# Mathematical Tripos Part IB: Lent 2010

# Numerical Analysis – Lecture 6[1]

**Back to the general case...** Typically, forming $L$ involves differentiation, integration and linear combination of function values. Since

$$\frac{\mathrm{d}}{\mathrm{d}x}(x-\theta)_+^k = k(x-\theta)_+^{k-1}, \qquad \int_a^x (t-\theta)_+^k \,\mathrm{d}t = \frac{1}{k+1}[(x-\theta)_+^{k+1} - (a-\theta)_+^{k+1}],$$

the exchange of $L$ with integration is justified in these cases. Similarly for differentiation and, trivially, for linear combinations.

**Theorem** Suppose that $K$ doesn't change sign in $(a,b)$ and that $f \in \mathrm{C}^{k+1}[a,b]$. Then

$$L(f) = \frac{1}{k!}\left[\int_a^b K(\theta)\,\mathrm{d}\theta\right] f^{(k+1)}(\xi) \quad \text{for some} \quad \xi \in (a,b).$$

**Proof.** Let $K \geq 0$. Then

$$L(f) \geq \frac{1}{k!}\int_a^b K(\theta) \min_{x\in[a,b]} f^{(k+1)}(x)\,\mathrm{d}\theta = \frac{1}{k!}\left(\int_a^b K(\theta)\,\mathrm{d}\theta\right) \min_{x\in[a,b]} f^{(k+1)}(x).$$

Likewise $L(f) \leq \frac{1}{k!}\left[\int_a^b K(\theta)\,\mathrm{d}\theta\right] \max_{x\in[a,b]} f^{(k+1)}(x)$, consequently

$$\min_{x\in[a,b]} f^{(k+1)}(x) \leq \frac{L[f]}{\frac{1}{k!}\int_a^b K(\theta)\,\mathrm{d}\theta} \leq \max_{x\in[a,b]} f^{(k+1)}(x)$$

and the required result follows from the intermediate value theorem. Similar analysis is true in the case $K \leq 0$. □

**Function norms:** We can measure the 'size' of function $g$ in various manners. Particular importance is afforded to the *1-norm* $\|g\|_1 = \int_a^b |f(x)|\,\mathrm{d}x$, the *2-norm* $\|g\|_2 = \left\{\int_a^b [g(x)]^2\,\mathrm{d}x\right\}^{1/2}$ and the *∞-norm* $\|g\|_\infty = \max_{x\in[a,b]} |g(x)|$.

**Back to our example** We have $K \geq 0$ and $\int_0^2 K(\theta)\,\mathrm{d}\theta = \frac{2}{3}$. Consequently $L(f) = \frac{1}{2!} \times \frac{2}{3} f'''(\xi) = \frac{1}{3} f'''(\xi)$ for some $\xi \in (0,2)$. We deduce in particular that $|L(f)| \leq \frac{1}{3}\|f'''\|_\infty$.

Likewise we can easily deduce from $\left|\int_a^b f(x)g(x)\,\mathrm{d}x\right| \leq \|g\|_\infty \|f\|_1$ that

$$|L(f)| \leq \frac{1}{k!}\|K\|_1 \|f^{(k+1)}\|_\infty \qquad \text{and} \qquad |L(f)| \leq \frac{1}{k!}\|K\|_\infty \|f^{(k+1)}\|_1.$$

This is valid also when $K$ changes sign. Moreover, the *Cauchy–Schwarz inequality*

$$\left|\int_a^b f(x)g(x)\,\mathrm{d}x\right| \leq \|f\|_2 \|g\|_2$$

implies the inequality

$$|L(f)| \leq \frac{1}{k!}\|K\|_2 \|f^{(k+1)}\|_2.$$

All these provide a very powerful means to bound the size of the error in our approximation procedures and verify how well 'polynomial assumptions' translate to arbitrary functions in $\mathrm{C}^{k+1}[a,b]$.

[1] Corrections and suggestions to these notes should be emailed to A.Iserles@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartIB/Handouts.html.

# 4 Ordinary differential equations

We wish to approximate the exact solution of the *ordinary differential equation (ODE)*

$$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}), \qquad t \geq 0, \tag{4.1}$$

where $\boldsymbol{y} \in \mathbb{R}^N$ and the function $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is sufficiently 'nice'. (In principle, it is enough for $\boldsymbol{f}$ to be Lipschitz to ensure that the solution exists and is unique. Yet, for simplicity, we henceforth assume that $\boldsymbol{f}$ is analytic: in other words, we are always able to expand locally into Taylor series.) The equation (4.1) is accompanied by the initial condition $\boldsymbol{y}(0) = \boldsymbol{y}_0$.
Our purpose is to approximate $\boldsymbol{y}_{n+1} \approx \boldsymbol{y}(t_{n+1})$, $n = 0, 1, \ldots$, where $t_m = mh$ and the *time step* $h > 0$ is small, from $\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ and equation (4.1).

## 4.1 One-step methods

**A one-step method** is a map $\boldsymbol{y}_{n+1} = \boldsymbol{\varphi}_h(t_n, \boldsymbol{y}_n)$, i.e. an algorithm which allows $\boldsymbol{y}_{n+1}$ to depend only on $t_n$, $\boldsymbol{y}_n$, $h$ and the ODE (4.1).

**The Euler method:** We know $\boldsymbol{y}$ and its slope $\boldsymbol{y}'$ at $t = 0$ and wish to approximate $\boldsymbol{y}$ at $t = h > 0$. The most obvious approach is to truncate $\boldsymbol{y}(h) = \boldsymbol{y}(0) + h\boldsymbol{y}'(0) + \frac{1}{2}h^2\boldsymbol{y}''(0) + \cdots$ at the $h^2$ term. Since $\boldsymbol{y}'(0) = \boldsymbol{f}(t_0, \boldsymbol{y}_0)$, this procedure approximates $\boldsymbol{y}(h) \approx \boldsymbol{y}_0 + h\boldsymbol{f}(t_0, \boldsymbol{y}_0)$ and we thus set $\boldsymbol{y}_1 = \boldsymbol{y}_0 + h\boldsymbol{f}(t_0, \boldsymbol{y}_0)$.
By the same token, we may advance from $h$ to $2h$ by letting $\boldsymbol{y}_2 = \boldsymbol{y}_1 + h\boldsymbol{f}(t_1, \boldsymbol{y}_1)$. In general, we obtain the *Euler method*

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n), \qquad n = 0, 1, \ldots. \tag{4.2}$$

**Convergence:** Let $t^* > 0$ be given. We say that a method, which for every $h > 0$ produces the solution sequence $\boldsymbol{y}_n = \boldsymbol{y}_n(h)$, $n = 0, 1, \ldots, \lfloor t^*/h \rfloor$, *converges* if, as $h \to 0$ and $n_k(h)h \overset{k\to\infty}{\longrightarrow} t$, it is true that $\boldsymbol{y}_{n_k} \to \boldsymbol{y}(t)$, the exact solution of (4.1), uniformly for $t \in [0, t^*]$.

***Theorem*** *Suppose that* $\boldsymbol{f}$ *satisfies the Lipschitz condition: there exists* $\lambda \geq 0$ *such that*

$$\|\boldsymbol{f}(t, \boldsymbol{v}) - \boldsymbol{f}(t, \boldsymbol{w})\| \leq \lambda \|\boldsymbol{v} - \boldsymbol{w}\|, \qquad t \in [0, t^*], \quad \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^N.$$

*Then the Euler method (4.2) converges.*

**Proof.** Let $\boldsymbol{e}_n = \boldsymbol{y}_n - \boldsymbol{y}(t_n)$, the error at step $n$, where $0 \leq n \leq t^*/h$. Thus,

$$\boldsymbol{e}_{n+1} = \boldsymbol{y}_{n+1} - \boldsymbol{y}(t_{n+1}) = [\boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)] - [\boldsymbol{y}(t_n) + h\boldsymbol{y}'(t_n) + \mathcal{O}(h^2)].$$

By the Taylor theorem, the $\mathcal{O}(h^2)$ term can be bounded uniformly for all $[0, t^*]$ (in the underlying norm $\|\cdot\|$) by $ch^2$, where $c > 0$. Thus, using (4.1) and the triangle inequality,

$$\begin{aligned} \|\boldsymbol{e}_{n+1}\| &\leq \|\boldsymbol{y}_n - \boldsymbol{y}(t_n)\| + h\|\boldsymbol{f}(t_n, \boldsymbol{y}_n) - \boldsymbol{f}(t_n, \boldsymbol{y}(t_n))\| + ch^2 \\ &\leq \|\boldsymbol{y}_n - \boldsymbol{y}(t_n)\| + h\lambda\|\boldsymbol{y}_n - \boldsymbol{y}(t_n)\| + ch^2 = (1 + h\lambda)\|\boldsymbol{e}_n\| + ch^2. \end{aligned}$$

Consequently, by induction,

$$\|\boldsymbol{e}_{n+1}\| \leq (1 + h\lambda)^m \|\boldsymbol{e}_{n+1-m}\| + ch^2 \sum_{j=0}^{m-1} (1 + h\lambda)^j, \qquad m = 0, 1, \ldots, n + 1.$$

In particular, letting $m = n + 1$ and bearing in mind that $\boldsymbol{e}_0 = \boldsymbol{0}$, we have

$$\|\boldsymbol{e}_{n+1}\| \leq ch^2 \sum_{j=0}^{n} (1 + h\lambda)^j = ch^2 \frac{(1 + h\lambda)^{n+1} - 1}{(1 + h\lambda) - 1} \leq \frac{ch}{\lambda}(1 + h\lambda)^{n+1}.$$

For small $h > 0$ it is true that $0 < 1 + h\lambda \leq e^{h\lambda}$. This and $(n + 1)h \leq t^*$ imply that $(1 + h\lambda)^{n+1} \leq e^{t^*\lambda}$, therefore $\|\boldsymbol{e}_n\| \leq \frac{ce^{t^*\lambda}}{\lambda} h \overset{h\to 0}{\longrightarrow} 0$ uniformly for $0 \leq nh \leq t^*$ and the theorem is true. □