Mathematical Tripos Part IB: Easter 2001

Numerical Analysis – Detailed Example 3a [1]

Jacobi and Gauss–Seidel methods

Jacobi and Gauss–Seidel methods for an approximate solution of

$$\begin{pmatrix} 4 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix} \quad \text{with the starting vector } x^0 = (1,1,1)^T$$

1) The Jacobi method:

$$\begin{aligned} x_1^{k+1} &= \tfrac{1}{4}\,(5 \qquad -2x_2^k - x_3^k) \\ x_2^{k+1} &= \tfrac{1}{5}\,(4 - 2x_1^k \qquad -2x_3^k) \\ x_3^{k+1} &= \tfrac{1}{6}\,(7 - \ x_1^k - 2x_2^k) \end{aligned}$$

with $x^0 = (1,1,1)^T$, the first two iterations are

$$\begin{array}{llllllll} x_1^1 & = \frac{1}{4}\,(5 - 2\cdot 1 - 1\cdot 1) & = \frac{1}{2} & & x_1^2 & = \frac{1}{4}\,(5 - 2\cdot 0 - 1\cdot \frac{2}{3}) & = \frac{13}{12} \\ x_2^1 & = \frac{1}{5}\,(4 - 2\cdot 1 - 2\cdot 1) & = 0 & \Rightarrow & x_2^2 & = \frac{1}{5}\,(4 - 2\cdot \frac{1}{2} - 2\cdot \frac{2}{3}) & = \frac{1}{3} \\ x_3^1 & = \frac{1}{6}\,(7 - 1\cdot 1 - 2\cdot 1) & = \frac{2}{3} & & x_3^2 & = \frac{1}{6}\,(7 - 1\cdot \frac{1}{2} - 2\cdot 0) & = \frac{13}{12} \end{array}$$

2) The Gauss–Seidel method:

$$\begin{aligned} x_1^{k+1} &= \tfrac{1}{4}\,(5 \qquad -2x_2^k \quad -x_3^k) \\ x_2^{k+1} &= \tfrac{1}{5}\,(4 - 2x_1^{k+1} \qquad -2x_3^k) \\ x_3^{k+1} &= \tfrac{1}{6}\,(7 \ - x_1^{k+1} - 2x_2^{k+1}) \end{aligned}$$

with $x^0 = (1,1,1)^T$, the first two iterations are

$$\begin{array}{llllllll} x_1^1 & = & \frac{1}{4}\,(5 - 2\cdot 1 - 1\cdot 1) = & \frac{1}{2} & & x_1^2 & = \frac{1}{4}\,(5 - 2\cdot \frac{1}{5} - 1\cdot \frac{61}{60}) & = \frac{43}{48} & = 0.8958 \\ x_2^1 & = & \frac{1}{5}\,(4 - 2\cdot \frac{1}{2} - 2\cdot 1) = & \frac{1}{5} & \Rightarrow & x_2^2 & = \frac{1}{5}\,(4 - 2\cdot \frac{43}{48} - 2\cdot \frac{61}{60}) & = \frac{7}{200} & = 0.035 \\ x_3^1 & = & \frac{1}{6}\,(7 - 1\cdot \frac{1}{2} - 2\cdot \frac{1}{5}) = & \frac{61}{60} & & x_3^2 & = \frac{1}{6}\,(7 - 1\cdot \frac{43}{48} - 2\cdot \frac{7}{200}) & = \frac{7241}{7200} & = 1.0057 \end{array}$$

3) The exact solution is $x^* = (1,0,1)^T$.

4) Both methods converge (because A is positive definite and diagonally dominant), the Gauss–Seidel method much faster.

[1] Corrections and suggestions to these notes should be emailed to `a.shadrin@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartIB/Handouts.html`.