Mathematical Tripos Part IB: Easter 1999

Numerical Analysis – Detailed Example 2[1]

LU factorization with pivoting

Let

$$A = A_0 = \begin{bmatrix} -3 & 2 & 3 & -1 \\ 6 & -2 & -6 & 0 \\ -9 & 4 & 10 & 3 \\ 12 & -4 & -13 & -5 \end{bmatrix}, \qquad \boldsymbol{i} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

($\boldsymbol{i}$ holds the order of rows.) Hence, assembling in A_k bits of L, U and the 'real' A_k

$$\text{Pivot} = (A_0)_{4,1}, \qquad A_0^{\text{new}} = \begin{bmatrix} 12 & -4 & -13 & -5 \\ 6 & -2 & -6 & 0 \\ -9 & 4 & 10 & 3 \\ -3 & 2 & 3 & -1 \end{bmatrix}, \qquad \boldsymbol{i} = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\boldsymbol{\ell}_1 = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{3}{4} \\ -\frac{1}{4} \end{bmatrix}, \qquad \boldsymbol{u}_1^{\mathrm{T}} = [12, -4, -13, -5], \qquad A_1 = \left[\begin{array}{c|ccc} 12 & 4 & -13 & -5 \\ \hline \frac{1}{2} & 0 & \frac{1}{2} & \frac{5}{2} \\ -\frac{3}{4} & 1 & \frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{9}{4} \end{array}\right],$$

$$\text{Pivot} = (A_1)_{3,2}, \qquad A_1^{\text{new}} = \left[\begin{array}{c|ccc} 12 & -4 & -13 & -5 \\ \hline -\frac{3}{4} & 1 & \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{5}{2} \\ -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{9}{4} \end{array}\right], \qquad \boldsymbol{i} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix},$$

$$\boldsymbol{\ell}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{u}_2^{\mathrm{T}} = [0, 1, \tfrac{1}{4}, -\tfrac{3}{4}], \qquad A_2 = \left[\begin{array}{cc|cc} 12 & -4 & -13 & -5 \\ -\frac{3}{4} & 1 & \frac{1}{4} & -\frac{3}{4} \\ \hline \frac{1}{2} & 0 & \frac{1}{2} & \frac{5}{2} \\ -\frac{1}{4} & 1 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right],$$

$$\text{Pivot} = (A_2)_{3,3}, \qquad A_2^{\text{new}} = A_2^{\text{old}}, \qquad \boldsymbol{i} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix},$$

$$\boldsymbol{\ell}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \qquad \boldsymbol{u}_3^{\mathrm{T}} = [0, 0, \tfrac{1}{2}, \tfrac{5}{2}], \qquad A_3 = \left[\begin{array}{ccc|c} 12 & -4 & -13 & -5 \\ -\frac{3}{4} & 1 & \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{5}{2} \\ \hline -\frac{1}{4} & 1 & -1 & 1 \end{array}\right],$$

$$\boldsymbol{\ell}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{u}_3^{\mathrm{T}} = [0, 0, 0, 1].$$

Hence

$$A = PLU = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ -\frac{1}{4} & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 12 & -4 & -13 & -5 \\ 0 & 1 & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

[1] Corrections and suggestions to these notes should be emailed to a.shadrin@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartIB/Handouts.html.