0 < \gamma \le 1$, $\gamma \neq n/m, n$ and $m^2 + n^2 > 0$,*

$$\begin{aligned}
a_{m,n} = & \frac{(-1)^n \mathrm{i} m\gamma \exp\left(-\frac{\mathrm{i}\pi^2 m^2}{4\omega}\right)}{2\pi^{\frac{1}{2}}(\gamma^2 m^2 - n^2)(-\mathrm{i}\omega)^{\frac{1}{2}}} \\
& \times \Bigg\{ \mathrm{e}^{-\mathrm{i}\pi\gamma m} \left[\mathrm{erf}\left((1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\mathrm{i}\pi m}{2(-\mathrm{i}\omega)^{\frac{1}{2}}} \right) + \mathrm{erf}\left((1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}} - \frac{\mathrm{i}\pi m}{2(-\mathrm{i}\omega)^{\frac{1}{2}}} \right) \right] \\
& - \mathrm{e}^{\mathrm{i}\pi\gamma m} \left[\mathrm{erf}\left((1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}} - \frac{\mathrm{i}\pi m}{2(-\mathrm{i}\omega)^{\frac{1}{2}}} \right) + \mathrm{erf}\left((1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\mathrm{i}\pi m}{2(-\mathrm{i}\omega)^{\frac{1}{2}}} \right) \right] \Bigg\} \\
& + \frac{(-1)^m \mathrm{i} n \exp\left(-\frac{\mathrm{i}\pi^2 n^2}{4\omega\gamma^2}\right)}{2\pi^{\frac{1}{2}}(\gamma^2 m^2 - n^2)(-\mathrm{i}\omega)^{\frac{1}{2}}} \\
& \times \Bigg\{ \mathrm{e}^{-\mathrm{i}\pi n/\gamma} \left[\mathrm{erf}\left((1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}} - \frac{\mathrm{i}\pi n}{2\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}} \right) - \mathrm{erf}\left((1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}} - \frac{\mathrm{i}\pi n}{2\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}} \right) \right] \\
& - \mathrm{e}^{\mathrm{i}\pi n/\gamma} \left[\mathrm{erf}\left((1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\mathrm{i}\pi n}{2\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}} \right) - \mathrm{erf}\left((1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\mathrm{i}\pi n}{2\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}} \right) \right] \Bigg\}
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
= & \frac{(-1)^n \mathrm{i} m\gamma \exp\left(-\frac{\mathrm{i}\pi^2 m^2}{4\omega}\right)}{2\pi^{\frac{1}{2}}(\gamma^2 m^2 - n^2)(-\mathrm{i}\omega)^{\frac{1}{2}}} \\
& \times \Bigg\{ \mathrm{e}^{-\mathrm{i}\pi\gamma m} \left[\mathrm{erf}\left(\frac{-\mathrm{i}\pi m}{2(-\mathrm{i}\omega)^{\frac{1}{2}}} + (1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}} \right) - \mathrm{erf}\left(\frac{-\mathrm{i}\pi m}{2(-\mathrm{i}\omega)^{\frac{1}{2}}} - (1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}} \right) \right]
\end{aligned}$$

$$
\begin{aligned}
&- e^{i\pi\gamma m}\left[\operatorname{erf}\left(\frac{-i\pi m}{2(-i\omega)^{\frac{1}{2}}}+(1-\gamma)(-i\omega)^{\frac{1}{2}}\right)-\operatorname{erf}\left(\frac{-i\pi m}{2(-i\omega)^{\frac{1}{2}}}-(1+\gamma)(-i\omega)^{\frac{1}{2}}\right)\right]\Bigg\}\\
&+\frac{(-1)^m in\exp\left(-\frac{i\pi^2 n^2}{4\omega\gamma^2}\right)}{2\pi^{\frac{1}{2}}(\gamma^2m^2-n^2)(-i\omega)^{\frac{1}{2}}}\\
&\times\Bigg\{e^{-i\pi n/\gamma}\left[\operatorname{erf}\left(\frac{-i\pi n}{2\gamma(-i\omega)^{\frac{1}{2}}}+(1-\gamma)(-i\omega)^{\frac{1}{2}}\right)-\operatorname{erf}\left(\frac{-i\pi n}{2\gamma(-i\omega)^{\frac{1}{2}}}+(1+\gamma)(-i\omega)^{\frac{1}{2}}\right)\right]\\
&- e^{i\pi n/\gamma}\left[\operatorname{erf}\left(\frac{-i\pi n}{2\gamma(-i\omega)^{\frac{1}{2}}}-(1+\gamma)(-i\omega)^{\frac{1}{2}}\right)-\operatorname{erf}\left(\frac{-i\pi n}{2\gamma(-i\omega)^{\frac{1}{2}}}-(1-\gamma)(-i\omega)^{\frac{1}{2}}\right)\right]\Bigg\},
\end{aligned}
\tag{2.2}
$$

while for $\gamma = n/m$ for $\gamma \in (0,1]$, $1 \le n \le m$, we have

$$
\begin{aligned}
a_{m,m\gamma}=&\frac{\exp\left(-\frac{i\pi^2m^2}{4\omega}\right)}{8m\gamma\omega\pi^{\frac{1}{2}}(-i\omega)^{\frac{1}{2}}}\Bigg[(\pi^2m^2+2i\omega+2\pi m\omega(1+\gamma))\operatorname{erf}\left((1+\gamma)(-i\omega)^{\frac{1}{2}}-\frac{i\pi m}{2(-i\omega)^{\frac{1}{2}}}\right)\\
&-(\pi^2m^2+2i\omega-2\pi m\omega(1+\gamma))\operatorname{erf}\left((1+\gamma)(-i\omega)^{\frac{1}{2}}+\frac{i\pi m}{2(-i\omega)^{\frac{1}{2}}}\right)\\
&-(\pi^2m^2+2i\omega+2\pi m\omega(1-\gamma))\operatorname{erf}\left((1-\gamma)(-i\omega)^{\frac{1}{2}}-\frac{i\pi m}{2(-i\omega)^{\frac{1}{2}}}\right)\\
&+(\pi^2m^2+2i\omega-2\pi m\omega(1-\gamma))\operatorname{erf}\left((1-\gamma)(-i\omega)^{\frac{1}{2}}+\frac{i\pi m}{2(-i\omega)^{\frac{1}{2}}}\right)\Bigg]\\
&-\frac{(-1)^{m(1+\gamma)}e^{i\omega(1+\gamma^2)}\sin(2\gamma\omega)}{\gamma\omega}
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
=&\frac{\exp\left(-\frac{i\pi^2m^2}{4\omega}\right)}{8m\gamma\omega\pi^{\frac{1}{2}}(-i\omega)^{\frac{1}{2}}}\Bigg[(\pi^2m^2+2i\omega+2\pi m\omega(1+\gamma))\operatorname{erf}\left(\frac{-i\pi m}{2(-i\omega)^{\frac{1}{2}}}+(1+\gamma)(-i\omega)^{\frac{1}{2}}\right)\\
&+(\pi^2m^2+2i\omega-2\pi m\omega(1+\gamma))\operatorname{erf}\left(\frac{-i\pi m}{2(-i\omega)^{\frac{1}{2}}}-(1+\gamma)(-i\omega)^{\frac{1}{2}}\right)\\
&-(\pi^2m^2+2i\omega+2\pi m\omega(1-\gamma))\operatorname{erf}\left(\frac{-i\pi m}{2(-i\omega)^{\frac{1}{2}}}+(1-\gamma)(-i\omega)^{\frac{1}{2}}\right)\\
&-(\pi^2m^2+2i\omega-2\pi m\omega(1-\gamma))\operatorname{erf}\left(\frac{-i\pi m}{2(-i\omega)^{\frac{1}{2}}}-(1-\gamma)(-i\omega)^{\frac{1}{2}}\right)\Bigg]\\
&-\frac{(-1)^{m(1+\gamma)}e^{i\omega(1+\gamma^2)}\sin(2\gamma\omega)}{\gamma\omega}.
\end{aligned}
\tag{2.4}
$$

Proof. The detailed proof is given in Appendix A. □

Note the different forms of identical expressions: (2.1) and (2.3) are more convenient when ω is large, while (2.2) and (2.4) are more convenient for large m and n large.

For $d_{m,n}$, we have the similar formulae to $a_{m,n}$.

Theorem 2. *With respect to the modified Fourier basis, when* $0 < \gamma \le 1$, $\gamma \ne (2n+1)/(2m+1)$, *the coefficient* $d_{m,n}$ *of (1.4) is*

$$
\begin{aligned}
d_{m,n} = & \frac{-(-1)^n\gamma(2m+1)\exp\left(-\frac{\mathrm{i}\pi^2(2m+1)^2}{16\omega}\right)}{\pi^{\frac12}(4\gamma^2m^2+4\gamma^2m+\gamma^2-4n^2-4n-1)(-\mathrm{i}\omega)^{\frac12}} \\
& \times\Bigg\{\mathrm{e}^{-\mathrm{i}\pi\gamma(m+\frac12)}\Bigg[\mathrm{erf}\left((1-\gamma)(-\mathrm{i}\omega)^{\frac12}+\frac{\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac12}}\right) \\
& \qquad +\mathrm{erf}\left((1+\gamma)(-\mathrm{i}\omega)^{\frac12}-\frac{\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac12}}\right)\Bigg] \\
& \quad +\mathrm{e}^{\mathrm{i}\pi\gamma(m+\frac12)}\Bigg[\mathrm{erf}\left((1-\gamma)(-\mathrm{i}\omega)^{\frac12}-\frac{\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac12}}\right) \\
& \qquad +\mathrm{erf}\left((1+\gamma)(-\mathrm{i}\omega)^{\frac12}+\frac{\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac12}}\right)\Bigg]\Bigg\} \\
& +\frac{(-1)^m(2n+1)\exp\left(-\frac{\mathrm{i}\pi^2(2n+1)^2}{16\omega\gamma^2}\right)}{\pi^{\frac12}(4\gamma^2m^2+4\gamma^2m+\gamma^2-4n^2-4n-1)(-\mathrm{i}\omega)^{\frac12}} \\
& \times\Bigg\{\mathrm{e}^{-\mathrm{i}\pi(n+\frac12)/\gamma}\Bigg[\mathrm{erf}\left((1+\gamma)(-\mathrm{i}\omega)^{\frac12}-\frac{\mathrm{i}\pi(2n+1)}{4\gamma(-\mathrm{i}\omega)^{\frac12}}\right) \\
& \qquad -\mathrm{erf}\left((1-\gamma)(-\mathrm{i}\omega)^{\frac12}-\frac{\mathrm{i}\pi(2n+1)}{4\gamma(-\mathrm{i}\omega)^{\frac12}}\right)\Bigg] \\
& \quad +\mathrm{e}^{\mathrm{i}\pi(n+\frac12)/\gamma}\Bigg[\mathrm{erf}\left((1+\gamma)(-\mathrm{i}\omega)^{\frac12}+\frac{\mathrm{i}\pi(2n+1)}{4\gamma(-\mathrm{i}\omega)^{\frac12}}\right) \\
& \qquad -\mathrm{erf}\left((1-\gamma)(-\mathrm{i}\omega)^{\frac12}+\frac{\mathrm{i}\pi(2n+1)}{4\gamma(-\mathrm{i}\omega)^{\frac12}}\right)\Bigg]\Bigg\} \qquad (2.5) \\
= & \frac{-(-1)^n\gamma(2m+1)\exp\left(-\frac{\mathrm{i}\pi^2(2m+1)^2}{16\omega}\right)}{\pi^{\frac12}(4\gamma^2m^2+4\gamma^2m+\gamma^2-4n^2-4n-1)(-\mathrm{i}\omega)^{\frac12}} \\
& \times\Bigg\{\mathrm{e}^{-\mathrm{i}\pi\gamma(m+\frac12)}\Bigg[\mathrm{erf}\left(\frac{-\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac12}}+(1+\gamma)(-\mathrm{i}\omega)^{\frac12}\right) \\
& \qquad -\mathrm{erf}\left(\frac{-\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac12}}-(1-\gamma)(-\mathrm{i}\omega)^{\frac12}\right)\Bigg] \\
& \quad -\mathrm{e}^{\mathrm{i}\pi\gamma(m+\frac12)}\Bigg[\mathrm{erf}\left(\frac{-\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac12}}-(1+\gamma)(-\mathrm{i}\omega)^{\frac12}\right) \\
& \qquad -\mathrm{erf}\left(\frac{-\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac12}}+(1-\gamma)(-\mathrm{i}\omega)^{\frac12}\right)\Bigg]\Bigg\} \\
& +\frac{(-1)^m(2n+1)\exp\left(-\frac{\mathrm{i}\pi^2(2n+1)^2}{16\omega\gamma^2}\right)}{\pi^{\frac12}(4\gamma^2m^2+4\gamma^2m+\gamma^2-4n^2-4n-1)(-\mathrm{i}\omega)^{\frac12}} \\
& \times\Bigg\{\mathrm{e}^{-\mathrm{i}\pi(n+\frac12)/\gamma}\Bigg[\mathrm{erf}\left(\frac{-\mathrm{i}\pi(2n+1)}{4\gamma(-\mathrm{i}\omega)^{\frac12}}+(1+\gamma)(-\mathrm{i}\omega)^{\frac12}\right)
\end{aligned}
$$

$$-\text{erf}\left(\frac{-\mathrm{i}\pi(2n+1)}{4\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\Bigg]$$

$$-\,\mathrm{e}^{\mathrm{i}\pi(n+\frac{1}{2})/\gamma}\left[\text{erf}\left(\frac{-\mathrm{i}\pi(2n+1)}{4\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right.$$

$$\left.-\text{erf}\left(\frac{-\mathrm{i}\pi(2n+1)}{4\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right]\Bigg\}. \tag{2.6}$$

If $\gamma = \frac{2n+1}{2m+1}$ *and* $0 < \gamma \leq 1$*, the expressions are as follows*

$$d_{m,(m+\frac{1}{2})\gamma-\frac{1}{2}} = -\frac{\exp\left(-\frac{\mathrm{i}\pi^2(2m+1)^2}{16\omega}\right)}{16\gamma(2m+1)\omega\pi^{\frac{1}{2}}(-\mathrm{i}\omega)^{\frac{1}{2}}}$$

$$\Bigg[-(\pi^2(2m+1)^2+8\mathrm{i}\omega+8\pi\omega\left(m+\frac{1}{2}\right)(1+\gamma))\text{erf}\left((1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}-\frac{\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)$$

$$+(\pi^2(2m+1)^2+8\mathrm{i}\omega-8\pi\omega\left(m+\frac{1}{2}\right)(1+\gamma))\text{erf}\left((1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}+\frac{\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)$$

$$+(\pi^2(2m+1)^2+8\mathrm{i}\omega+8\pi\omega\left(m+\frac{1}{2}\right)(1-\gamma))\text{erf}\left((1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}-\frac{\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)$$

$$-(\pi^2(2m+1)^2+8\mathrm{i}\omega-8\pi\omega\left(m+\frac{1}{2}\right)(1-\gamma))\text{erf}\left((1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}+\frac{\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)\Bigg]$$

$$-\frac{(-1)^{(m+\frac{1}{2})(1+\gamma)-1}\mathrm{i}\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}\cos(2\gamma\omega)}{\gamma\omega} \tag{2.7}$$

$$= \frac{\exp\left(-\frac{\mathrm{i}\pi^2(2m+1)^2}{16\omega}\right)}{16\gamma(2m+1)\omega\pi^{\frac{1}{2}}(-\mathrm{i}\omega)^{\frac{1}{2}}}$$

$$\Bigg[(\pi^2(2m+1)^2+8\mathrm{i}\omega+8\pi\omega\left(m+\frac{1}{2}\right)(1+\gamma))\text{erf}\left(\frac{-\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)$$

$$+(\pi^2(2m+1)^2+8\mathrm{i}\omega-8\pi\omega\left(m+\frac{1}{2}\right)(1+\gamma))\text{erf}\left(\frac{-\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)$$

$$-(\pi^2(2m+1)^2+8\mathrm{i}\omega+8\pi\omega\left(m+\frac{1}{2}\right)(1-\gamma))\text{erf}\left(\frac{-\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)$$

$$-(\pi^2(2m+1)^2+8\mathrm{i}\omega-8\pi\omega\left(m+\frac{1}{2}\right)(1-\gamma))\text{erf}\left(\frac{-\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\Bigg]$$

$$-\frac{(-1)^{(m+\frac{1}{2})(1+\gamma)-1}\mathrm{i}\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}\cos(2\gamma\omega)}{\gamma\omega}. \tag{2.8}$$

Proof. Appendix B shows the proof. □

As for the $a_{m,n}$s, we present two equivalent expressions for $d_{m,n}$: (2.5) and (2.7) for large ω, (2.6) and (2.8) for large $m+n$.

Equivalent expressions in the case $\gamma > 1$ can be derived from the above. Since

$$\mathcal{I}_\omega^\gamma[\psi_m, \phi_n] = \int_{-1}^{1}\int_{-1}^{1} \psi_m(x)\phi_n(y)\mathrm{e}^{\mathrm{i}\omega(x-\gamma y)^2}\mathrm{d}x\mathrm{d}y$$

$$= \int_{-1}^{1}\int_{-1}^{1} \psi_m(x)\phi_n(y)\mathrm{e}^{\mathrm{i}\omega\gamma^2(x/\gamma-y)^2}\mathrm{d}x\mathrm{d}y$$
$$= \int_{-1}^{1}\int_{-1}^{1} \phi_n(x)\psi_m(y)\mathrm{e}^{\mathrm{i}\omega\gamma^2(x-y/\gamma)^2}\mathrm{d}x\mathrm{d}y = \mathcal{I}_{\omega\gamma^2}^{\gamma^{-1}}[\phi_n, \psi_m],$$

we get

$$\mathcal{I}_\omega^\gamma[\psi_m, \phi_n] = \mathcal{I}_{\omega\gamma^2}^{\gamma^{-1}}[\phi_n, \psi_m].$$

The expansions of $I_\omega^{\gamma^{-1}}[\phi_n, \psi_m]$ can be obtained by exchanging n and m and replacing ω by $\frac{\omega}{\gamma^2}$ in the expansions of $a_{m,n}$ and $d_{m,n}$.

3 Asymptotic estimation of the matrix

According to `http://dlmf.nist.gov/7.12`, the asymptotic expansion for the error function is

$$\operatorname{erf} u \sim 1 - \frac{\mathrm{e}^{-u^2}}{\pi^{1/2}} \sum_{k=0}^{\infty} (-1)^k \frac{(\frac{1}{2})_k}{u^{2k+1}}, \quad |\mathrm{ph}(u)| < \frac{3\pi}{4}, \quad |u| \to \infty. \tag{3.1}$$

For each expression of $a_{m,n}$ and $d_{m,n}$, the first step is to determine the range of the principal value of the phase $\mathrm{ph}(u)$.

Consider the expressions (2.2) and (2.4) for large m, n. Denote the argument of the error functions in $a_{m,n}$ by u, in the form

$$u = \frac{-\mathrm{i}\pi m}{2(-\mathrm{i}\omega)^{\frac{1}{2}}} + \beta(-\mathrm{i}\omega)^{\frac{1}{2}}, \quad \text{or} \quad \frac{-\mathrm{i}\pi n}{2\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}} + \beta(-\mathrm{i}\omega)^{\frac{1}{2}}, \quad \beta = \pm(1 \pm \gamma), \quad \gamma \in (0, 1]. \tag{3.2}$$

Since $\sqrt{-\mathrm{i}} = \frac{\sqrt{2}}{2}(1 - \mathrm{i})$, the variable u in (3.2) can be simplified as

$$\begin{aligned} u &= \operatorname{Re} u + \mathrm{i}\operatorname{Im} u, \quad \operatorname{Im} u = -\operatorname{Re} u, \\ \operatorname{Re} u &= \frac{\sqrt{2}(\pi m + 2\beta\omega)}{4\sqrt{\omega}} \qquad \text{or} \qquad \frac{\sqrt{2}(\pi n + 2\beta\omega\gamma)}{4\sqrt{\omega}\gamma}. \end{aligned}$$

Changing m and n in (3.2) to $m+\frac{1}{2}$ and $n+\frac{1}{2}$ in (2.6) and (2.8) yields similar form for $d_{m,n}$. It is observed that the principal value $\mathrm{ph}(u)$ is either $-\frac{\pi}{4}$ or $\frac{3\pi}{4}$ because $\tan\frac{\operatorname{Im}(u)}{\operatorname{Re}(u)} = -1$. When $m, n \to \infty$, the condition $\operatorname{Re} u > 0$ means that the principal value is $-\frac{\pi}{4}$. Thus, the asymptotic formula (3.1) can be applied to estimate the asymptotics for the case of $m, n \to \infty$ in (2.2), (2.4), (2.6) and (2.8).

On the other hand, once ω approaches infinity, we focus on the arguments of the expressions (2.1),(2.3),(2.5) and (2.7). Denoting the argument u as

$$u = |\beta|(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\delta}{(-\mathrm{i}\omega)^{\frac{1}{2}}},$$

where $\delta = \pm\frac{\mathrm{i}\pi m}{2}$ or $\pm\frac{\mathrm{i}\pi n}{2\gamma}$, once ω tends to infinity the phases of the arguments are still $-\frac{\pi}{4}$. It holds by the asymptotic expression in (3.1) that

$$\operatorname{erf}\left(|\beta|(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\delta}{(-\mathrm{i}\omega)^{\frac{1}{2}}}\right) \sim 1 - \frac{\mathrm{e}^{-\left(|\beta|(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\delta}{(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)^2}}{\pi^{1/2}\left(|\beta|(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\delta}{(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)} + \mathcal{O}\left(\omega^{-\frac{3}{2}}\right). \tag{3.3}$$

Based on these asymptotic formulæ, we explore the asymptotics of $a_{m,n}$ and $d_{m,n}$.

3.1 Asymptotics for $a_{m,n}$

3.1.1 The decay rate of $a_{m,n}$ for $m\gamma \neq n$, $\gamma \in (0,1]$

To determine the asymptotic behaviour of these coefficients, we separate the argument into two regimes: large ω, or large m and n. Firstly, with large ω, the formula (3.3) shows that

$$a_{m,n} \sim \frac{(-1)^n 2m\gamma \sin(\pi\gamma m)\exp\Bigl(-\frac{\mathrm{i}\pi^2 m^2}{4\omega}\Bigr)}{\pi^{\frac{1}{2}}(\gamma^2 m^2 - n^2)(-\mathrm{i}\omega)^{\frac{1}{2}}} + \mathcal{O}\bigl(\omega^{-1}\bigr), \qquad m\gamma \notin \mathbb{Z}, \tag{3.4}$$

$$\begin{aligned} a_{m,n} \sim \frac{-\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}}{2\pi(\gamma^2 m^2 - n^2)\omega} \Biggl\{ & (-1)^{n+\gamma m} m\gamma \Biggl(\frac{\mathrm{e}^{-\mathrm{i}2\gamma\omega + \mathrm{i}\pi m(1-\gamma)}}{(1-\gamma)+\frac{\pi m}{2\omega}} \\ & + \frac{\mathrm{e}^{\mathrm{i}2\gamma\omega - \mathrm{i}\pi m(1+\gamma)}}{(1+\gamma)-\frac{\pi m}{2\omega}} - \frac{\mathrm{e}^{-\mathrm{i}2\gamma\omega - \mathrm{i}\pi m(1-\gamma)}}{(1-\gamma)-\frac{\pi m}{2\omega}} - \frac{\mathrm{e}^{\mathrm{i}2\gamma\omega + \mathrm{i}\pi m(1+\gamma)}}{(1+\gamma)+\frac{\pi m}{2\omega}} \Biggr) \\ & + (-1)^m n \Biggl[\cos\frac{\pi n}{\gamma} \Biggl(\frac{\mathrm{e}^{\mathrm{i}2\gamma\omega + \mathrm{i}\pi n(1+\gamma)/\gamma}}{(1+\gamma)+\frac{\pi n}{2\gamma\omega}} - \frac{\mathrm{e}^{-\mathrm{i}2\gamma\omega + \mathrm{i}\pi n(1-\gamma)/\gamma}}{(1-\gamma)+\frac{\pi n}{2\gamma\omega}} \\ & - \frac{\mathrm{e}^{\mathrm{i}2\gamma\omega - \mathrm{i}\pi n(1+\gamma)/\gamma}}{(1+\gamma)-\frac{\pi n}{2\gamma\omega}} + \frac{\mathrm{e}^{-\mathrm{i}2\gamma\omega + \mathrm{i}\pi n(1-\gamma)/\gamma}}{(1-\gamma)-\frac{\pi n}{2\gamma\omega}} \Biggr) - \mathrm{i}\sin\frac{\pi n}{\gamma} \\ & \Biggl(\frac{\mathrm{e}^{\mathrm{i}2\gamma\omega + \mathrm{i}\pi n(1+\gamma)/\gamma}}{(1+\gamma)+\frac{\pi n}{2\gamma\omega}} - \frac{\mathrm{e}^{-\mathrm{i}2\gamma\omega + \mathrm{i}\pi n(1-\gamma)/\gamma}}{(1-\gamma)+\frac{\pi n}{2\gamma\omega}} + \frac{\mathrm{e}^{\mathrm{i}2\gamma\omega - \mathrm{i}\pi n(1+\gamma)/\gamma}}{(1+\gamma)-\frac{\pi n}{2\gamma\omega}} \\ & - \frac{\mathrm{e}^{-\mathrm{i}2\gamma\omega + \mathrm{i}\pi n(1-\gamma)/\gamma}}{(1-\gamma)-\frac{\pi n}{2\gamma\omega}} \Biggr) \Biggr] \Biggr\} + \mathcal{O}\bigl(\omega^{-2}\bigr), \qquad m\gamma \in \mathbb{Z}, \end{aligned} \tag{3.5}$$

for all $m \geq 0$ and $n \geq 0$.

Next consider the asymptotics for large m and n. A good strategy is to consider this along the lines $n = \kappa m$, where $\kappa \in [0, \infty]$. This boils down to three cases: (1) $m \gg \omega^{\frac{1}{2}}$, $n = 0$, (2) $m = 0$, $n \gg \omega^{\frac{1}{2}}$, and (3) $m, n \gg \omega^{\frac{1}{2}}$, $\lim_{m\to\infty} \frac{n}{m} = \kappa \neq 0, \gamma, \infty$.

If m or n is substantially greater than ω, the expressions (2.2) and (2.4) for $a_{m,n}$ can be used to estimate the asymptotics. Let

$$\begin{aligned} t &= (-\mathrm{i}\omega)^{\frac{1}{2}}, \\ z &= \frac{2t}{-\mathrm{i}\pi m} \quad \text{or} \quad \frac{2\gamma t}{-\mathrm{i}\pi n}. \end{aligned}$$

Then the arguments of the error function in (2.2) and (2.4) are $\frac{1}{z} \pm (1 \pm \gamma)t$ where

$$\begin{aligned} \frac{1}{z} &= \frac{-\mathrm{i}\pi m}{2(-\mathrm{i}\omega)^{\frac{1}{2}}} \quad \text{or} \quad \frac{-\mathrm{i}\pi n}{2\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}, \\ \frac{2t}{z} &= -\mathrm{i}\pi m \quad \text{or} \quad \frac{2t\gamma}{z} = -\mathrm{i}\pi n, \\ \mathrm{e}^{-\frac{\mathrm{i}\pi^2 m^2}{4\omega}} &= \mathrm{e}^{z^{-2}} \quad \text{or} \quad \mathrm{e}^{-\frac{\mathrm{i}\pi^2 n^2}{4\gamma^2\omega}} = \mathrm{e}^{z^{-2}}. \end{aligned}$$

Combining the above definitions with the binomial series formula

$$(1+v)^n = 1 + nv + \frac{n(n-1)}{2!}v^2 + \frac{n(n-1)(n-2)}{3!}v^3 + \cdots, \quad |v| < 1,$$

we get

$$\operatorname{erf}\left(\frac{1}{z} - (1+\gamma)t\right) \sim 1 - \frac{\mathrm{e}^{-\left(\frac{1}{z}-(1+\gamma)t\right)^2}}{\pi^{\frac{1}{2}}} \left\{ z + (1+\gamma)tz^2 + \left[(1+\gamma)^2 t^2 - \frac{1}{2}\right] z^3 \right.$$

$$
\begin{aligned}
&+\left[(1+\gamma)^2t^2-\frac{3}{2}\right](1+\gamma)tz^4+\left[(1+\gamma)^4t^4-3(1+\gamma)^2t^2+\frac{3}{4}\right]z^5\\
&+\left[(1+\gamma)^4t^4-5(1+\gamma)^2t^2+\frac{15}{4}\right](1+\gamma)tz^6\\
&+\left[(1+\gamma)^6t^6-\frac{15}{2}(1+\gamma)^4t^4+\frac{45}{4}(1+\gamma)^2t^2-\frac{15}{8}\right]z^7+O\big(z^{-8}\big)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{erf}\left(\frac{1}{z}+(1+\gamma)t\right)\sim 1-\frac{\mathrm{e}^{-\left(\frac{1}{z}+(1+\gamma)t\right)^2}}{\pi^{\frac{1}{2}}}&\Bigg\{z-(1+\gamma)tz^2+\left[(1+\gamma)^2t^2-\frac{1}{2}\right]z^3\\
&-\left[(1+\gamma)^2t^2-\frac{3}{2}\right](1+\gamma)tz^4+\left[(1+\gamma)^4t^4-3(1+\gamma)^2t^2+\frac{3}{4}\right]z^5\\
&-\left[(1+\gamma)^4t^4-5(1+\gamma)^2t^2+\frac{15}{4}\right](1+\gamma)tz^6\\
&+\left[(1+\gamma)^6t^6-\frac{15}{2}(1+\gamma)^4t^4+\frac{45}{4}(1+\gamma)^2t^2-\frac{15}{8}\right]z^7+O\big(z^{-8}\big),
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{erf}\left(\frac{1}{z}-(1-\gamma)t\right)\sim 1-\frac{\mathrm{e}^{-\left(\frac{1}{z}-(1-\gamma)t\right)^2}}{\pi^{\frac{1}{2}}}&\Bigg\{z+(1-\gamma)tz^2+\left[(1-\gamma)^2t^2-\frac{1}{2}\right]z^3\\
&+\left[(1-\gamma)^2t^2-\frac{3}{2}\right](1-\gamma)tz^4+\left[(1-\gamma)^4t^4-3(1-\gamma)^2t^2+\frac{3}{4}\right]z^5\\
&+\left[(1-\gamma)^4t^4-5(1-\gamma)^2t^2+\frac{15}{4}\right](1-\gamma)tz^6\\
&+\left[(1-\gamma)^6t^6-\frac{15}{2}(1-\gamma)^4t^4+\frac{45}{4}(1-\gamma)^2t^2-\frac{15}{8}\right]z^7+O\big(z^{-8}\big)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{erf}\left(\frac{1}{z}+(1-\gamma)t\right)\sim 1-\frac{\mathrm{e}^{-\left(\frac{1}{z}+(1-\gamma)t\right)^2}}{\pi^{\frac{1}{2}}}&\Bigg\{z-(1-\gamma)tz^2+\left[(1-\gamma)^2t^2-\frac{1}{2}\right]z^3\\
&-\left[(1-\gamma)^2t^2-\frac{3}{2}\right](1-\gamma)tz^4+\left[(1-\gamma)^4t^4-3(1-\gamma)^2t^2+\frac{3}{4}\right]z^5\\
&-\left[(1-\gamma)^4t^4-5(1-\gamma)^2t^2+\frac{15}{4}\right](1-\gamma)tz^6\\
&+\left[(1-\gamma)^6t^6-\frac{15}{2}(1-\gamma)^4t^4+\frac{45}{4}(1-\gamma)^2t^2-\frac{15}{8}\right]z^7+O\big(z^{-8}\big)\Bigg\}.
\end{aligned}
\tag{3.6}
$$

Substituting all this into $a_{m,n}$, we examine the asymptotics of $a_{m,n}$ for each case.

1. $m \gg \omega^{\frac{1}{2}}$ **and** $n = 0$

Setting $n = 0$ in (2.2) yields

$$
\begin{aligned}
a_{m,0}=&\frac{\mathrm{i}\exp\!\left(-\frac{\mathrm{i}\pi^2m^2}{4\omega}\right)}{2\pi^{\frac{1}{2}}\gamma m(-\mathrm{i}\omega)^{\frac{1}{2}}}\\
&\times\Bigg\{\mathrm{e}^{-\mathrm{i}\pi\gamma m}\left[\operatorname{erf}\left(\frac{-\mathrm{i}\pi m}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\operatorname{erf}\left(\frac{-\mathrm{i}\pi m}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right]\\
&-\mathrm{e}^{\mathrm{i}\pi\gamma m}\left[\operatorname{erf}\left(\frac{-\mathrm{i}\pi m}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right)-\operatorname{erf}\left(\frac{-\mathrm{i}\pi m}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right]\Bigg\}
\end{aligned}
$$

$$= \frac{\mathrm{i}e^{1/z^2}}{2\pi^{\frac{1}{2}}\gamma m t}\Big\{e^{2\gamma t/z}\Big[\mathrm{erf}\Big(\frac{1}{z}+(1+\gamma)t\Big)-\mathrm{erf}\Big(\frac{1}{z}-(1-\gamma)t\Big)\Big]$$
$$- e^{-2\gamma t/z}\Big(\mathrm{erf}\Big(\frac{1}{z}+(1-\gamma)t\Big)-\mathrm{erf}\Big(\frac{1}{z}-(1+\gamma)t\Big)\Big]\Big\}.$$

Expanding the error function erf for large $m \gg \omega^{1/2}$ using the formula (3.6), we have

$$e^{1/z^2}\Big\{e^{2\gamma t/z}\Big[\mathrm{erf}\Big(\frac{1}{z}+(1+\gamma)t\Big)-\mathrm{erf}\Big(\frac{1}{z}-(1-\gamma)t\Big)\Big]$$
$$- e^{-2\gamma t/z}\Big(\mathrm{erf}\Big(\frac{1}{z}+(1-\gamma)t\Big)-\mathrm{erf}\Big(\frac{1}{z}-(1+\gamma)t\Big)\Big]\Big\}$$
$$\sim \frac{2(-1)^m}{\pi^{1/2}}e^{\mathrm{i}\omega(1+\gamma^2)}\Big\{-2\mathrm{i}\sin(2\gamma\omega)z$$
$$-\Big[\Big((1+\gamma)^2t^2-\frac{1}{2}\Big)e^{\mathrm{i}2\omega\gamma}-\Big((1-\gamma)^2t^2-\frac{1}{2}\Big)e^{-\mathrm{i}2\omega\gamma}\Big]z^3+\mathcal{O}(z^5)\Big\}.$$

Finally, it can be deduced that for $m \geq 1$,

$$a_{m,0} \sim \frac{\mathrm{i}(-1)^m}{\gamma}e^{\mathrm{i}\omega(1+\gamma^2)}\Big\{\frac{4\sin(2\omega\gamma)}{\pi^2}\frac{1}{m^2}$$
$$-\frac{8\omega}{\pi^4}\left[-2\omega(1+\gamma^2)\sin(2\gamma\omega)+4\mathrm{i}\gamma\omega\cos(2\gamma\omega)+\mathrm{i}\sin(2\gamma\omega)\right]\frac{1}{m^4}+\mathcal{O}\Big(\frac{1}{m^6}\Big)\Big\}. \quad (3.7)$$

2. $m = 0$ **and** $n \gg \omega^{\frac{1}{2}}$

Setting $m = 0$ in (2.2), we have

$$a_{0,n} = \frac{\mathrm{i}\exp\Big(-\frac{\mathrm{i}\pi^2n^2}{4\gamma^2\omega}\Big)}{2\pi^{\frac{1}{2}}n(-\mathrm{i}\omega)^{\frac{1}{2}}}$$
$$\times\Big\{e^{-\mathrm{i}\pi n/\gamma}\Big[\mathrm{erf}\Big(\frac{-\mathrm{i}\pi n}{2\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\Big)-\mathrm{erf}\Big(\frac{-\mathrm{i}\pi n}{2\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\Big)\Big]$$
$$- e^{\mathrm{i}\pi n/\gamma}\Big[\mathrm{erf}\Big(\frac{-\mathrm{i}\pi n}{2\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\Big)-\mathrm{erf}\Big(\frac{-\mathrm{i}\pi n}{2\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\Big)\Big]\Big\}$$
$$= \frac{\mathrm{i}e^{z^{-2}}}{2\pi^{1/2}nt}\Big\{e^{2t/z}\Big[\mathrm{erf}\Big(\frac{1}{z}+(1+\gamma)t\Big)-\mathrm{erf}\Big(\frac{1}{z}+(1-\gamma)t\Big)\Big]$$
$$- e^{-2t/z}\Big[\mathrm{erf}\Big(\frac{1}{z}-(1-\gamma)t\Big)-\mathrm{erf}\Big(\frac{1}{z}-(1+\gamma)t\Big)\Big]\Big\},$$

hence

$$e^{1/z^2}\Big\{e^{2t/z}\Big[\mathrm{erf}\Big(\frac{1}{z}+(1+\gamma)t\Big)-\mathrm{erf}\Big(\frac{1}{z}+(1-\gamma)t\Big)\Big]$$
$$- e^{-2t/z}\Big[\mathrm{erf}\Big(\frac{1}{z}-(1-\gamma)t\Big)-\mathrm{erf}\Big(\frac{1}{z}-(1+\gamma)t\Big)\Big]\Big\}$$
$$\sim \frac{2(-1)^n}{\pi^{\frac{1}{2}}}e^{\mathrm{i}\omega(1+\gamma^2)}\{-2\mathrm{i}\sin(2\gamma\omega)z$$

$$- \left[\left((1+\gamma)^2 t^2 - \frac{1}{2} \right) \mathrm{e}^{\mathrm{i}2\omega\gamma} - \left((1-\gamma)^2 t^2 - \frac{1}{2} \right) \mathrm{e}^{-\mathrm{i}2\omega\gamma} \right] z^3 + \mathcal{O}(z^5) \Big\} .$$

Thus, we deduce

$$a_{0,n} \sim (-1)^n \mathrm{i} \mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)} \left(\frac{4\gamma}{\pi^2} \sin(2\gamma\omega) \frac{1}{n^2} - \frac{8\gamma^3\omega}{\pi^4} \left\{ -2\omega(1+\gamma^2)\sin(2\gamma\omega) + 4\mathrm{i}\gamma\omega\cos(2\gamma\omega) + \mathrm{i}\sin(2\gamma\omega) \right\} \frac{1}{n^4} + \mathcal{O}(n^{-6}) \right) \tag{3.8}$$

for $n \geq 1$.

3. **Letting** $\kappa > 0$ **be a rational number,** $n = \kappa m$, $\kappa \neq \gamma$

   An important observation to note, following at once from (2.2), is that

$$a_{m,n} = \frac{1}{\gamma^2 m^2 - n^2} [(-1)^n \gamma^2 m^2 a_{m,0} - (-1)^m n^2 a_{0,n}], \qquad m, n \in \mathbb{Z}_+. \tag{3.9}$$

   We conclude, using (3.9), that for $m, n \geq 1$, $\gamma^2 m^2 \neq n^2$, the expansion is

$$a_{m,n} \sim \frac{8(-1)^{m+n} \mathrm{i}\gamma\omega \mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}}{\pi^4} \left\{ -2\omega(1+\gamma^2)\sin(2\gamma\omega) + 4\mathrm{i}\gamma\omega\cos(2\gamma\omega) + \mathrm{i}\sin(2\gamma\omega) \right\} \frac{1}{m^2 n^2} + \cdots . \tag{3.10}$$

   It follows at once that for bounded $\kappa > 0$, $a_{m,\kappa m} = \mathcal{O}(m^{-4})$.

3.1.2 The decay rate of $a_{m,n}$ for $m\gamma = n$, $1 \leq n \leq m$, $0 < \gamma \leq 1$

Two conditions are considered: one is large ω, the other is large m provided that $\gamma m = n$. For large ω, taking the expansion (3.3), the form (2.3) can be approximated by

$$\begin{aligned} a_{m,m\gamma} \sim & \frac{\exp\left(-\frac{\mathrm{i}\pi^2 m^2}{4\omega}\right)}{8\gamma m\omega\pi^{\frac{1}{2}}(-\mathrm{i}\omega)^{\frac{1}{2}}} \Big[(\pi^2 m^2 + 2\mathrm{i}\omega + 2\pi m(1+\gamma)\omega) \left(1 + \mathcal{O}\left(\omega^{-\frac{1}{2}}\right)\right) \\ & -(\pi^2 m^2 + 2\mathrm{i}\omega - 2\pi m(1+\gamma)\omega) \left(1 + \mathcal{O}\left(\omega^{-\frac{1}{2}}\right)\right) \\ & - (\pi^2 m^2 + 2\mathrm{i}\omega + 2\pi m(1-\gamma)\omega) \left(1 + \mathcal{O}\left(\omega^{-\frac{1}{2}}\right)\right) \\ & + (\pi^2 m^2 + 2\mathrm{i}\omega - 2\pi m(1-\gamma)\omega) \left(1 + \mathcal{O}\left(\omega^{-\frac{1}{2}}\right)\right) \Big] \\ & + \frac{(-1)^{m(1+\gamma)}\mathrm{i}}{2\gamma} \left[\exp\left(\mathrm{i}\omega(1+\gamma)^2\right) - \exp\left(\mathrm{i}\omega(1-\gamma)^2\right) \right] \frac{1}{\omega} \\ = & \frac{\pi^{\frac{1}{2}} \exp\left(-\frac{\mathrm{i}\pi^2 m^2}{4\omega}\right)}{(-\mathrm{i}\omega)^{\frac{1}{2}}} + \mathcal{O}(\omega^{-1}). \end{aligned} \tag{3.11}$$

Secondly, if $m\gamma = n$ and m approaches infinity for fixed ω, we expand the error functions in (2.4) using the expansions (3.6),

$$a_{m,m\gamma} = \frac{(-1)^{m+n} \mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}}{4m\pi\gamma\omega(-\mathrm{i}\omega)^{\frac{1}{2}}} \Big\{ - \left(\pi^2 m^2 + 2\mathrm{i}\omega\right) \mathrm{e}^{\mathrm{i}2\gamma\omega} \Big[z + \left[(1+\gamma)^2 t^2 - \frac{1}{2} \right] z^3 + \left[(1+\gamma)^4 t^4 - 3(1+\gamma)^2 t^2 + \frac{3}{4} \right] z^5$$

$$
\begin{aligned}
&+\left[(1+\gamma)^6t^6-\frac{15}{2}(1+\gamma)^4t^4+\frac{45}{4}(1+\gamma)^2t^2-\frac{15}{8}\right]z^7\Bigg]\\
&+2\pi m\omega e^{i2\gamma\omega}(1+\gamma)\left[(1+\gamma)tz^2+\left[(1+\gamma)^2t^2-\frac{3}{2}\right](1+\gamma)tz^4\right.\\
&\left.+\left[(1+\gamma)^4t^4-5(1+\gamma)^2t^2+\frac{15}{4}\right](1+\gamma)tz^6\right]+\mathcal{O}\big(z^8\big)\Bigg\}\\
&+\left(\pi^2m^2+2i\omega\right)e^{-i2\gamma\omega}\Bigg[z+\left[(1-\gamma)^2t^2-\frac{1}{2}\right]z^3\\
&+\left[(1-\gamma)^4t^4-3(1-\gamma)^2t^2+\frac{3}{4}\right]z^5\\
&+\left[(1-\gamma)^6t^6-\frac{15}{2}(1-\gamma)^4t^4+\frac{45}{4}(1-\gamma)^2t^2-\frac{15}{8}\right]z^7\Bigg]\\
&-2\pi m\omega e^{-i2\gamma\omega}(1-\gamma)\left[(1-\gamma)tz^2+\left[(1-\gamma)^2t^2-\frac{3}{2}\right](1-\gamma)tz^4\right.\\
&\left.+\left[(1-\gamma)^4t^4-5(1-\gamma)^2t^2+\frac{15}{4}\right](1-\gamma)tz^6\right]+\mathcal{O}\big(z^8\big)\Bigg\}\\
&-\frac{(-1)^{m+n}e^{i\omega(1+\gamma^2)}\sin(2\gamma\omega)}{\gamma\omega}\\
=&\frac{(-1)^{m+n}e^{i\omega(1+\gamma^2)}}{4m\pi\gamma\omega(-i\omega)^{\frac{1}{2}}}\Big\{-\left(\pi^2m^2+2i\omega\right)2i\sin(2\gamma\omega)z\\
&+2\pi m\omega\left[(1+\gamma)^2e^{i2\gamma\omega}-(1-\gamma)^2e^{-i2\gamma\omega}\right]tz^2\\
&-\left(\pi^2m^2+2i\omega\right)\left[\left((1+\gamma)^2t^2-\frac{1}{2}\right)e^{i2\gamma\omega}-\left((1-\gamma)^2t^2-\frac{1}{2}\right)e^{-i2\gamma\omega}\right]z^3\\
&+2\pi m\omega\left[(1+\gamma)^2\left((1+\gamma)^2t^2-\frac{3}{2}\right)e^{i2\gamma\omega}-(1-\gamma)^2\left((1-\gamma)^2t^2-\frac{3}{2}\right)e^{-i2\gamma\omega}\right]tz^4\\
&-\left(\pi^2m^2+2i\omega\right)\Bigg[\left((1+\gamma)^4t^4-3(1+\gamma)^2t^2+\frac{3}{4}\right)e^{i2\gamma\omega}\\
&-\left((1-\gamma)^4t^4-3(1-\gamma)^2t^2+\frac{3}{4}\right)e^{-i2\gamma\omega}\Bigg]z^5\\
&+2\pi m\omega\Bigg[(1+\gamma)^2\left((1+\gamma)^4t^4-5(1+\gamma)^2t^2+\frac{15}{4}\right)e^{i2\gamma\omega}\\
&-(1-\gamma)^2\left((1-\gamma)^4t^4-5(1-\gamma)^2t^2+\frac{15}{4}\right)e^{-i2\gamma\omega}\Bigg]tz^6\\
&-\left(\pi^2m^2+2i\omega\right)\Bigg[\left((1+\gamma)^6t^6-\frac{15}{2}(1+\gamma)^4t^4+\frac{45}{4}(1+\gamma)^2t^2-\frac{15}{8}\right)e^{i2\gamma\omega}\\
&-\left((1-\gamma)^6t^6-\frac{15}{2}(1-\gamma)^4t^4+\frac{45}{4}(1-\gamma)^2t^2-\frac{15}{8}\right)e^{-i2\gamma\omega}\Bigg]z^7+\mathcal{O}\big(z^8\big)\Big\}\\
&-\frac{(-1)^{m+n}e^{i\omega(1+\gamma^2)}\sin(2\gamma\omega)}{\gamma\omega}\\
\sim&\frac{-(-1)^{m+n}e^{i\omega(1+\gamma^2)}8\omega}{\pi^4m^4\gamma}\left(\left(2i\gamma^2\omega+2i\omega+1\right)\sin(2\gamma\omega)+4\gamma\omega\cos(2\gamma\omega)\right)\\
&-\frac{(-1)^{m+n}e^{i\omega(1+\gamma^2)}32\omega^2}{\pi^6m^6\gamma}\left(\left(4i\gamma^4\omega^2+24i\gamma^2\omega^2+12\gamma^2\omega+12\omega+4i\omega^2-3i\right)\sin(2\gamma\omega)\right.\\
&\left.+(16\gamma^3\omega^2-24i\gamma\omega+16\gamma\omega^2)\cos(2\gamma\omega)\right)+\mathcal{O}\big(m^{-8}\big). \qquad (3.12)
\end{aligned}
$$

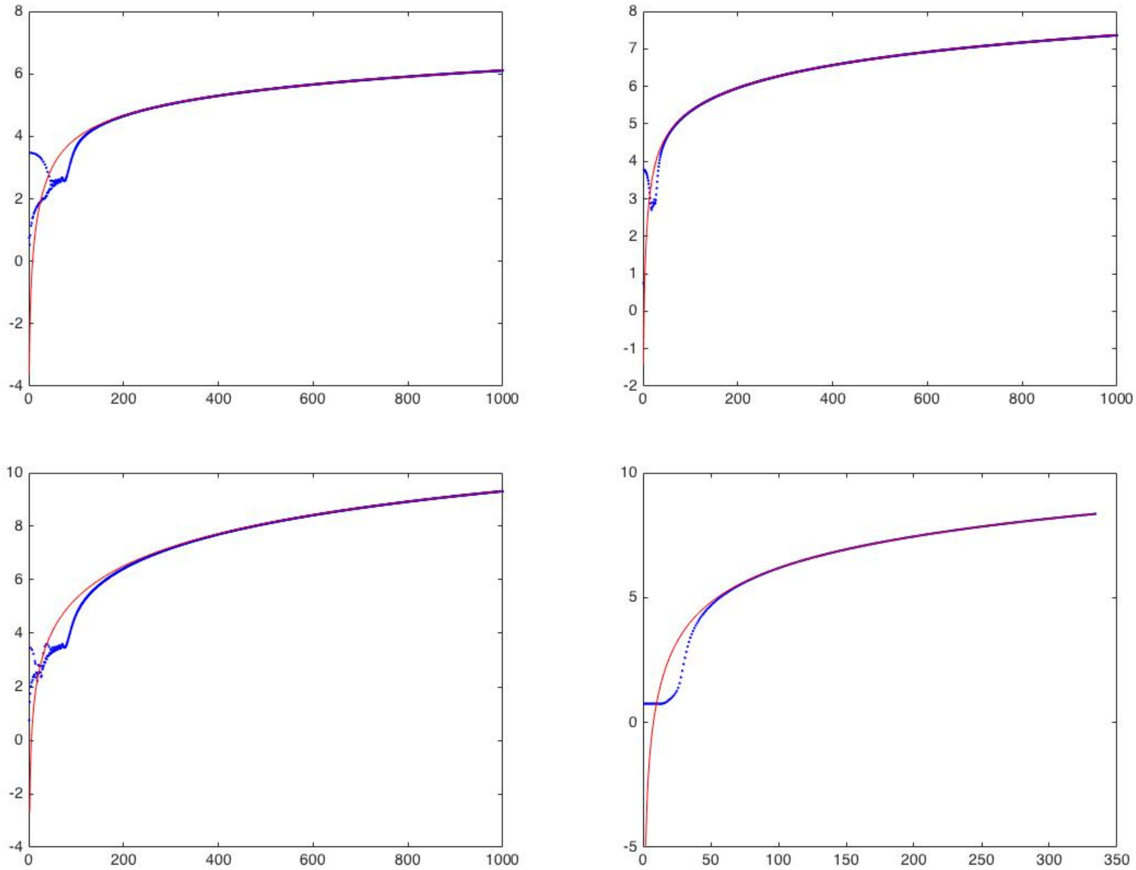

Figure 3.1: $-\log_{10}|a_{m,n}|$ (the blue dotted line) and its asymptotic approximation (the red line), for $n = 0$ (the top left), $m = 0$ (the top right), $m = n$ (the bottom left) and $\gamma m = n$ (the bottom right) with $\gamma = \frac{1}{3}$, $m, n \in [1, 1000]$, and $\omega = 100$.

Summarising the above analysis, we have presented the asymptotics for $a_{m,n}$ in six cases. The validity of these asymptotics is illustrated by plotting the asymptotic results for $a_{m,n}$ against the the exact expressions. With $\gamma = \frac{1}{3}$ and $\omega = 100$, the curves $-\log_{10}|a_{m,n}|$ (the blue dotted line) and its asymptotic expression (the red line) are displayed in Fig. 3.1 for $n = 0$ (the top left, the asymptotic (3.7)), $m = 0$ (the top right, the asymptotic expression(3.8)), $m = n$ (the bottom left, the asymptotic expression (3.10)) and $\gamma m = n$ (the bottom right, the asymptotic expression (3.12)) for $m, n \in \{1, \dots, 1000\}$.

On the other hand, for fixed m, n, if ω approaches infinity, the decay of $a_{m,n}$ is consistent with the estimations (3.4, 3.5, 3.11). Taking $m = 2$, $n = 7$ in (3.4), $m = 3$, $n = 4$ for (3.5) and $m = 3$, $n = 1$ in (3.11) with $\gamma = \frac{1}{3}$, the accurate value $a_{m,n}$ (the blue dotted line) and its asymptotics (the red line) are shown in Fig. 3.2 for ω from 1 to 1000.

3.2 Asymptotics for $d_{m,n}$

Similarly to $a_{m,n}$, we will analyse $d_{m,n}$ for six cases:

1. large ω, fixed m, n and $\lim_{m\to\infty} \frac{2n+1}{2m+1} \neq \gamma$;

2. $d_{m,0}$ for $m \gg \omega^{\frac{1}{2}}$;

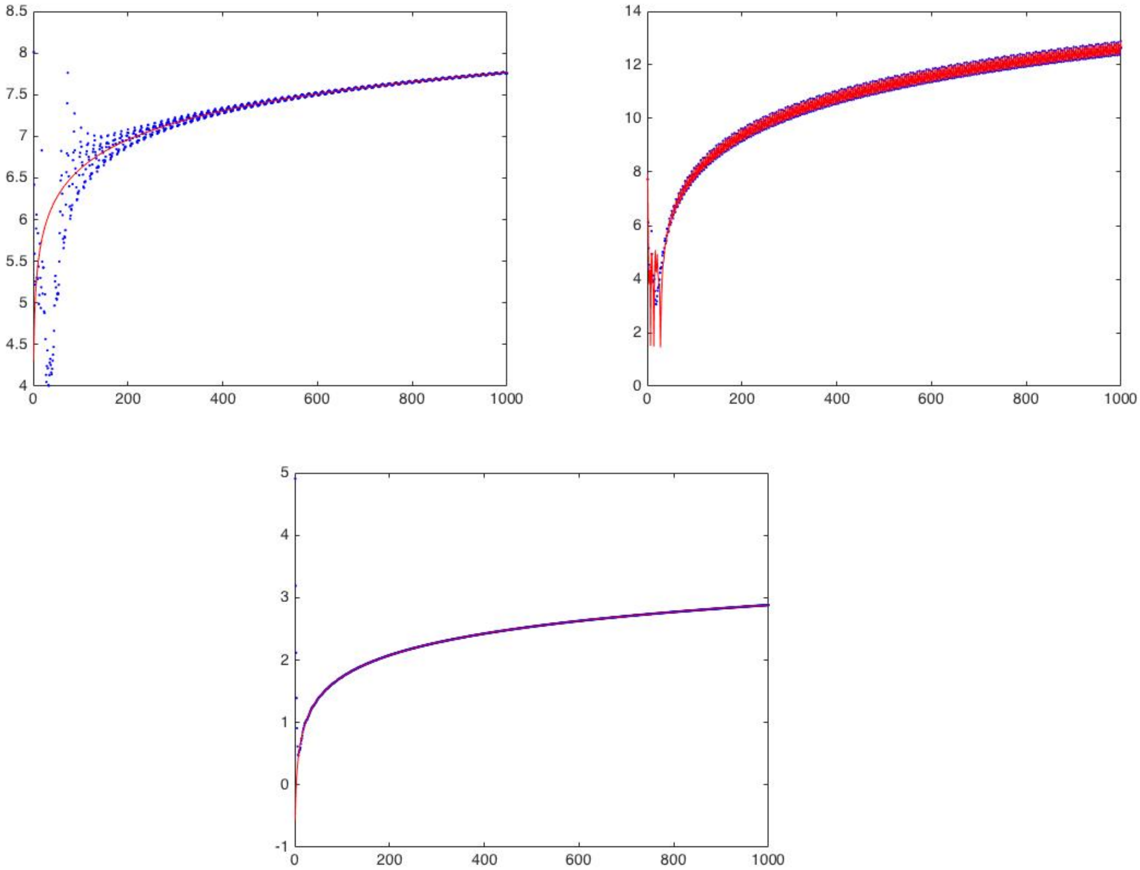

Figure 3.2: $-\log_{10}|a_{m,n}(\omega)|$ (the blue dotted line) and its asymptotic approximation(the red line), for $m=2$, $n=7$ (the top left), $m=3$, $n=4$ (the top right), $m=3$, $n=1$ (the bottom) with $\gamma=\frac{1}{3}$, $\omega\in[1,1000]$.

3. $d_{0,n}$ for $n \gg \omega^{\frac{1}{2}}$;

4. $d_{m,n}$, $\lim_{m\to\infty}\frac{2n+1}{2m+1}=\kappa$, $\kappa\neq 0,\infty,\gamma$ for $m\gg\omega^{\frac{1}{2}}$ and $n\gg\omega^{\frac{1}{2}}$;

5. large ω, fixed m, n and $\lim_{m\to\infty}\frac{2n+1}{2m+1}=\gamma$; and

6. $d_{m,n}$ and $\lim_{m\to\infty}\frac{2n+1}{2m+1}=\gamma$ for $m\gg\omega^{\frac{1}{2}}$ and $n\gg\omega^{\frac{1}{2}}$.

Applying (3.3) to (2.5), for case (1), we have

$$d_{m,n}\sim\frac{-4(-1)^n\gamma(2m+1)\cos\!\left(\pi\gamma\left(m+\frac{1}{2}\right)\right)\exp\!\left(-\frac{\mathrm{i}\pi^2(2m+1)^2}{16\omega}\right)}{\pi^{\frac{1}{2}}(4\gamma^2m^2+4\gamma^2m+\gamma^2-4n^2-4n-1)(-\mathrm{i}\omega)^{\frac{1}{2}}}+O\!\left(\omega^{-1}\right),\qquad(3.13)$$

for $\gamma\left(m+\frac{1}{2}\right)\neq k+\frac{1}{2}$ and $k\in\mathbb{Z}_+$. But when $\gamma\left(m+\frac{1}{2}\right)=k+\frac{1}{2}$, $k\in\mathbb{Z}_+$, it holds that

$$\begin{aligned}d_{m,n}\ \sim\ &\frac{\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}}{\pi(4\gamma^2m^2+4\gamma^2m+\gamma^2-4n^2-4n-1)\omega}\Big\{(-1)^{n+\gamma\left(m+\frac{1}{2}\right)-\frac{1}{2}}\left(2m+1\right)\gamma\\ &\left[\frac{\mathrm{e}^{-\mathrm{i}2\gamma\omega-\mathrm{i}\pi\left(m+\frac{1}{2}\right)(1-\gamma)}}{(1-\gamma)-\frac{\pi\left(m+\frac{1}{2}\right)}{2\omega}}+\frac{\mathrm{e}^{\mathrm{i}2\gamma\omega+\mathrm{i}\pi\left(m+\frac{1}{2}\right)(1+\gamma)}}{(1+\gamma)+\frac{\pi\left(m+\frac{1}{2}\right)}{2\omega}}\right.\end{aligned}$$

$$
-\frac{\mathrm{e}^{-\mathrm{i}2\gamma\omega+\mathrm{i}\pi\left(m+\frac{1}{2}\right)(1-\gamma)}}{(1-\gamma)+\frac{\pi\left(m+\frac{1}{2}\right)}{2\omega}}-\frac{\mathrm{e}^{\mathrm{i}2\gamma\omega-\mathrm{i}\pi\left(m+\frac{1}{2}\right)(1+\gamma)}}{(1+\gamma)-\frac{\pi\left(m+\frac{1}{2}\right)}{2\omega}}\Bigg]
$$

$$
+\mathrm{i}(-1)^m(2n+1)\Bigg[\cos\frac{\pi(n+\frac{1}{2})}{\gamma}\Bigg(\frac{\mathrm{e}^{-\mathrm{i}2\gamma\omega-\mathrm{i}\pi\left(n+\frac{1}{2}\right)(1-\gamma)/\gamma}}{(1-\gamma)-\frac{\pi\left(n+\frac{1}{2}\right)}{2\gamma\omega}}
$$

$$
-\frac{\mathrm{e}^{\mathrm{i}2\gamma\omega-\mathrm{i}\pi\left(n+\frac{1}{2}\right)(1+\gamma)/\gamma}}{(1+\gamma)-\frac{\pi\left(n+\frac{1}{2}\right)}{2\gamma\omega}}+\frac{\mathrm{e}^{-\mathrm{i}2\gamma\omega+\mathrm{i}\pi\left(n+\frac{1}{2}\right)(1-\gamma)/\gamma}}{(1-\gamma)+\frac{\pi\left(n+\frac{1}{2}\right)}{2\gamma\omega}}-\frac{\mathrm{e}^{\mathrm{i}2\gamma\omega+\mathrm{i}\pi\left(n+\frac{1}{2}\right)(1+\gamma)/\gamma}}{(1+\gamma)+\frac{\pi\left(n+\frac{1}{2}\right)}{2\gamma\omega}}\Bigg)
$$

$$
+\mathrm{i}\sin\frac{\pi(n+\frac{1}{2})}{\gamma}\Bigg(\frac{\mathrm{e}^{-\mathrm{i}2\gamma\omega-\mathrm{i}\pi\left(n+\frac{1}{2}\right)(1-\gamma)/\gamma}}{(1-\gamma)-\frac{\pi\left(n+\frac{1}{2}\right)}{2\gamma\omega}}-\frac{\mathrm{e}^{\mathrm{i}2\gamma\omega-\mathrm{i}\pi\left(n+\frac{1}{2}\right)(1+\gamma)/\gamma}}{(1+\gamma)-\frac{\pi\left(n+\frac{1}{2}\right)}{2\gamma\omega}}
$$

$$
-\frac{\mathrm{e}^{-\mathrm{i}2\gamma\omega+\mathrm{i}\pi\left(n+\frac{1}{2}\right)(1-\gamma)/\gamma}}{(1-\gamma)+\frac{\pi\left(n+\frac{1}{2}\right)}{2\gamma\omega}}+\frac{\mathrm{e}^{\mathrm{i}2\gamma\omega+\mathrm{i}\pi\left(n+\frac{1}{2}\right)(1+\gamma)/\gamma}}{(1+\gamma)+\frac{\pi\left(n+\frac{1}{2}\right)}{2\gamma\omega}}\Bigg)\Bigg]\Bigg\}+\mathcal{O}\big(\omega^{-2}\big). \tag{3.14}
$$

In case (2), denote

$$
\frac{1}{z}=\frac{-\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}},\quad t=(-\mathrm{i}\omega)^{\frac{1}{2}},\quad \frac{2t}{z}=-\mathrm{i}\pi\left(m+\frac{1}{2}\right). \tag{3.15}
$$

Note that if $m \gg \omega^{\frac{1}{2}}$, $n=0$, the condition $\gamma(2m+1)\neq 1$ is valid subject to the premise $\gamma(2m+1)\neq 2n+1$ for non-negative integers m and n, especially $n=0$. Thus, with (3.15), the formula (2.6) yields

$$
\begin{aligned}
d_{m,0}-C_1 &= \frac{-\gamma(2m+1)\exp\left(-\frac{\mathrm{i}\pi^2(2m+1)^2}{16\omega}\right)}{\pi^{\frac{1}{2}}(4\gamma^2m^2+4\gamma^2m+\gamma^2-1)(-\mathrm{i}\omega)^{\frac{1}{2}}}\\
&\times\Bigg\{\mathrm{e}^{-\mathrm{i}\pi\gamma(m+\frac{1}{2})}\Bigg[\mathrm{erf}\left(\frac{-\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\mathrm{erf}\left(\frac{-\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\Bigg]\\
&-\mathrm{e}^{\mathrm{i}\pi\gamma(m+\frac{1}{2})}\Bigg[\mathrm{erf}\left(\frac{-\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\mathrm{erf}\left(\frac{-\mathrm{i}\pi(2m+1)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\Bigg]\Bigg\}\\
&\sim\frac{-\gamma(2m+1)\mathrm{e}^{1/z^2}}{\pi^{\frac{1}{2}}(4\gamma^2m^2+4\gamma^2m+\gamma^2-1)(-\mathrm{i}\omega)^{\frac{1}{2}}}\\
&\times\Bigg\{\mathrm{e}^{2\gamma t/z}\Bigg[\mathrm{erf}\left(\frac{1}{z}+(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\mathrm{erf}\left(\frac{1}{z}-(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\Bigg]\\
&-\mathrm{e}^{-2\gamma t/z}\Bigg[\mathrm{erf}\left(\frac{1}{z}-(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\mathrm{erf}\left(\frac{1}{z}+(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\Bigg]\Bigg\},
\end{aligned}
$$

where C_1 is a constant

$$
\begin{aligned}
C_1 &= \frac{(-1)^m\exp\left(-\frac{\mathrm{i}\pi^2}{16\omega\gamma^2}\right)}{\pi^{\frac{1}{2}}(4\gamma^2m^2+4\gamma^2m+\gamma^2-1)(-\mathrm{i}\omega)^{\frac{1}{2}}}\\
&\quad\times\Bigg\{\mathrm{e}^{-\mathrm{i}\pi/(2\gamma)}\Bigg[\mathrm{erf}\left(\frac{-\mathrm{i}\pi}{4\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\mathrm{erf}\left(\frac{-\mathrm{i}\pi}{4\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\Bigg]\\
&\quad-\mathrm{e}^{\mathrm{i}\pi/(2\gamma)}\Bigg[\mathrm{erf}\left(\frac{-\mathrm{i}\pi}{4\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\mathrm{erf}\left(\frac{-\mathrm{i}\pi}{4\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\Bigg]\Bigg\}
\end{aligned}
$$

$$\sim \quad \frac{1}{\gamma^2(2m+1)^2}.$$

Using the formula (3.6), we get

$$\begin{aligned}
& \mathrm{e}^{1/z^2}\left\{\mathrm{e}^{2\gamma t/z}\left[\operatorname{erf}\left(\frac{1}{z}+(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\operatorname{erf}\left(\frac{1}{z}-(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right]\right. \\
& \qquad \left.-\mathrm{e}^{-2\gamma t/z}\left[\operatorname{erf}\left(\frac{1}{z}-(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\operatorname{erf}\left(\frac{1}{z}+(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right]\right\} \\
\sim \; & \frac{-2\mathrm{i}(-1)^m\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}}{\pi^{\frac{1}{2}}}\left\{2\cos(2\gamma\omega)z\right. \\
& \left.+\left[\left((1+\gamma)^2t^2-\frac{1}{2}\right)\mathrm{e}^{\mathrm{i}2\gamma\omega}+\left((1-\gamma)^2t^2-\frac{1}{2}\right)\mathrm{e}^{-\mathrm{i}2\gamma\omega}\right]z^3+\mathcal{O}\big(z^5\big)\right\}.
\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}
d_{m,0}-C_1 \quad \sim \quad & \frac{(-1)^m 16\mathrm{i}\gamma(2m+1)\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}}{\pi(\gamma^2(2m+1)^2-1)}\left\{\frac{\mathrm{i}\cos(2\gamma\omega)}{\pi(2m+1)}+\frac{8\omega}{\pi^3(2m+1)^3}\right. \\
& \left.\big[2\mathrm{i}\omega(1+\gamma^2)\cos(2\gamma\omega)-4\gamma\omega\sin(2\gamma\omega)+\cos(2\gamma\omega)\big]\right\}+\mathcal{O}\big((2m+1)^{-6}\big).
\end{aligned} \tag{3.16}$$

Case (3). When $m=0$ and $n \gg \omega^{\frac{1}{2}}$, letting

$$\frac{1}{z}=\frac{-\mathrm{i}\pi(2n+1)}{4\gamma\,(-\mathrm{i}\omega)^{\frac{1}{2}}}, \qquad \frac{2t\gamma}{z}=-\mathrm{i}\pi\left(n+\frac{1}{2}\right), \qquad \frac{2t}{z}=\frac{-\mathrm{i}\pi}{\gamma}\left(n+\frac{1}{2}\right), \tag{3.17}$$

the formula (2.6) becomes

$$\begin{aligned}
d_{0,n}+C_2 = \; & \frac{(2n+1)\exp\Big(-\frac{\mathrm{i}\pi^2(2n+1)^2}{16\omega\gamma^2}\Big)}{\pi^{\frac{1}{2}}(\gamma^2-4n^2-4n-1)(-\mathrm{i}\omega)^{\frac{1}{2}}} \\
& \times\left\{\mathrm{e}^{-\mathrm{i}\pi(n+\frac{1}{2})/\gamma}\left[\operatorname{erf}\left(\frac{-\mathrm{i}\pi(2n+1)}{4\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\operatorname{erf}\left(\frac{-\mathrm{i}\pi(2n+1)}{4\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right]\right. \\
& \left.-\mathrm{e}^{\mathrm{i}\pi(n+\frac{1}{2})/\gamma}\left[\operatorname{erf}\left(\frac{-\mathrm{i}\pi(2n+1)}{4\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\operatorname{erf}\left(\frac{-\mathrm{i}\pi(2n+1)}{4\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right]\right\} \\
\sim \; & \frac{(2n+1)\mathrm{e}^{1/z^2}}{\pi^{\frac{1}{2}}(\gamma^2-4n^2-4n-1)(-\mathrm{i}\omega)^{\frac{1}{2}}} \\
& \times\left\{\mathrm{e}^{2t/z}\left[\operatorname{erf}\left(\frac{1}{z}+(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\operatorname{erf}\left(\frac{1}{z}+(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right]\right. \\
& \left.-\mathrm{e}^{-2t/z}\left[\operatorname{erf}\left(\frac{1}{z}-(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\operatorname{erf}\left(\frac{1}{z}-(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right]\right\},
\end{aligned}$$

where the constant C_2 is

$$\begin{aligned}
C_2 \quad = \quad & \frac{(-1)^n\gamma\exp\Big(-\frac{\mathrm{i}\pi^2}{16\omega}\Big)}{\pi^{\frac{1}{2}}(\gamma^2-4n^2-4n-1)(-\mathrm{i}\omega)^{\frac{1}{2}}} \\
& \times\left\{\mathrm{e}^{-\mathrm{i}\pi\gamma/2}\left[\operatorname{erf}\left(\frac{-\mathrm{i}\pi}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\operatorname{erf}\left(\frac{-\mathrm{i}\pi}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right]\right. \\
& \left.-\mathrm{e}^{\mathrm{i}\pi\gamma/2}\left[\operatorname{erf}\left(\frac{-\mathrm{i}\pi}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}-(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\operatorname{erf}\left(\frac{-\mathrm{i}\pi}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}+(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right]\right\}.
\end{aligned}$$

Using (3.6) with (3.17) gives

$$\begin{aligned}
&\mathrm{e}^{1/z^2}\left\{\mathrm{e}^{2t/z}\left[\mathrm{erf}\left(\frac{1}{z}+(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\mathrm{erf}\left(\frac{1}{z}+(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right]\right.\\
&\left.-\,\mathrm{e}^{-2t/z}\left[\mathrm{erf}\left(\frac{1}{z}-(1+\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)-\mathrm{erf}\left(\frac{1}{z}-(1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}}\right)\right]\right\}\\
\sim\;&\frac{-2\mathrm{i}(-1)^n\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}}{\pi^{\frac{1}{2}}}\{2\cos(2\gamma\omega)z\\
&+\left[\left((1+\gamma)^2t^2-\frac{1}{2}\right)\mathrm{e}^{\mathrm{i}2\gamma\omega}+\left((1-\gamma)^2t^2-\frac{1}{2}\right)\mathrm{e}^{-\mathrm{i}2\gamma\omega}\right]z^3+\mathcal{O}\big(z^5\big)\Big\},
\end{aligned}$$

therefore

$$\begin{aligned}
d_{0,n}+C_2\;\sim\;&\frac{-(-1)^n16\mathrm{i}\gamma(2n+1)\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}}{\pi(\gamma^2-(2n+1)^2)}\left\{\frac{\mathrm{i}\cos(2\gamma\omega)}{\pi(2n+1)}+\frac{8\gamma^2\omega}{\pi^3(2n+1)^3}\right.\\
&\left.\big[2\mathrm{i}\omega(1+\gamma^2)\cos(2\gamma\omega)-4\gamma\omega\sin(2\gamma\omega)+\cos(2\gamma\omega)\big]\right\}+\mathcal{O}\big((2n+1)^{-6}\big).
\end{aligned}\tag{3.18}$$

As for case (4) with $m\gg\omega^{\frac{1}{2}}$, $n\gg\omega^{\frac{1}{2}}$ and $\kappa\neq 0,\gamma,\infty$, $d_{m,n}$ in (2.6) with (3.17) and (3.18) satisfies

$$\begin{aligned}
d_{m,n}\;=\;&\frac{(-1)^n(\gamma^2(2m+1)^2-1)}{\gamma^2(2m+1)^2-(2n+1)^2}\left[d_{m,0}-C_1(m)\right]\\
&+\frac{(-1)^m(\gamma^2-(2n+1)^2)}{\gamma^2(2m+1)^2-(2n+1)^2}\left[d_{0,n}+C_2(n)\right]\\
\sim\;&\frac{-128(-1)^{m+n}\mathrm{i}\gamma\omega\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}}{\pi^4(2m+1)^2(2n+1)^2}\big[2\mathrm{i}\omega(1+\gamma^2)\cos(2\gamma\omega)-4\gamma\omega\sin(2\gamma\omega)+\cos(2\gamma\omega)\big]+\cdots.
\end{aligned}\tag{3.19}$$

In cases (5) and (6) with $\lim_{m\to\infty}\frac{2n+1}{2m+1}=\kappa=\gamma$, we are concerned with the expressions (2.7) and (2.8). Firstly, when $\omega\gg m$ in case (5), using the asymptotic formula (3.3) in (2.7), we derive the asymptotic behaviour

$$d_{m,(m+\frac{1}{2})\gamma-\frac{1}{2}}\sim\frac{\exp\left(-\frac{\mathrm{i}\pi^2(2m+1)^2}{16\omega}\right)\pi^{\frac{1}{2}}}{(-\mathrm{i}\omega)^{\frac{1}{2}}}+O\big(\omega^{-1}\big).\tag{3.20}$$

On the other hand, if m and n go to infinity in case (6), we consider the expression (2.8). Using the expansions of error function (3.6) and the definitions (3.15) inside (2.8),

$$\begin{aligned}
d_{m,(m+\frac{1}{2})\gamma-\frac{1}{2}}=\;&\frac{\mathrm{e}^{1/z^2}}{16\gamma(2m+1)\omega\pi^{\frac{1}{2}}(-\mathrm{i}\omega)^{\frac{1}{2}}}\\
&\left[(\pi^2(2m+1)^2+8\mathrm{i}\omega+8\pi\omega\left(m+\frac{1}{2}\right)(1+\gamma))\mathrm{erf}\left(\frac{1}{z}+(1+\gamma)t\right)\right.\\
&+(\pi^2(2m+1)^2+8\mathrm{i}\omega-8\pi\omega\left(m+\frac{1}{2}\right)(1+\gamma))\mathrm{erf}\left(\frac{1}{z}-(1+\gamma)t\right)\\
&-(\pi^2(2m+1)^2+8\mathrm{i}\omega+8\pi\omega\left(m+\frac{1}{2}\right)(1-\gamma))\mathrm{erf}\left(\frac{1}{z}+(1-\gamma)t\right)\\
&\left.-(\pi^2(2m+1)^2+8\mathrm{i}\omega-8\pi\omega\left(m+\frac{1}{2}\right)(1-\gamma))\mathrm{erf}\left(\frac{1}{z}-(1-\gamma)t\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(-1)^{\left(m+\frac{1}{2}\right)(1+\gamma)-1}\mathrm{i}\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}\cos(2\gamma\omega)}{\gamma\omega} \\
\sim\ & \frac{(-1)^{\left(m+\frac{1}{2}\right)(1+\gamma)-1}\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}}{8(2m+1)\pi\gamma\omega(-\mathrm{i}\omega)^{\frac{1}{2}}} \\
&\Bigg\{ \left(\pi^2(2m+1)^2+8\mathrm{i}\omega\right)\mathrm{e}^{\mathrm{i}2\gamma\omega}\Bigg(z + \left[(1+\gamma)^2t^2-\frac{1}{2}\right]z^3 \\
&+ \left[(1+\gamma)^4t^4-3(1+\gamma)^2t^2+\frac{3}{4}\right]z^5 \\
&+ \left[(1+\gamma)^6t^6-\frac{15}{2}(1+\gamma)^4t^4+\frac{45}{4}(1+\gamma)^2t^2-\frac{15}{8}\right]z^7\Bigg) \\
&-8\pi\omega\left(m+\frac{1}{2}\right)(1+\gamma)\mathrm{e}^{\mathrm{i}2\gamma\omega}\Bigg((1+\gamma)tz^2+\left[(1+\gamma)^2t^2-\frac{3}{2}\right](1+\gamma)tz^4 \\
&+ \left[(1+\gamma)^4t^4-5(1+\gamma)^2t^2+\frac{15}{4}\right](1+\gamma)tz^6\Bigg) \\
&+ \left(\pi^2(2m+1)^2+8\mathrm{i}\omega\right)\mathrm{e}^{-\mathrm{i}2\gamma\omega}\Bigg(z + \left[(1-\gamma)^2t^2-\frac{1}{2}\right]z^3 \\
&+ \left[(1-\gamma)^4t^4-3(1-\gamma)^2t^2+\frac{3}{4}\right]z^5 \\
&+ \left[(1-\gamma)^6t^6-\frac{15}{2}(1-\gamma)^4t^4+\frac{45}{4}(1-\gamma)^2t^2-\frac{15}{8}\right]z^7\Bigg) \\
&-8\pi\omega\left(m+\frac{1}{2}\right)(1-\gamma)\mathrm{e}^{-\mathrm{i}2\gamma\omega}\Bigg((1-\gamma)tz^2+\left[(1-\gamma)^2t^2-\frac{3}{2}\right](1-\gamma)tz^4 \\
&+ \left[(1-\gamma)^4t^4-5(1-\gamma)^2t^2+\frac{15}{4}\right](1-\gamma)tz^6\Bigg) + \mathcal{O}\left(z^8\right)\Bigg\} \\
&- \frac{(-1)^{\left(m+\frac{1}{2}\right)(1+\gamma)-1}\mathrm{i}\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}\cos(2\gamma\omega)}{\gamma\omega} \\
=\ & \frac{(-1)^{\left(m+\frac{1}{2}\right)(1+\gamma)-1}\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}}{8(2m+1)\pi\gamma\omega(-\mathrm{i}\omega)^{\frac{1}{2}}}\left\{\left(\pi^2(2m+1)^2+8\mathrm{i}\omega\right)2\cos(2\gamma\omega)z\right. \\
&- 8\pi\omega\left(m+\frac{1}{2}\right)\left[(1+\gamma)^2\mathrm{e}^{\mathrm{i}2\gamma\omega}+(1-\gamma)^2\mathrm{e}^{-\mathrm{i}2\gamma\omega}\right]tz^2 \\
&+ \left(\pi^2(2m+1)^2+8\mathrm{i}\omega\right)\left[\left((1+\gamma)^2t^2-\frac{1}{2}\right)\mathrm{e}^{\mathrm{i}2\gamma\omega}+\left((1-\gamma)^2t^2-\frac{1}{2}\right)\mathrm{e}^{-\mathrm{i}2\gamma\omega}\right]z^3 \\
&- 8\pi\omega\left(m+\frac{1}{2}\right)\left[(1+\gamma)^2\left((1+\gamma)^2t^2-\frac{3}{2}\right)\mathrm{e}^{\mathrm{i}2\gamma\omega}\right. \\
&\qquad\qquad \left.+(1-\gamma)^2\left((1-\gamma)^2t^2-\frac{3}{2}\right)\mathrm{e}^{-\mathrm{i}2\gamma\omega}\right]tz^4 \\
&+ \left(\pi^2(2m+1)^2+8\mathrm{i}\omega\right)\left[\left((1+\gamma)^4t^4-3(1+\gamma)^2t^2+\frac{3}{4}\right)\mathrm{e}^{\mathrm{i}2\gamma\omega}\right. \\
&\qquad\qquad \left.+\left((1-\gamma)^4t^4-3(1-\gamma)^2t^2+\frac{3}{4}\right)\mathrm{e}^{-\mathrm{i}2\gamma\omega}\right]z^5 \\
&- 8\pi\omega\left(m+\frac{1}{2}\right)\left[(1+\gamma)^2\left((1+\gamma)^4t^4-5(1+\gamma)^2t^2+\frac{15}{4}\right)\mathrm{e}^{\mathrm{i}2\gamma\omega}\right. \\
&\qquad\qquad \left.+(1-\gamma)^2\left((1-\gamma)^4t^4-5(1-\gamma)^2t^2+\frac{15}{4}\right)\mathrm{e}^{-\mathrm{i}2\gamma\omega}\right]tz^6
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\pi^2(2m+1)^2+8\mathrm{i}\omega\right)\Bigg[\bigg((1+\gamma)^6t^6-\frac{15}{2}(1+\gamma)^4t^4+\frac{45}{4}(1+\gamma)^2t^2-\frac{15}{8}\bigg)\mathrm{e}^{\mathrm{i}2\gamma\omega}\\
&\qquad+\bigg((1-\gamma)^6t^6-\frac{15}{2}(1-\gamma)^4t^4+\frac{45}{4}(1-\gamma)^2t^2-\frac{15}{8}\bigg)\mathrm{e}^{-\mathrm{i}2\gamma\omega}\Bigg]z^7+\mathcal{O}\big(z^8\big)\Bigg\}\\
&-\frac{(-1)^{\left(m+\frac{1}{2}\right)(1+\gamma)-1}\mathrm{i}\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}\cos(2\gamma\omega)}{\gamma\omega}\\
\sim\;&\frac{(-1)^{\left(m+\frac{1}{2}\right)(1+\gamma)-1}\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}128\omega}{\gamma\pi^4(2m+1)^4}\left[(2\gamma^2\omega+2\omega-\mathrm{i})\cos(2\gamma\omega)+4\mathrm{i}\gamma\omega\sin(2\gamma\omega)\right]\\
&+\frac{(-1)^{\left(m+\frac{1}{2}\right)(1+\gamma)-1}\mathrm{e}^{\mathrm{i}\omega(1+\gamma^2)}2048\omega^2}{\gamma\pi^6(2m+1)^6}\\
&\qquad\qquad\left[\left(4\gamma^4\omega^2+24\gamma^2\omega^2-12\mathrm{i}\gamma^2\omega-12\mathrm{i}\omega+4\omega^2-3\right)\cos(2\gamma\omega)\right.\\
&\qquad\qquad\left.+\left(16\mathrm{i}\gamma^3\omega^2+24\gamma\omega+16\mathrm{i}\gamma\omega^2\right)\sin(2\gamma\omega)\right]+\mathcal{O}\big((2m+1)^{-8}\big).
\end{aligned}
\tag{3.21}
$$

We are interested in the numerical results for verifying the decay rate of the coefficients $d_{m,n}$ for large m, n or large ω. The curves $-\log_{10}|d_{m,n}|$ (blue dotted lines) and its asymptotic expressions (the red lines) are depicted in Fig. 3.3, for $n = 0$ (the top left, derived from (3.16)), $m = 0$ (the top right, obtained from (3.18)), $m = n$ (the bottom left, 3.19) and $\gamma(2m + 1) = 2n + 1$ (the bottom right, (3.21)) with increasing m, n. Note that oscillation occurs when m or n are less than 100. The reason is that when $m, n < \omega = 100$, the parameter ω dominates the behaviour of $d_{m,n}$. Thus, in this regime, we need to use large ω asymptotics in their analysis.

Fig. 3.4 illustrates our estimates for large ω, fixed m, n and $\gamma = \frac{1}{3}$, and they are in accordance with the exact decay rate of $d_{m,n}$. The top left red curve corresponds to the estimate of (3.13) with $m = 2$, $n = 7$, the top right for (3.14) with $m = 4$, $n = 5$ and the bottom for (3.20) with $m = 4$, $n = 1$, respectively.

4 Numerical results

The goal of the paper isan efficient calculation of the eigenvalues of the integral equation (1.1). The initial integral equation (1.1) has been transformed into two sub-systems of linear equations (1.5) and (1.6), where the structure of the matrices $\mathcal{A}$ and $\mathcal{D}$ plays a vital role in the determination of a suitable numerical method. Thus, firstly we display the images with scaled colours for the matrices $\mathcal{A} = (a_{m,n})_{1\le m,n\le 1000}$ and $\mathcal{D} = (d_{m,n})_{1\le m,n\le 1000}$ of dimension 1000 in Fig. 4.1 for $\gamma = \frac{1}{10}\frac{3}{10}, \frac{1}{2}$ and Fig. 4.2 for $\gamma = \frac{7}{10}, \frac{9}{10}, 1$, by taking the logarithmic (to base10) scale for the absolute value of each entry. The darker the colour, the smaller the values in $\mathcal{A}$ and $\mathcal{D}$. It is evident that the elements of the matrices become much smaller with increasing m and n. The entries in horizontal and vertical directions decay slower than along the diagonal direction, and this is consistent with the theoretical analysis of Section 3. Furthermore, it is observed that the area of the lighter region becomes larger, in other words, there are more larger values as γ increases from $\frac{1}{10}$ to 1. Although the maximum values for the matrices remain the same, the minimum values increase from 10^{-10} to 10^{-8} as γ increases from $\frac{1}{10}$ into 1.

To calculate the eigenvalues, the coefficient matrix of the infinite system (1.5) or (1.6) must be truncated in practical computation. Note that the hyperbolic-cross nature of

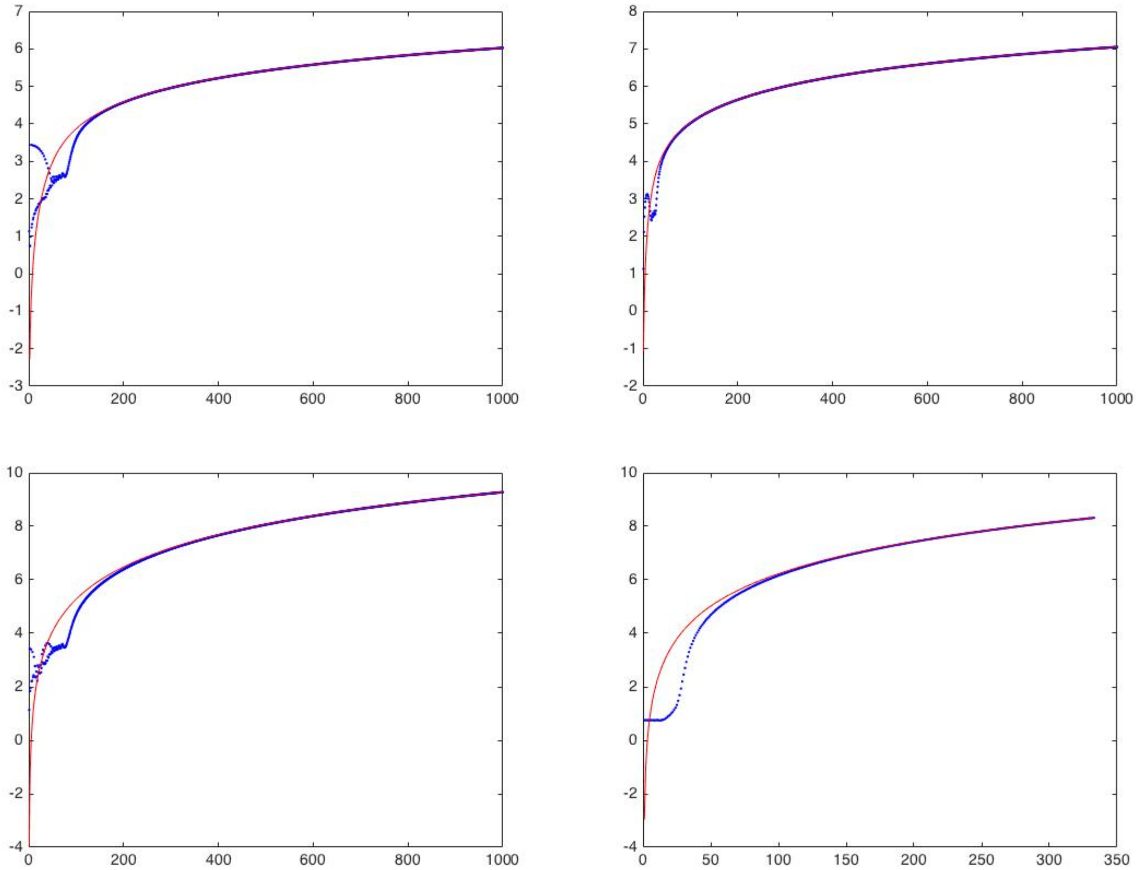

Figure 3.3: $-\log_{10}|d_{m,n}|$ (the blue dotted line) and its asymptotic approximation (the red line), for $n = 0$ (the top left), $m = 0$ (the top right), $m = n$ (the bottom left) and $m = 3n + 1$ (the bottom right) with $\gamma = \frac{1}{3}$ and $\omega = 100$.

the patterns of $\mathcal{A}$ and $\mathcal{D}$ exhibited in Figs 4.1 and 4.2 suggest the truncation technique proposed in (Brunner et al. 2011). Partitioning the matrix $\mathcal{A}$ in the form

$$\mathcal{A} = \left[\begin{array}{cc} \mathcal{A}_{1,1_{r\times r}} & \mathcal{A}_{1,2_{r\times\infty}} \\ \mathcal{A}_{2,1_{\infty\times r}} & \mathcal{A}_{2,2_{\infty\times\infty}} \end{array} \right],$$

it is observed that the entries of $\mathcal{A}_{2,2}$ are very small for sufficiently large r. Thus, we can set the sub-matrix $\mathcal{A}_{2,2}$ to zero. Correspondingly the matrix $\mathcal{A}$ is approximated by a new infinite matrix

$$\tilde{\mathcal{A}} = \left[\begin{array}{cc} \mathcal{A}_{1,1_{r\times r}} & \mathcal{A}_{1,2_{r\times\infty}} \\ \mathcal{A}_{2,1_{\infty\times r}} & \mathbf{0}_{\infty\times\infty} \end{array} \right]. \tag{4.1}$$

According to (Brunner et al. 2011, Theorem 7), the rank of the matrix $\tilde{\mathcal{A}}$ in (4.1) is $2r$ and its nonzero eigenvalues are the same as those of a $2r$ dimensional square matrix

$$\mathcal{B} = \left[\begin{array}{cc} \mathcal{A}_{1,1} & \mathcal{G}_1 \\ \mathcal{G}_2 & \mathbf{0} \end{array} \right]_{2r\times 2r},$$

such that $\mathcal{G}_1\mathcal{G}_2 = \mathcal{A}_{1,2}\mathcal{A}_{2,1}$. The four sub-matrices are all r square matrices. One simple choice is $\mathcal{G}_1 = \mathcal{G}$ and $\mathcal{G}_2 = \mathcal{I}$.

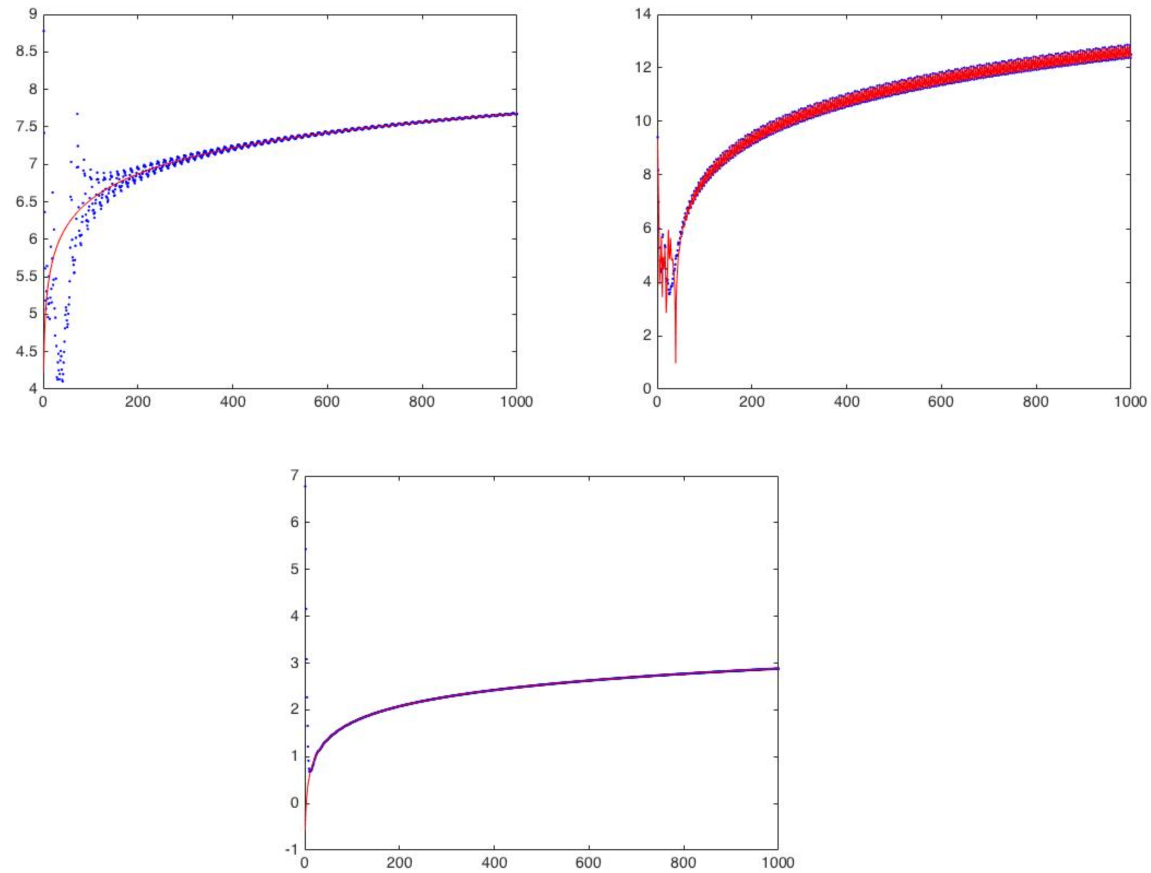

Figure 3.4: $-\log_{10}|d_{m,n}(\omega)|$ (the blue dotted line) and its asymptotic approximation (the red line), for $m = 2$, $n = 7$ (the top left), $m = 4$, $n = 5$ (the top right), $m = 4$, $n = 1$ (the bottom) with $\gamma = \frac{1}{3}$, $\omega \in [1, 1000]$.

We are also interested in the relationship between the eigenvalues and the parameter γ. The eigenvalues of the resulting linear system with $\omega = 100$ and $N = 1000$ for $\gamma = \frac{1}{10}, \frac{3}{10}, \frac{1}{2}, \frac{7}{10}, \frac{9}{10}, 1$, are plotted in Figs 4.3 and 4.4. Not all of the eigenvalues are visible since many are very small: the operator being compact, they accumulate at the origin. With increasing γ, $0 < \gamma < 1$, there are more eigenvalues visible as their magnitude increases.

The next step is to see how the eigenfunctions change with increasing γ in Fig. 4.5. Moreover, as γ increases, it seems that the eigenfunctions become less oscillatory. In particular, the eigenfunctions corresponding to the large eigenvalues (such as the first and second), become less oscillatory. The oscillatory nature of the eigenfunctions corresponding to small eigenvalues appears to be insensitive to γ.

5 Conclusions

Numerical simulation of laser resonators requires the knowledge of the eigenvalues and eigenfunctions of an oscillatory integral of the first kind with associated constant $\gamma \in (0, 1)$. This paper presents an effective, explicit algorithm for the computation of the discretised matrix and its corresponding eigenvalues. This algorithm commences from an infinite matrix and then considers in some detail the asymptotic properties of the matrix entries.

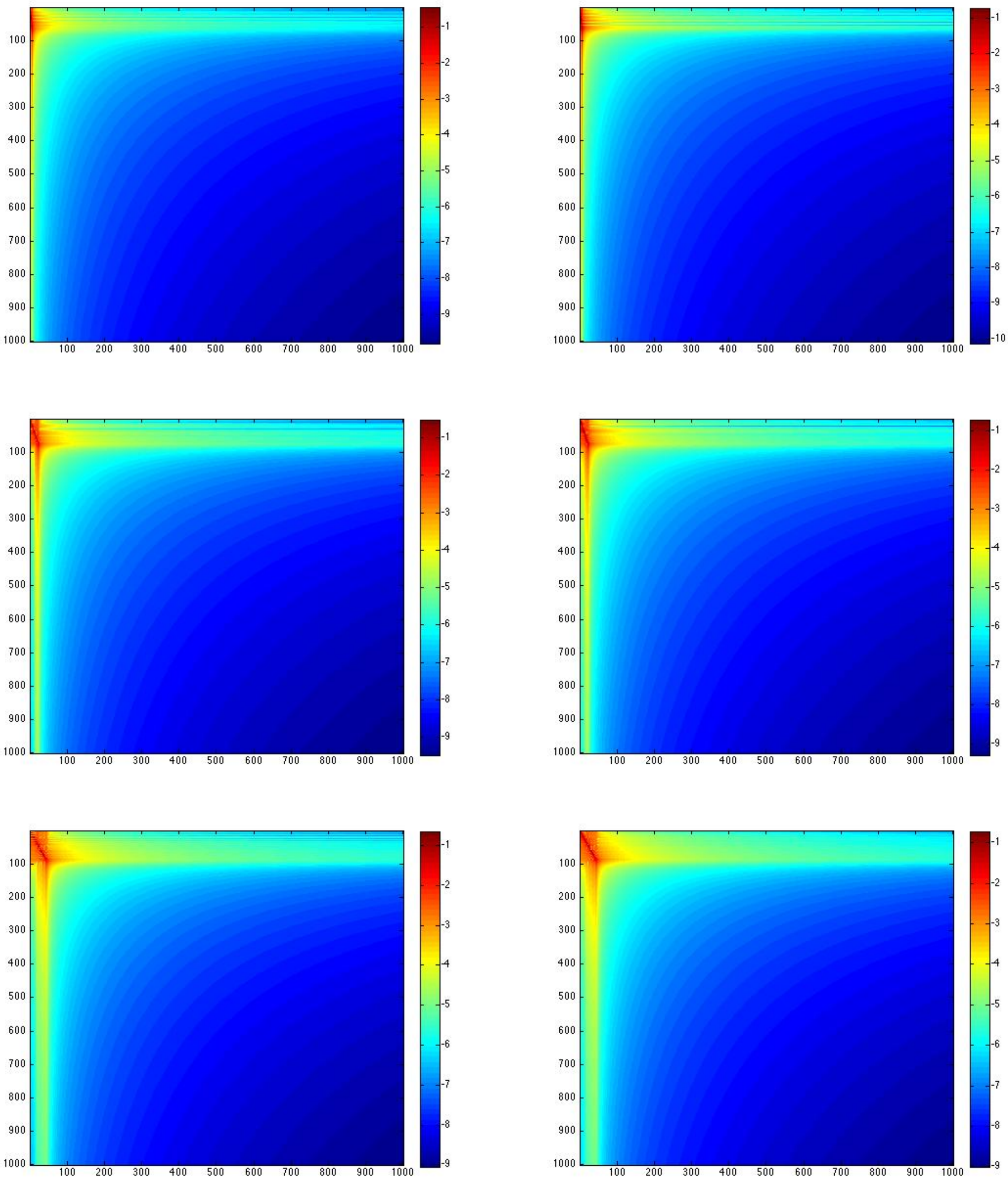

Figure 4.1: Images of matrices $\mathcal{A} = (a_{m,n})_{0 \le m,n < 1000}$ (left) and $\mathcal{D} = (d_{m,n})_{0 \le m,n < 1000}$ (right) in the form $\log_{10} |\cdot|$ with $\gamma = \frac{1}{10}$ (top), $\gamma = \frac{3}{10}$ (middle) and $\gamma = \frac{1}{2}$ (bottom) for $\omega = 100$.

Knowledge of these entries allows for a finite-dimensional truncation if the matrix which reduces the computational cost, while delivering desired accuracy. Moreover, asymptotic analysis shows that the decay of the entries occurs in a pattern which allows us to use the hyperbolic cross for a further substantial reduction in the computational outlay. Our asymptotic analysis is fully confirmed by our numerical experiments.

We note that mathematical – as distinct from numerical or asymptotic – analysis of

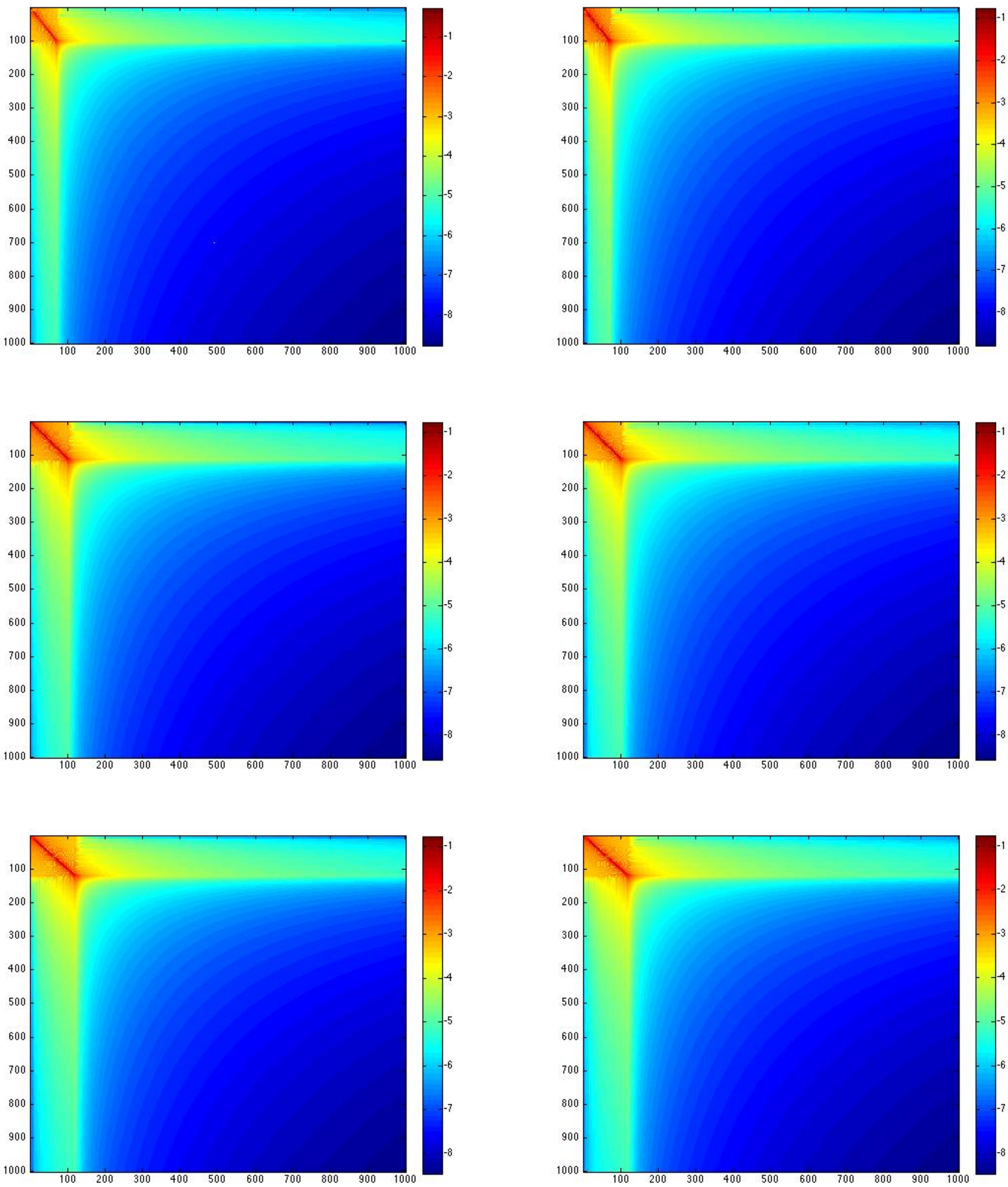

Figure 4.2: Images of matrices $\mathcal{A} = (a_{m,n})_{0\le m,n<1000}$ (left) and $\mathcal{D} = (d_{m,n})_{0\le m,n<1000}$ (right) in the form $\log_{10}|\cdot|$ with $\gamma = \frac{7}{10}$ (top), $\gamma = \frac{9}{10}$ (middle) and $\gamma = 1$ (bottom) for $\omega = 100$.

the spectra of the operator (1.1) is virtually nonexistent. Even the more familiar Fox–Li operator (the case $\gamma = 1$) represents a challenge for spectral theory (Böttcher, Grudsky & Iserles 2012).

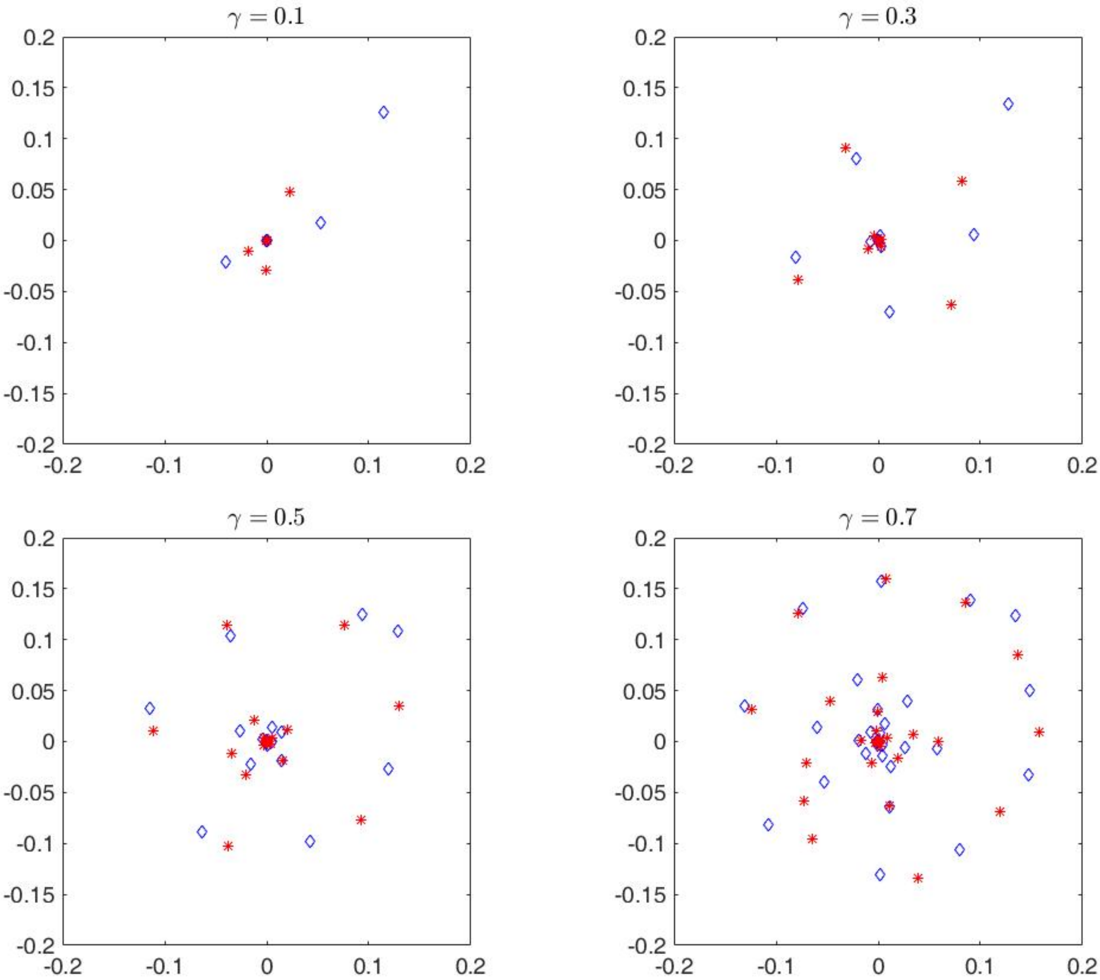

Figure 4.3: The eigenvalues of the integral operator with $\omega = 100$ for $\gamma = \frac{1}{10}, \frac{3}{10}, \frac{1}{2}, \frac{7}{10}$. The blue diamond denotes the 'cos' coefficients; The red stars denotes the 'sin' coefficients.

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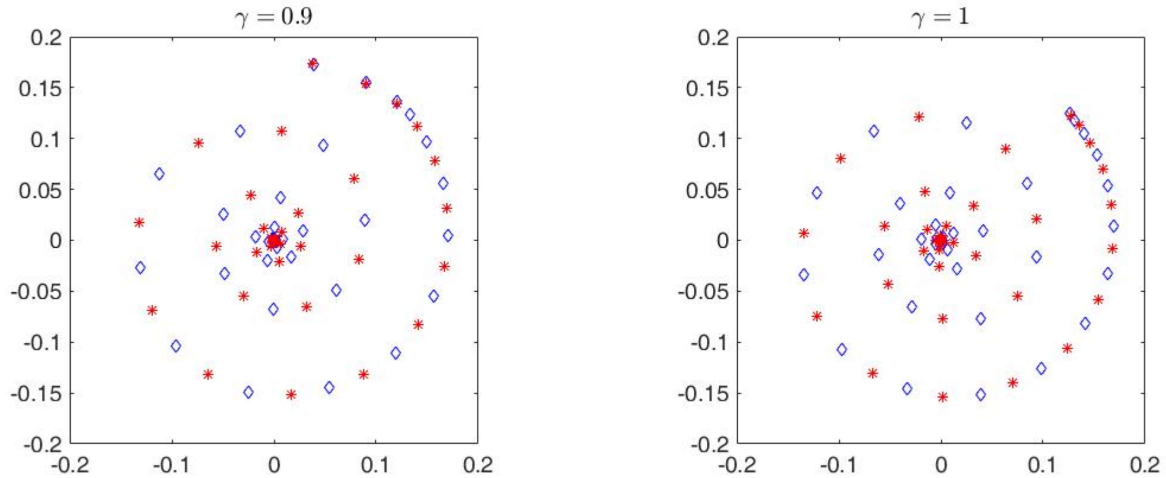

Figure 4.4: The eigenvalues of the integral operator with $\omega = 100$ for $\gamma = \frac{9}{10}, 1$. The blue diamond stands for the 'cos' coefficients; The red stars denotes the 'sin' coefficients.

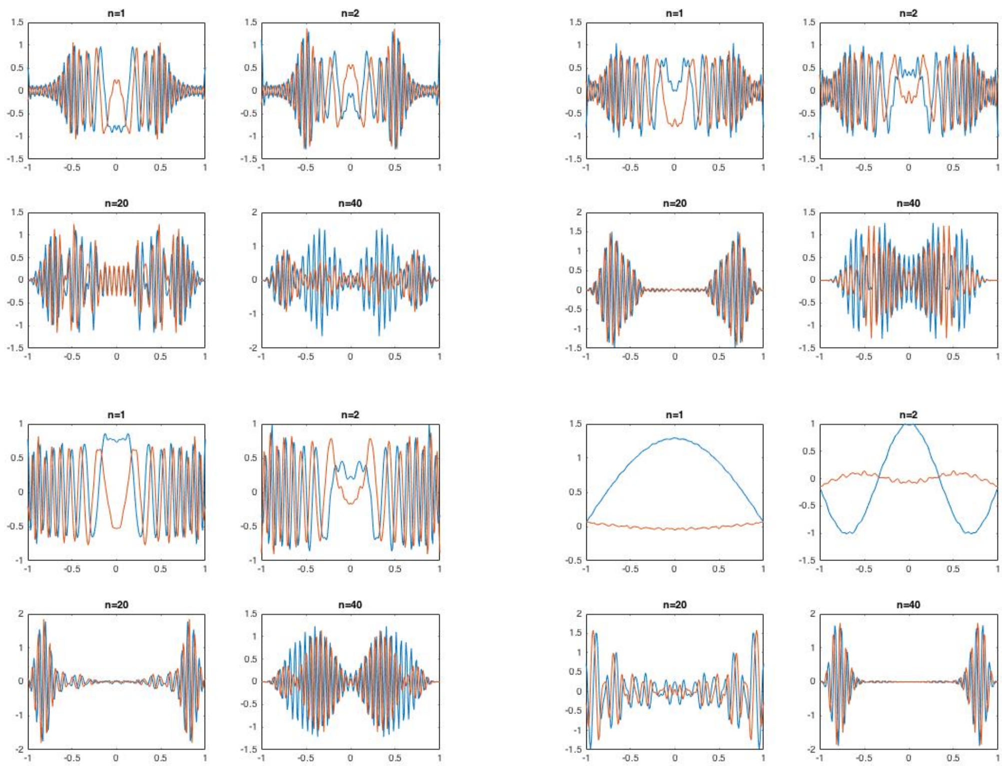

Figure 4.5: Real and imaginary parts of the eigenfunctions corresponding to the first, second, twentieth and fortieth 'cosine' eigenvalues, respectively, with $\omega = 100$ for $\gamma = \frac{1}{10}$ (top left), $\gamma = \frac{1}{2}$ (top right), $\gamma = \frac{7}{10}$ (bottom left) and $\gamma = 1$ (bottom right).

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A Proof of Theorem 1: explicit expressions for $a_{m,n}$

Proof. Recalling that $\phi_m(x) = \cos m\pi x$, we let $a_{m,n} = \mathcal{I}_\omega[\cos m\pi x, \cos n\pi y]$. Given $\gamma \in (0,1]$ and $\gamma m \neq n$ (which incidentally requires that m and n are not both zero simultaneously) it holds that

$$\begin{aligned}
\int_{-1}^{1} \cos(\pi m(t+\gamma y))\cos(\pi n y)\mathrm{d}y &= \frac{2(-1)^n m\gamma\sin(\pi\gamma m)}{\pi(\gamma^2 m^2 - n^2)}\cos(\pi m t),\\
\int_{-1}^{(1-t)/\gamma} \cos(\pi m(t+\gamma y))\cos(\pi n y)\mathrm{d}y &= \frac{(-1)^m n\cos\frac{\pi n}{\gamma}}{\pi(\gamma^2 m^2 - n^2)}\sin\frac{\pi n t}{\gamma} - \frac{(-1)^m n\sin\frac{\pi n}{\gamma}}{\pi(\gamma^2 m^2 - n^2)}\cos\frac{\pi n t}{\gamma}\\
&\quad - \frac{(-1)^n m\gamma\cos(\pi\gamma m)}{\pi(\gamma^2 m^2 - n^2)}\sin(\pi m t)\\
&\quad + \frac{(-1)^n m\gamma\sin(\pi\gamma m)}{\pi(\gamma^2 m^2 - n^2)}\cos(\pi m t),\\
\int_{-(1+t)/\gamma}^{1} \cos(\pi m(t+\gamma y))\cos(\pi n y)\mathrm{d}y &= -\frac{(-1)^m n\cos\frac{\pi n}{\gamma}}{\pi(\gamma^2 m^2 - n^2)}\sin\frac{\pi n t}{\gamma} - \frac{(-1)^m n\sin\frac{\pi n}{\gamma}}{\pi(\gamma^2 m^2 - n^2)}\cos\frac{\pi n t}{\gamma}\\
&\quad + \frac{(-1)^n m\gamma\cos(\pi\gamma m)}{\pi(\gamma^2 m^2 - n^2)}\sin(\pi m t)\\
&\quad + \frac{(-1)^n m\gamma\sin(\pi\gamma m)}{\pi(\gamma^2 m^2 - n^2)}\cos(\pi m t).
\end{aligned}$$

When $\gamma = n/m$ and $1 \le n \le m$, we have

$$\begin{aligned}
\int_{-1}^{1} \cos(\pi m(t+\gamma y))\cos(\pi n y)\mathrm{d}y &= \cos(\pi m t),\\
\int_{-1}^{(1-t)/\gamma} \cos(\pi m(t+\gamma y))\cos(\pi n y)\mathrm{d}y &= -\frac{m}{2n}t\cos(\pi m t) + \frac{m+n}{2n}\cos(\pi m t)\\
&\quad - \frac{1}{2\pi n}\sin(\pi m t),\\
\int_{-(1+t)/\gamma}^{1} \cos(\pi m(t+\gamma y))\cos(\pi n y)\mathrm{d}y &= \frac{m}{2n}t\cos(\pi m t) + \frac{m+n}{2n}\cos(\pi m t) + \frac{1}{2\pi n}\sin(\pi m t).
\end{aligned}$$

Note in particular that the term $a_{0,0}$ has the exact expression

$$\begin{aligned}
a_{0,0} = \int_{-1}^{1}\int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega(x-\gamma y)^2}\mathrm{d}y\mathrm{d}x = -\frac{\mathrm{e}^{-2\mathrm{i}\omega\gamma}}{\omega\gamma(-\mathrm{i}\omega)^{\frac{1}{2}}}\Big[&-\mathrm{i}(-\mathrm{i}\omega)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\omega+4\mathrm{i}\omega\gamma+\mathrm{i}\gamma^2\omega} + \mathrm{i}(-\mathrm{i}\omega)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\omega+\mathrm{i}\gamma^2\omega}\\
&- \omega\pi^{\frac{1}{2}}\mathrm{e}^{2\mathrm{i}\omega\gamma}\mathrm{erf}\left(\frac{\mathrm{i}\omega(1-\gamma)}{(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)(1-\gamma) + \omega\pi^{\frac{1}{2}}\mathrm{e}^{2\mathrm{i}\omega\gamma}\mathrm{erf}\left(\frac{\mathrm{i}\omega(\gamma+1)}{(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)(1+\gamma)\Big].
\end{aligned}$$

Let

$$C_{a,b}(\alpha) := \int_a^b \cos(\alpha t)\mathrm{e}^{\mathrm{i}\omega t^2}\mathrm{d}t = \frac{\pi^{\frac{1}{2}}\exp\left(-\frac{\mathrm{i}\alpha^2}{4\omega}\right)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}\left[-\mathrm{erf}\left(a(-\mathrm{i}\omega)^{\frac{1}{2}} - \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)\right.$$
$$- \mathrm{erf}\left(a(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right) + \mathrm{erf}\left(b(-\mathrm{i}\omega)^{\frac{1}{2}} - \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)$$
$$\left. + \mathrm{erf}\left(b(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)\right],$$

$$S_{a,b}(\alpha) := \int_a^b \sin(\alpha t)\mathrm{e}^{\mathrm{i}\omega t^2}\mathrm{d}t = \frac{\mathrm{i}\pi^{\frac{1}{2}}\exp\left(-\frac{\mathrm{i}\alpha^2}{4\omega}\right)}{4(-\mathrm{i}\omega)^{\frac{1}{2}}}\left[\mathrm{erf}\left(a(-\mathrm{i}\omega)^{\frac{1}{2}} - \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)\right.$$
$$- \mathrm{erf}\left(a(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right) - \mathrm{erf}\left(b(-\mathrm{i}\omega)^{\frac{1}{2}} - \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)$$
$$\left. + \mathrm{erf}\left(b(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)\right],$$

$$C_{1,a,b}(\alpha) := \int_a^b t\cos(\alpha t)\mathrm{e}^{\mathrm{i}\omega t^2}\mathrm{d}t = \frac{\mathrm{i}}{2\omega}\left[\cos(a\alpha)\exp(\mathrm{i}\omega a^2) - \cos(b\alpha)\exp(\mathrm{i}\omega b^2)\right]$$
$$+ \frac{\pi^{\frac{1}{2}}\alpha\exp\left(-\frac{\mathrm{i}\alpha^2}{4\omega}\right)}{8\omega(-\mathrm{i}\omega)^{\frac{1}{2}}}\left[\mathrm{erf}\left(a(-\mathrm{i}\omega)^{\frac{1}{2}} - \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)\right.$$
$$- \mathrm{erf}\left(a(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right) - \mathrm{erf}\left(b(-\mathrm{i}\omega)^{\frac{1}{2}} - \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)$$
$$\left. + \mathrm{erf}\left(b(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)\right].$$

We observe that

$$C_{1-\gamma,1+\gamma}(\alpha) = C_{-1-\gamma,-1+\gamma}(\alpha)\,,$$
$$S_{1-\gamma,1+\gamma}(\alpha) = -S_{-1-\gamma,-1+\gamma}(\alpha)\,,$$
$$C_{1,1-\gamma,1+\gamma}(\alpha) = -C_{1,-1-\gamma,-1+\gamma}(\alpha)$$

and

$$C_{-1+\gamma,1-\gamma}(\alpha) = \frac{\pi^{\frac{1}{2}}\exp\left(-\frac{\mathrm{i}\alpha^2}{4\omega}\right)}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\left[\mathrm{erf}\left((1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}} - \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)\right.$$
$$\left. + \mathrm{erf}\left((1-\gamma)(-\mathrm{i}\omega)^{\frac{1}{2}} + \frac{\mathrm{i}\alpha}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}\right)\right].$$

For $0 < \gamma \le 1$, $\gamma \ne n/m$ and n and $m^2 + n^2 > 0$,

$$a_{m,n} = \frac{(-1)^n m\gamma\sin(\pi\gamma m)}{\pi(\gamma^2 m^2 - n^2)}[2C_{-1+\gamma,1-\gamma}(\pi m) + C_{1-\gamma,1+\gamma}(\pi m) + C_{-1-\gamma,-1+\gamma}(\pi m)]$$
$$- \frac{(-1)^n m\gamma\cos(\pi\gamma m)}{\pi(\gamma^2 m^2 - n^2)}[S_{1-\gamma,1+\gamma}(\pi m) - S_{-1-\gamma,-1+\gamma}(\pi m)]$$

$$
\begin{aligned}
&- \frac{(-1)^m n \sin \frac{\pi n}{\gamma}}{\pi(\gamma^2 m^2 - n^2)} \left[C_{1-\gamma,1+\gamma}\left(\frac{\pi n}{\gamma}\right) + C_{-1-\gamma,-1+\gamma}\left(\frac{\pi n}{\gamma}\right) \right] \\
&+ \frac{(-1)^m n \cos \frac{\pi n}{\gamma}}{\pi(\gamma^2 m^2 - n^2)} \left[S_{1-\gamma,1+\gamma}\left(\frac{\pi n}{\gamma}\right) - S_{-1-\gamma,-1+\gamma}\left(\frac{\pi n}{\gamma}\right) \right].
\end{aligned}
$$

On the other hand, when $\gamma = n/m$ for $\gamma \in (0,1]$, $1 \leq n \leq m$, we have

$$
\begin{aligned}
a_{m,m\gamma} &= C_{-1+\gamma,1-\gamma}(\pi m) - \frac{1}{2\gamma} \left[C_{1,1-\gamma,1+\gamma}(\pi m) - C_{1,-1-\gamma,-1+\gamma}(\pi m) \right] + \\
&\quad \frac{1+\gamma}{2\gamma} \left[C_{1-\gamma,1+\gamma}(\pi m) + C_{-1-\gamma,-1+\gamma}(\pi m) \right] \\
&\quad - \frac{1}{2\pi m\gamma} \left[S_{1-\gamma,1+\gamma}(\pi m) - S_{-1-\gamma,-1+\gamma}(\pi m) \right] \\
&= C_{-1+\gamma,1-\gamma}(\pi m) - \frac{1}{\gamma} C_{1,1-\gamma,1+\gamma}(\pi m) + \frac{1+\gamma}{\gamma} C_{1-\gamma,1+\gamma}(\pi m) - \frac{1}{\pi m\gamma} S_{1-\gamma,1+\gamma}(\pi m).
\end{aligned}
$$

It is now trivial to deduce Theorem 1. □

B Proof of Theorem 2: explicit expressions for $d_{m,n}$

Proof. Following a similar line of reasoning for $d_{m,n}$, we firstly consider the case of $0 < \gamma \leq 1$ and $(2m+1)\gamma \neq 2n+1$, where m and n are non-negative integers. Similarly, it holds that

$$
\begin{aligned}
&\int_{-1}^{1} \sin\left(\pi\left(m+\frac{1}{2}\right)(t+\gamma y)\right) \sin\left(\pi\left(n+\frac{1}{2}\right)y\right) \mathrm{d}y \\
&= \frac{-4(-1)^n \gamma(2m+1) \cos\left(\left(m+\frac{1}{2}\right)\pi\gamma\right)}{\pi(4\gamma^2 m^2 + 4\gamma^2 m + \gamma^2 - 4n^2 - 4n - 1)} \cos\left(\left(m+\frac{1}{2}\right)\pi t\right), \\
&\int_{-1}^{(1-t)/\gamma} \sin\left(\pi\left(m+\frac{1}{2}\right)(t+\gamma y)\right) \sin\left(\pi\left(n+\frac{1}{2}\right)y\right) \mathrm{d}y \\
&= \frac{-2(-1)^n \gamma(2m+1) \cos\left(\left(m+\frac{1}{2}\right)\pi\gamma\right)}{\pi(4\gamma^2 m^2 + 4\gamma^2 m + \gamma^2 - 4n^2 - 4n - 1)} \cos\left(\left(m+\frac{1}{2}\right)\pi t\right) \\
&\quad + \frac{-2(-1)^n \gamma(2m+1) \sin\left(\left(m+\frac{1}{2}\right)\pi\gamma\right)}{\pi(4\gamma^2 m^2 + 4\gamma^2 m + \gamma^2 - 4n^2 - 4n - 1)} \sin\left(\left(m+\frac{1}{2}\right)\pi t\right) \\
&\quad + \frac{2(-1)^m (2n+1) \cos\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\gamma}\right)}{\pi(4\gamma^2 m^2 + 4\gamma^2 m + \gamma^2 - 4n^2 - 4n - 1)} \cos\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\gamma} t\right) \\
&\quad + \frac{2(-1)^m (2n+1) \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\gamma}\right)}{\pi(4\gamma^2 m^2 + 4\gamma^2 m + \gamma^2 - 4n^2 - 4n - 1)} \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\gamma} t\right), \\
&\int_{-(1+t)/\gamma}^{1} \sin\left(\pi\left(m+\frac{1}{2}\right)(t+\gamma y)\right) \sin\left(\pi\left(n+\frac{1}{2}\right)y\right) \mathrm{d}y \\
&= \frac{-2(-1)^n \gamma(2m+1) \cos\left(\left(m+\frac{1}{2}\right)\pi\gamma\right)}{\pi(4\gamma^2 m^2 + 4\gamma^2 m + \gamma^2 - 4n^2 - 4n - 1)} \cos\left(\left(m+\frac{1}{2}\right)\pi t\right) \\
&\quad + \frac{2(-1)^n \gamma(2m+1) \sin\left(\left(m+\frac{1}{2}\right)\pi\gamma\right)}{\pi(4\gamma^2 m^2 + 4\gamma^2 m + \gamma^2 - 4n^2 - 4n - 1)} \sin\left(\left(m+\frac{1}{2}\right)\pi t\right) \\
&\quad + \frac{2(-1)^m (2n+1) \cos\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\gamma}\right)}{\pi(4\gamma^2 m^2 + 4\gamma^2 m + \gamma^2 - 4n^2 - 4n - 1)} \cos\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\gamma} t\right)
\end{aligned}
$$

$$+\frac{-2(-1)^m(2n+1)\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\gamma}\right)}{\pi(4\gamma^2m^2+4\gamma^2m+\gamma^2-4n^2-4n-1)}\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\gamma}t\right).$$

On the other hand, if $\gamma \in (0,1]$ and $(2m+1)\gamma = 2n+1$, where m, n are nonnegative integers, the corresponding integrals are

$$\int_{-1}^{1}\sin\left(\pi\left(m+\frac{1}{2}\right)\left(t+\frac{2n+1}{2m+1}y\right)\right)\sin\left(\pi\left(n+\frac{1}{2}\right)y\right)\mathrm{d}y = \cos\left(\left(m+\frac{1}{2}\right)\pi t\right),$$

$$\int_{-1}^{\frac{(1-t)(2m+1)}{2n+1}}\sin\left(\pi\left(m+\frac{1}{2}\right)\left(t+\frac{2n+1}{2m+1}y\right)\right)\sin\left(\pi\left(n+\frac{1}{2}\right)y\right)\mathrm{d}y$$
$$= -\frac{2m+1}{4n+2}t\cos\left(\left(m+\frac{1}{2}\right)\pi t\right)+\frac{m+n+1}{2n+1}\cos\left(\left(m+\frac{1}{2}\right)\pi t\right)$$
$$-\frac{1}{(2n+1)\pi}\sin\left(\left(m+\frac{1}{2}\right)\pi t\right),$$

$$\int_{\frac{-(1+t)(2m+1)}{2n+1}}^{1}\sin\left(\pi\left(m+\frac{1}{2}\right)\left(t+\frac{2n+1}{2m+1}y\right)\right)\sin\left(\pi\left(n+\frac{1}{2}\right)y\right)\mathrm{d}y$$
$$= \frac{2m+1}{4n+2}t\cos\left(\left(m+\frac{1}{2}\right)\pi t\right)+\frac{m+n+1}{2n+1}\cos\left(\left(m+\frac{1}{2}\right)\pi t\right)$$
$$+\frac{1}{(2n+1)\pi}\sin\left(\left(m+\frac{1}{2}\right)\pi t\right).$$

Based on the definition (1.4), the coefficient $d_{m,n}$ can be represented in the form

$$\begin{aligned}
d_{m,n} &= \mathcal{I}_\omega\left[\sin\left(m+\frac{1}{2}\right)\pi x, \sin\left(n+\frac{1}{2}\right)\pi y\right] \\
&= \frac{-4(-1)^n\gamma(2m+1)\cos\left(\left(m+\frac{1}{2}\right)\pi\gamma\right)}{\pi(4\gamma^2m^2+4\gamma^2m+\gamma^2-4n^2-4n-1)} \\
&\quad \left[C_{-1+\gamma,1-\gamma}\left(\left(m+\frac{1}{2}\right)\pi\right)+C_{1-\gamma,1+\gamma}\left(\left(m+\frac{1}{2}\right)\pi\right)\right] \\
&\quad +\frac{-4(-1)^n\gamma(2m+1)\sin\left(\left(m+\frac{1}{2}\right)\pi\gamma\right)}{\pi(4\gamma^2m^2+4\gamma^2m+\gamma^2-4n^2-4n-1)}S_{1-\gamma,1+\gamma}\left(\left(m+\frac{1}{2}\right)\pi\right) \\
&\quad +\frac{4(-1)^m(2n+1)\cos\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\gamma}\right)}{\pi(4\gamma^2m^2+4\gamma^2m+\gamma^2-4n^2-4n-1)}C_{1-\gamma,1+\gamma}\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\gamma}\right) \\
&\quad +\frac{4(-1)^m(2n+1)\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\gamma}\right)}{\pi(4\gamma^2m^2+4\gamma^2m+\gamma^2-4n^2-4n-1)}S_{1-\gamma,1+\gamma}\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\gamma}\right).
\end{aligned}$$

When $\gamma = \frac{2n+1}{2m+1}$ and $0 < \gamma \le 1$, we get

$$\begin{aligned}
d_{m,\left(m+\frac{1}{2}\right)\gamma-\frac{1}{2}} &= C_{-1+\gamma,1-\gamma}\left(\left(m+\frac{1}{2}\right)\pi\right)+\frac{1}{\gamma}C_{1,-1-\gamma,-1+\gamma}\left(\left(m+\frac{1}{2}\right)\pi\right) \\
&\quad +\frac{1+\gamma}{\gamma}C_{-1-\gamma,-1+\gamma}\left(\left(m+\frac{1}{2}\right)\pi\right)+\frac{1}{\left(m+\frac{1}{2}\right)\gamma\pi}S_{-1\gamma,-1+\gamma}\left(\left(m+\frac{1}{2}\right)\pi\right).
\end{aligned}$$

Theorem 2 now follows. □