

Highly oscillatory time-dependent trapped wave equation

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Abstract

We explore a new asymptotic-numerical solver for the time-dependent wave equation with an interaction term that is oscillating in time with a very high frequency. The method involves representing the solution as an asymptotic series in inverse powers of the oscillation frequency. Using these new schemes, high accuracy is achieved at a low computational cost. The salient features of the new approach are highlighted by a numerical example.

1 Introduction

In this paper, we consider the time-dependent wave equation [1]

$$\partial_t^2 u(x, t) = \partial_x^2 u(x, t) - g(x, t)u(x, t) \quad x \in \mathbb{R}, \quad t \geq 0, \quad (1.1)$$

where $u(x, t)$ is the wavefunction and $g(x, t)$ is a real-valued highly oscillatory term. The initial and boundary conditions are

$$\begin{aligned} u(x, 0) &= \phi(x), & x &\in [-L, L], \\ \partial_t u(x, 0) &= \Phi(x), & x &\in [-L, L], \\ u(-L, t) &= u(L, t) \equiv 0, & t &\geq 0, \\ \partial_t u(-L, t) &= \partial_t u(L, t) \equiv 0, & t &\geq 0. \end{aligned}$$

Such equations are considered when computing scattering frequencies [7]. Highly oscillatory interaction terms present a difficulty for numerical simulations as a small time step and fine space discretisation are typically required to obtain an accurate solution. The present contribution involves representing the solution as an asymptotic series in inverse powers of the temporal oscillation frequency of $g(x, t)$. The coefficients of the terms in the series are independent of the oscillation frequency and hence, the computational effort in determining these coefficients and the associated asymptotic series is greatly reduced. The coefficients are determined from recursive or partial differential equations and the solutions of the latter do not depend on the oscillation frequency of $g(x, t)$. Consequently, the proposed methods achieve high accuracies despite using extremely large time steps. In principle, it might appear that the error

of the computation is independent of the frequency of oscillation but, actually, the situation is often even better! The more rapid the oscillation, the smaller the error of the numerical scheme. An example illustrates salient features of underlying the method.

Before we describe our numerical approach, it is important to discuss briefly the well posedness of (1.1). This is a linear hyperbolic equation and, in general, we expect it to be well posed, but it is always a good idea to conform this.

Letting $v(x, t) = \partial_t u(x, t)$, we obtain a first-order hyperbolic system

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ \partial_x^2 u - gu \end{bmatrix}, \quad (1.2)$$

with the above initial and boundary conditions. Denote by \mathcal{E} the semigroup associated with the wave equation – in other words, we consider the solution of the standard *wave equation*

$$\partial_t \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} \tilde{v} \\ \partial_x^2 \tilde{u} \end{bmatrix}$$

with the same initial conditions and zero Dirichlet boundary conditions by

$$\begin{bmatrix} \tilde{u}(t) \\ \tilde{v}(t) \end{bmatrix} = \mathcal{E}(t) \begin{bmatrix} \tilde{u}(0) \\ \tilde{v}(0) \end{bmatrix}, \quad t \geq 0,$$

we recall that $\|\mathcal{E}(t)\| \leq 1$ in the H_1 Sobolev norm on u , which is identical to the standard L_2 norm on $\begin{bmatrix} u \\ v \end{bmatrix}$.

Applying the *Duhamel principle* to (1.2), we have

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \mathcal{E}(t) \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} - \int_0^t \mathcal{E}(t-\tau) \begin{bmatrix} 0 \\ g(x, \tau)u(x, \tau) \end{bmatrix} d\tau$$

therefore, by the integral form of the *Grönwall Lemma* and bearing in mind that $\|\mathcal{E}(t)\| \leq 1$,

$$\begin{aligned} & \left\| \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \right\| \\ & \leq \left\| \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} \right\| + \left\| \int_0^t \mathcal{E}(\tau)\mathcal{E}(t-\tau) \begin{bmatrix} 0 & 0 \\ g(x, \tau) & 0 \end{bmatrix} \exp\left(\int_\tau^t \mathcal{E}(\tau-\xi) \begin{bmatrix} 0 & 0 \\ g(x, \xi) & 0 \end{bmatrix} d\xi\right) d\tau \right\| \\ & \leq \left\| \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} \right\| + \int_0^t \left\| \begin{bmatrix} 0 \\ g(x, \tau) \end{bmatrix} \right\| \exp\left(\int_\tau^t \|\mathcal{E}(\tau-\xi)\| \left\| \begin{bmatrix} 0 \\ g(x, \xi) \end{bmatrix} \right\| d\xi\right) d\tau \\ & \leq \left\| \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} \right\| + \int_0^t |g(x, \tau)| \exp\left(\int_\tau^t |g(x, \xi)| d\xi\right) d\tau \end{aligned}$$

which, provided that $\max_{t \geq 0} \|g(\cdot, t)\|_{L_\infty(-L, L)}$ is bounded, demonstrates that the solution of (1.2) is uniformly L_2 -bounded in every compact interval $t \in [0, T]$. Hence well posedness.

2 The proposed solution

We consider equation (1.1) with

$$g(x, t) = \sum_{n=-\infty}^{\infty} a_n(x, t)e^{in\omega t}.$$

where $\omega \gg 1$ is the frequency of interest.

The first step is to rearrange the equation as in [1],

$$\begin{aligned} \partial_t u(x, t) &= v(x, t), \\ \partial_t v(x, t) &= \partial_x^2 u(x, t) - g(x, t)u(x, t). \end{aligned} \quad (2.1)$$

The initial and boundary conditions are given by

$$\begin{aligned} u(x, 0) &= \phi(x), & x &\in [-L, L], \\ v(x, 0) &= \Phi(x), & x &\in [-L, L], \\ u(-L, t) &= u(L, t) \equiv 0, & t &\geq 0, \\ v(-L, t) &= v(L, t) \equiv 0, & t &\geq 0. \end{aligned}$$

To expand asymptotically, we assume that there exist functions $p_{0,0}(x, t)$, $p_{1,0}(x, t)$ and $p_{r,n}(x, t)$, $r \geq 2$, $n \in \mathbb{Z}$, and $q_{0,0}(x, t)$ and $q_{r,n}(x, t)$, $r \geq 1$, $n \in \mathbb{Z}$ such that

$$u(x, t) = p_{0,0}(x, t) + \frac{1}{\omega} p_{1,0}(x, t) + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{n=-\infty}^{\infty} p_{r,n}(x, t)e^{in\omega t}$$

and

$$v(x, t) = q_{0,0}(x, t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=-\infty}^{\infty} q_{r,n}(x, t)e^{in\omega t}. \quad (2.2)$$

In other words, we expand u and v in *modulated Fourier expansions* [3]. Differentiating term-by-term, we have

$$\begin{aligned} \partial_t u(x, t) &= \partial_t p_{0,0}(x, t) + \frac{1}{\omega} \left[\partial_t p_{1,0}(x, t) + \sum_{n=-\infty}^{\infty} in p_{2,n}(x, t)e^{in\omega t} \right] \\ &+ \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{n=-\infty}^{\infty} (\partial_t p_{r,n}(x, t) + in p_{r+1,n}(x, t))e^{in\omega t}, \end{aligned} \quad (2.3)$$

$$\begin{aligned} \partial_t v(x, t) &= \partial_t q_{0,0}(x, t) + \sum_{n=-\infty}^{\infty} in q_{1,n}(x, t)e^{in\omega t} \\ &+ \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=-\infty}^{\infty} (\partial_t q_{r,n}(x, t) + in q_{r+1,n}(x, t))e^{in\omega t} \\ \partial_x^2 u(x, t) &= \partial_x^2 p_{0,0}(x, t) + \frac{1}{\omega} \partial_x^2 p_{1,0}(x, t) + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{n=-\infty}^{\infty} \partial_x^2 p_{r,n}(x, t)e^{in\omega t}. \end{aligned} \quad (2.4)$$

Once equations (2.2), (2.3) and (2.4) are substituted into (2.1), we have

$$\begin{aligned}
& \partial_t p_{0,0}(x, t) + \frac{1}{\omega} \left[\partial_t p_{1,0}(x, t) + \sum_{n=-\infty}^{\infty} i n p_{2,n}(x, t) e^{i n \omega t} \right] \\
& + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{n=-\infty}^{\infty} [\partial_t p_{r,n}(x, t) + i n p_{r+1,n}(x, t)] e^{i n \omega t} \\
= & q_{0,0}(x, t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=-\infty}^{\infty} q_{r,n}(x, t) e^{i n \omega t}, \\
& \partial_t q_{0,0}(x, t) + \sum_{n=-\infty}^{\infty} i n q_{1,n}(x, t) e^{i n \omega t} + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=-\infty}^{\infty} [\partial_t q_{r,n}(x, t) + i n q_{r+1,n}(x, t)] e^{i n \omega t} \\
= & \partial_x^2 p_{0,0}(x, t) + \frac{1}{\omega} \partial_x^2 p_{1,0}(x, t) + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{n=-\infty}^{\infty} \partial_x^2 p_{r,n}(x, t) e^{i n \omega t} \tag{2.5} \\
& - \sum_{n=-\infty}^{\infty} a_n(x, t) e^{i n \omega t} p_{0,0}(x, t) - \frac{1}{\omega} \sum_{n=-\infty}^{\infty} a_n(x, t) e^{i n \omega t} p_{1,0}(x, t) \\
& - \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k(x, t) p_{r,n-k}(x, t) e^{i n \omega t}.
\end{aligned}$$

We next define coefficients in two levels. Firstly we consider orders of magnitude (inverse powers of ω – signified by the values of r), and then frequencies (values of n) within each order of magnitude.

- The first level is when $r = 0$.

When $n = 0$ then

$$\begin{aligned}
\partial_t p_{0,0}(x, t) &= q_{0,0}(x, t), \\
\partial_t q_{0,0}(x, t) &= \partial_x^2 p_{0,0}(x, t) - a_0(x, t) p_{0,0}(x, t). \tag{2.6}
\end{aligned}$$

The initial and boundary conditions are

$$\begin{aligned}
p_{0,0}(x, 0) &= \phi(x), & x &\in [-L, L], \\
q_{0,0}(x, 0) &= \Phi(x), & x &\in [-L, L], \\
p_{0,0}(-L, t) &= p_{0,0}(L, t), & t &\geq 0, \\
q_{0,0}(-L, t) &= q_{0,0}(L, t), & t &\geq 0.
\end{aligned}$$

When $n \neq 0$ then

$$q_{1,n}(x, t) = -\frac{1}{i n} a_n(x, t) p_{0,0}(x, t).$$

Note that ω plays no role in the formation of $p_{0,0}$ and $q_{1,n}$.

- In the case $r = 1$,

$$\begin{aligned}
\partial_t p_{1,0}(x, t) &= q_{1,0}(x, t), \\
\partial_t q_{1,0}(x, t) &= \partial_x^2 p_{1,0}(x, t) - a_0(x, t) p_{1,0}(x, t). \tag{2.7}
\end{aligned}$$

The initial and boundary conditions are

$$\begin{aligned}
p_{1,0}(x, 0) &= 0, & x \in [-L, L], \\
q_{1,0}(x, 0) &= -\sum_{n \neq 0} q_{1,n}(x, 0), & x \in [-L, L], \\
p_{1,0}(-L, t) &= p_{1,0}(L, t), & t \geq 0, \\
q_{1,0}(-L, t) &= q_{1,0}(L, t), & t \geq 0.
\end{aligned}$$

Moreover, for $n \neq 0$

$$\begin{aligned}
q_{2,n}(x, t) &= \frac{1}{in} [-a_n(x, t)p_{1,0}(x, t) - \partial_t q_{1,n}(x, t)], \\
p_{2,n}(x, t) &= \frac{1}{in} q_{1,n}(x, t).
\end{aligned}$$

- For $r = 2$

$$\begin{aligned}
\partial_t p_{2,0}(x, t) &= q_{2,0}(x, t), \\
\partial_t q_{2,0}(x, t) &= \partial_x^2 p_{2,0}(x, t) - \sum_k a_k(x, t)p_{2,-k}(x, t),
\end{aligned}$$

alternatively

$$\partial_t q_{2,0}(x, t) = \partial_x^2 p_{2,0}(x, t) - a_0(x, t)p_{2,0}(x, t) - \sum_{k \neq 0} a_k(x, t)p_{2,-k}(x, t), \quad (2.8)$$

with initial and boundary conditions

$$\begin{aligned}
p_{2,0}(x, 0) &= -\sum_{n \neq 0} p_{2,n}(x, 0), & x \in [-L, L], \\
q_{2,0}(x, 0) &= -\sum_{n \neq 0} q_{2,n}(x, 0), & x \in [-L, L], \\
p_{2,0}(-L, t) &= p_{2,0}(L, t), & t \geq 0, \\
q_{2,0}(-L, t) &= q_{2,0}(L, t), & t \geq 0.
\end{aligned}$$

Once $n \neq 0$, the recursion is

$$\begin{aligned}
q_{3,n}(x, t) &= \frac{1}{in} \left[-\partial_t q_{2,n}(x, t) + \partial_x^2 p_{2,n}(x, t) - \sum_{k=-\infty}^{\infty} a_k(x, t)p_{2,n-k}(x, t) \right], \\
p_{3,n}(x, t) &= \frac{1}{in} (q_{2,n}(x, t) - \partial_t p_{2,n}(x, t)).
\end{aligned}$$

Generalisation for all $r \geq 1$ is straightforward. The outcome, having truncated (2.2), is the scheme

$$\begin{aligned} u(x, t) &= p_{0,0}(x, t) + \frac{1}{\omega} p_{1,0}(x, t) + \sum_{r=2}^R \frac{1}{\omega^r} \sum_{n=-N}^N p_{r,n}(x, t) e^{in\omega t}, \\ v(x, t) &= q_{0,0}(x, t) + \sum_{r=1}^R \frac{1}{\omega^r} \sum_{n=-N}^N q_{r,n}(x, t) e^{in\omega t}, \end{aligned} \quad (2.9)$$

where $R \geq 1$ and $N \geq 1$ are given – typically R is fairly small, while N must be sufficiently large to encompass all relevant oscillating components of the solution.

At each stage we need to solve differential equations for $p_{r,0}$ and $q_{2,0}$, as well as straightforward recursions for $p_{r+1,n}, q_{r+1,n}$, $n \neq 0$. In each case ω is absent from the calculations: it reappears only once we assemble the $p_{r,n}$ s and $q_{r,n}$ s into the expansion (2.2). Thus, unlike standard numerical methods, the behaviour of the solution is immune to the frequency of oscillation.

It is essential to consider stability of standard numerical schemes for evolutionary PDEs. In tandem with consistency, stability is equivalent to convergence by virtue of the Lax Equivalence Theorem. This however is not the case with the numerical-asymptotic methods of this paper, since they do not involve time-stepping in a usual way. All we need is to ensure that the different PDEs that we need to solve along the way, e.g. (2.6), (2.7) and (2.8), are themselves well posed – and this follows at once from the well-posedness of (1.1) – and that they are discretised by stable numerical methods.

3 An example

To illustrate the proposed approach, we consider equation (1.1), where the $g(x, t)$ term is

$$g(x, t) = x^2 + \varepsilon \cos(\omega t)x^2.$$

The spatial domain considered is $[-L, L]$ and $L = 10$. The temporal domain of interest is $[0, T]$ and $T = 1$. The initial and boundary conditions of the problem are

$$\begin{aligned} u(x, 0) &= e^{-\frac{x^2}{2}}, & \partial_t u(x, 0) &= 0, \\ u(\pm L, t) &\equiv 0, & t &\geq 0, \end{aligned}$$

and

$$a_0(x) = x^2, \quad a_1(x) = \frac{\varepsilon}{2}x^2, \quad a_{-1}(x) = \frac{\varepsilon}{2}x^2.$$

As we note in the sequel, the time independence of a_n is not a fundamental restriction – by employing Magnus expansions [4, 6] the approach can also be applied at no additional expense when the a_n terms are time-dependent. For this example, matching

the asymptotic terms, we arrive at the explicit equations:

$$\begin{aligned}
p_{0,0}(x, t) &= e^{-\frac{x^2}{2}} \cos(t), \\
q_{0,0}(x, t) &= -e^{-\frac{x^2}{2}} \sin(t), \\
q_{1,1}(x, t) &= -\frac{1}{i} \frac{\varepsilon x^2}{2} e^{-\frac{x^2}{2}} \cos(t), \\
q_{1,-1}(x, t) &= \frac{1}{i} \frac{\varepsilon x^2}{2} e^{-\frac{x^2}{2}} \cos(t), \\
p_{1,0}(x, t) &= 0, \\
q_{2,1}(x, t) &= \frac{\varepsilon x^2}{2} e^{-\frac{x^2}{2}} \sin(t), \\
q_{2,-1}(x, t) &= -\frac{\varepsilon x^2}{2} e^{-\frac{x^2}{2}} \sin(t), \\
p_{2,1}(x, t) &= \frac{\varepsilon x^2}{2} e^{-\frac{x^2}{2}} \cos(t), \\
p_{2,-1}(x, t) &= \frac{\varepsilon x^2}{2} e^{-\frac{x^2}{2}} \cos(t),
\end{aligned}$$

and, finally, the two partial differential equations that we need to solve numerically,

$$\begin{aligned}
\partial_t p_{2,0}(x, t) &= q_{2,0}(x, t), \\
\partial_t q_{2,0}(x, t) &= \partial_x^2 p_{2,0}(x, t) - a_0(x) p_{2,0}(x, t) - a_1(x) p_{2,-1}(x, t) - a_{-1}(x) p_{2,1}(x, t).
\end{aligned}$$

The above equations are solved using the Strang splitting method [8] for time propagation, while employing the Fourier spectral method for the spatial derivative.

The structure of the equations enables efficient computation of the exponential terms. Two different splitting approaches may be employed. In the first case

$$\begin{bmatrix} p_{2,0}(x, h) \\ q_{2,0}(x, h) \end{bmatrix} = \Omega(x, h) \begin{bmatrix} p_{2,0}(x, 0) \\ q_{2,0}(x, 0) \end{bmatrix} + \int_0^h \Omega(x, h - \tau) \begin{bmatrix} 0 \\ f(x, \tau) \end{bmatrix} d\tau + \mathcal{O}(h^3).$$

where

$$f(x, t) = -[a_{-1}(x) p_{2,1}(x, t) + a_1(x) p_{2,-1}(x, t)] = -\frac{\varepsilon^2}{2} x^4 e^{-\frac{x^2}{2}} \cos(t),$$

and $\Omega(x, t)$ approximates the flow of the equations up to $\mathcal{O}(h^3)$ using a Strang splitting,

$$\begin{aligned}
\Omega(x, t) &= e^{\frac{A}{2}t} e^{Bt} e^{\frac{A}{2}t}, \\
A &= \begin{bmatrix} 0 & 1 \\ -a_0(x) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ \partial_x^2 & 0 \end{bmatrix}.
\end{aligned}$$

Note that in the case that a_0 is time-dependent, sampling it at midpoint of the interval (i.e. at $t/2$) suffices for $\mathcal{O}(t^3) = \mathcal{O}(h^3)$ accuracy. Effectively, this amounts to a

second order Magnus expansion on the vector field $A(s) + B$ where the quadrature $\int_0^t (A(s) + B) ds$ is approximated by $t(A(t/2) + B)$. The exponential of this expansion is then approximated via a Strang splitting.

The integral in this scheme is computed using Gaussian–Legendre integration. The error in the splitting method is $\mathcal{O}(h^3)$ so the order of the Gaussian quadrature should be selected bearing this in mind. For the present example, the following fourth-order Gauss–Legendre points are selected

$$c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}.$$

This completes the description of the first of the proposed asymptotic schemes for this example, which we will denote as A1.

Alternatively, following [2, 5], a Strang splitting can be applied directly to

$$\begin{bmatrix} p_{2,0}(x, r)' \\ q_{2,0}(x, s)' \\ r' \\ s' \end{bmatrix} = \begin{bmatrix} q_{2,0}(x, s) \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \partial_x^2 p_{2,0}(x, r) - a_0(x)p_{2,0}(x, r) + f(x, r) \\ 1 \\ 0 \end{bmatrix},$$

after expanding the system using auxiliary time variables s and r .

There are three parts to the implementation of each step of this splitting. Starting with $r = s = t$, only one of the time variables r and s advances in each step, while the other remains frozen. Effectively, in the first part $p_{2,0}(x, t)$ is advanced to $p_{2,0}(x, t_{1/2}) = p_{2,0}(x, t) + \frac{h}{2}q_{2,0}(x, t)$, where $t_{1/2} = t + h/2$. This is followed by $q_{2,0}(x, t)$ advancing to

$$q_{2,0}(x, t + h) = q_{2,0}(x, t) + h \left[\partial_x^2 p_{2,0}(x, t_{1/2}) - a_0(x)p_{2,0}(x, t_{1/2}) + f(x, t_{1/2}) \right],$$

followed by a repetition of the first part, which involves the advancement of $p_{2,0}(x, t_{1/2})$ to $p_{2,0}(x, t + h) = p_{2,0}(x, t_{1/2}) + \frac{h}{2}q_{2,0}(x, t + h)$. The resulting method, denoted A2, does not involve the Strang splittings in the integral and is computationally more efficient – requiring only two Fast Fourier Transforms (FFTs) per time step.

The solution with the asymptotic methods A1 and A2 is computed as follows

$$U_{\text{asym}} = p_{0,0}(x, t) + \frac{1}{\omega}p_{1,0}(x, t) + \frac{1}{\omega^2} \left(p_{2,-1}(x, t)e^{-i\omega t} + p_{2,0}(x, t) + p_{2,1}(x, t)e^{i\omega t} \right).$$

Table 1 compares the accuracy and costs of the proposed method A2 with the sixth-order method Φ_{11} ^[6] proposed in [1], and a direct application of Strang splitting to (2.1). The reference solution is computed using Φ_{11} ^[6] with 10^4 time steps. The number of spatial grid points is $M = 100$ for all computations.

When $\varepsilon = 1$, the error for A2 should be $\mathcal{O}(\omega^{-3})$ for sufficiently small h . In practice, we already see this behaviour for very large time steps in Table 1 – merely two to four time steps are adequate for reaching the asymptotic accuracy.

This highlights the low costs of the proposed schemes, which is particularly appealing in the context of large ω . For $\omega = 10$, the asymptotic accuracy ($\sim 10^{-3}$) is achieved in four FFTs (2 time steps) by A2 but requires more than 44 FFTs in the

	N h	1	2	4	8	16	32
		1.0000	0.5000	0.2500	0.1250	0.0625	0.0313
Number of FFTs	$\Phi_{11}^{[6]}$	22	44	88	176	352	704
	Strang	2	4	8	16	32	64
	A2	2	4	8	16	32	64
Error $\omega = 100$ $\varepsilon = 1$	$\Phi_{11}^{[6]}$	4.8e-01	6.4e-01	6.9e-02	4.1e-01	2.0e-02	3.1e-06
	Strang	1.3e+00	7.3e-01	7.0e-01	7.0e-01	1.1e+00	1.6e-04
	A2	1.4e-04	1.9e-05	5.9e-06	2.9e-06	2.3e-06	2.1e-06
Error $\omega = 10$ $\varepsilon = 1$	$\Phi_{11}^{[6]}$	2.3e-01	1.0e-02	2.7e-05	5.9e-07	9.6e-09	1.5e-10
	Strang	4.3e-01	4.2e-01	1.3e-02	2.2e-03	5.3e-04	1.3e-04
	A2	1.6e-02	4.5e-03	3.5e-03	3.3e-03	3.3e-03	3.2e-03

Table 1: [Large time steps] Error and cost comparison for the methods $\Phi_{11}^{[6]}$, Strang, and the second of the proposed asymptotic methods, denoted A2. N is the number of time steps ($h = T/N$, $T = 1$), while the number of FFTs required per step of $\Phi_{11}^{[6]}$, Strang and A2 are 22, 2 and 2, respectively.

context of $\Phi_{11}^{[6]}$. For $\omega = 100$, the asymptotic accuracy ($\sim 10^{-6}$) is achieved in 16 FFTs in contrast to 704 FFTs for $\Phi_{11}^{[6]}$.

In contrast to the sixth-order accuracy of $\Phi_{11}^{[6]}$, however, smaller time steps do not increase accuracy in the case of A2 once the asymptotic accuracy is achieved. Thus, once very high accuracy solutions are required for moderate to small ω , the method $\Phi_{11}^{[6]}$ remains an appealing candidate. Of course, once we desire higher accuracy, we are perfectly free to use a numerical-asymptotic solver (2.9) incorporating larger R .

Figures 3.1 and 3.2 compare the accuracy and efficiency of asymptotic methods A1 and A2 with $\Phi_{11}^{[6]}$ and Strang splitting of (2.1). These results are as expected as the $p_{r,n}(x, t)$ are functions of ε . Hence, the error is reduced for $\varepsilon < 1$ and increased for $\varepsilon > 1$.

Note that the error with respect to the reference solution is computed as the L^1 error,

$$\mathbf{err} = \frac{2L}{M} \sum_{k=1}^M |U_{\text{ref}}(x_k, t) - U_{\text{asym}}(x_k, t)|.$$

where M is the number spatial grid points.

4 Comments

We have described in this paper an asymptotic-numerical approach for the solution of wave equations with external high-oscillatory forcing. The results indicate that for large values of ω , the proposed method is significantly superior to the method reported in [1] or similar traditional time-stepping numerical methods. A major advantage of the proposed approach is that it works equally well regardless of the frequency of oscillation: this represents the main advantage of asymptotic analysis over more conventional time-stepping methods.

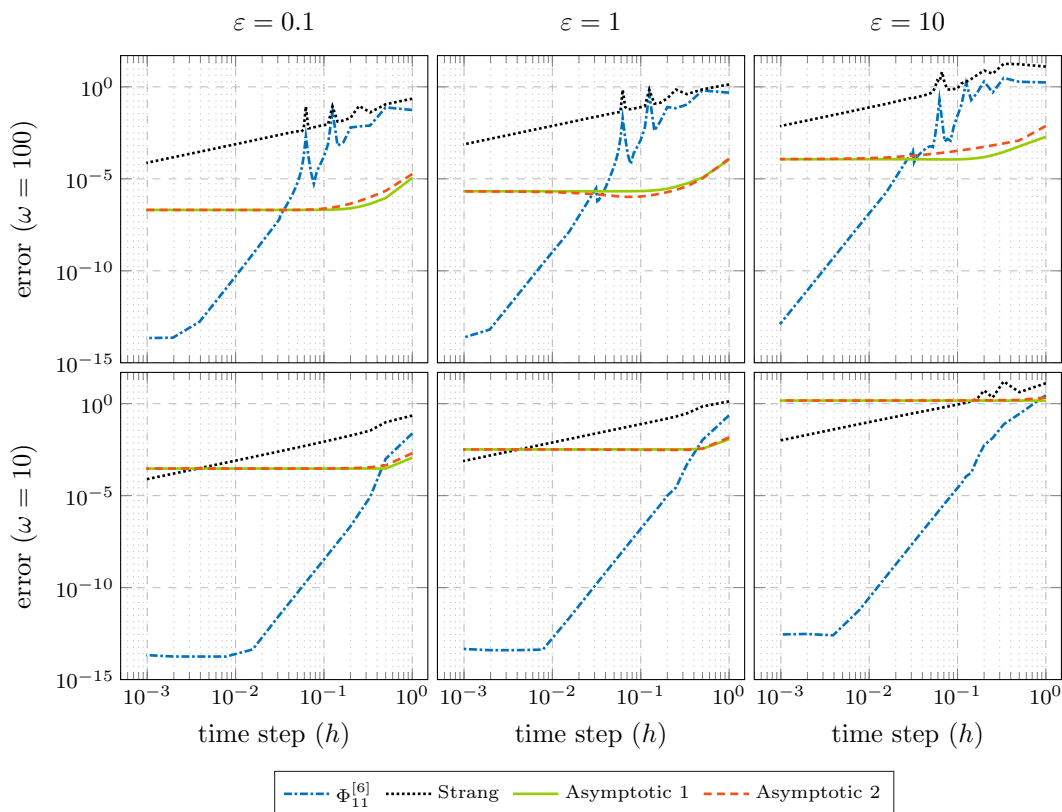


Figure 3.1: [Accuracy] Error as a function of time step size, h . Top row displays results for $\omega = 100$ and bottom row for $\omega = 10$.

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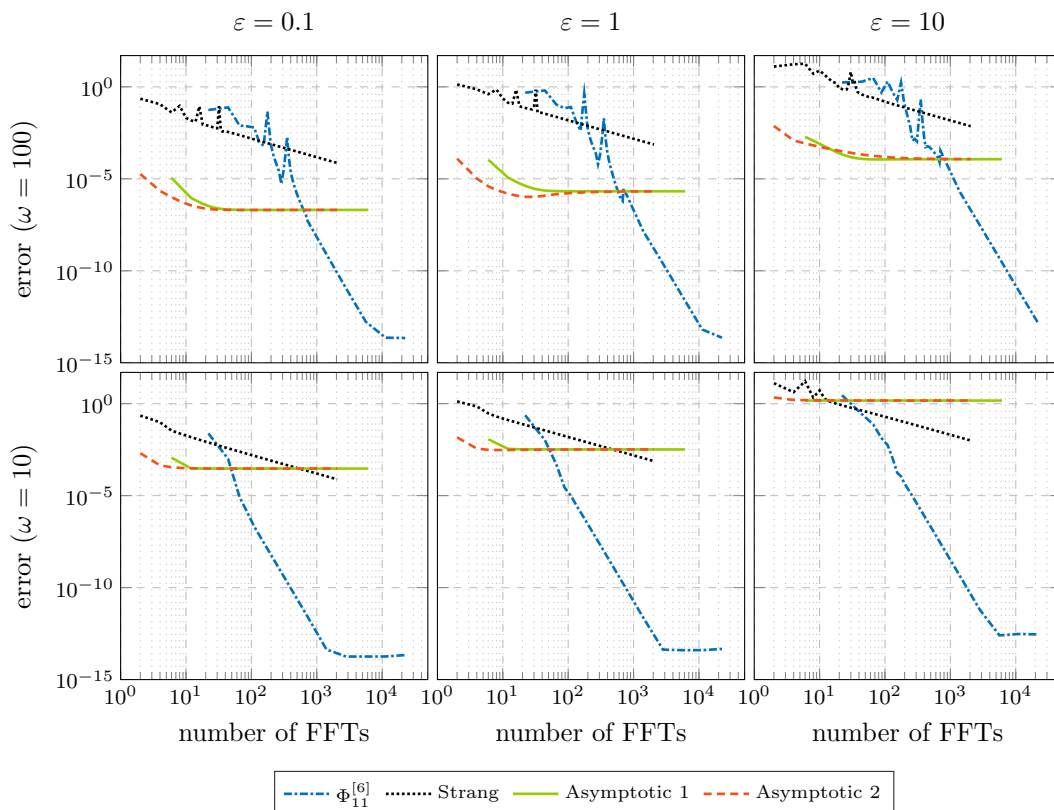


Figure 3.2: [Efficiency] Error compared to the total number of FFTs required in propagating till the final time, $T = 1$. Top row displays results for $\omega = 100$ and bottom row for $\omega = 10$.

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