Mathematical Tripos Part IB: Easter 1999

Numerical Analysis – Detailed Example 4[1]

QR factorization: Gram–Schmidt

Let

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \qquad \boldsymbol{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{a}_3 = \begin{bmatrix} 3 \\ 1 \\ -3 \\ 1 \end{bmatrix}.$$

Therefore

$$\boldsymbol{q}_1 = \frac{1}{\|\boldsymbol{a}_1\|}\boldsymbol{a}_1 = \frac{1}{5}\begin{bmatrix} 1 \\ -2 \\ 4 \\ -2 \end{bmatrix},$$

$$\boldsymbol{b} = \boldsymbol{a}_2 - (\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{a}_2)\boldsymbol{q}_1 = \frac{1}{25}\begin{bmatrix} 18 \\ 14 \\ 22 \\ 39 \end{bmatrix}, \qquad \boldsymbol{q}_2 = \frac{1}{\|\boldsymbol{b}\|}\boldsymbol{b} = \frac{\sqrt{101}}{505}\begin{bmatrix} 18 \\ 14 \\ 22 \\ 39 \end{bmatrix},$$

$$\boldsymbol{b} = \boldsymbol{a}_3 - (\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{a}_3)\boldsymbol{q}_1 - (\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{a}_3)\boldsymbol{q}_2 = \frac{1}{101}\begin{bmatrix} 326 \\ -27 \\ -129 \\ -68 \end{bmatrix}, \qquad \boldsymbol{q}_3 = \frac{1}{\|\boldsymbol{b}\|}\boldsymbol{b} = \frac{\sqrt{128270}}{128270}\begin{bmatrix} 326 \\ -27 \\ -129 \\ -68 \end{bmatrix}.$$

Therefore (to four significant digits)

$$\boldsymbol{q}_1 = \begin{bmatrix} 0.2000 \\ -0.4000 \\ 0.8000 \\ -0.4000 \end{bmatrix}, \qquad \boldsymbol{q}_2 = \begin{bmatrix} 0.3582 \\ 0.2786 \\ 0.4378 \\ 0.7761 \end{bmatrix}, \qquad \boldsymbol{q}_3 = \begin{bmatrix} 0.9102 \\ -0.0754 \\ -0.3602 \\ -0.1899 \end{bmatrix}.$$

To obtain a basis of $\mathbb{R}^4$ we set $\boldsymbol{b} = \boldsymbol{e}_1 - \sum_{k=1}^{s}(\boldsymbol{q}_k^{\mathrm{T}}\boldsymbol{e}_1)\boldsymbol{q}_k$ (any other unit vector will do) and let $\boldsymbol{q}_4 = \boldsymbol{b}/\|\boldsymbol{b}\|$, whence

$$Q = [\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4] = \begin{bmatrix} 0.2000 & 0.3582 & 0.9102 & 0.0561 \\ -0.4000 & 0.2786 & -0.0754 & -0.8699 \\ 0.8000 & 0.4378 & -0.3602 & -0.1964 \\ -0.4000 & 0.7761 & -0.1899 & 0.4490 \end{bmatrix},$$

$$R = Q^{\mathrm{T}}A = \begin{bmatrix} 5.0000 & 1.4000 & -2.6000 \\ 0 & 2.0100 & 0.8159 \\ 0 & 0 & 3.5460 \\ 0 & 0 & 0 \end{bmatrix}.$$

[1] Corrections and suggestions to these notes should be emailed to `A.Iserles@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartIB/Handouts.html`.