Mathematical Tripos Part IB: Easter 2001

Numerical Analysis – Detailed Example 4a[1]

QR factorization: Gram–Schmidt

Let

$$A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3], \quad \boldsymbol{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{a}_2 = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{a}_3 = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}.$$

Then

$$\boldsymbol{b}_1 = \boldsymbol{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \underbrace{\|\boldsymbol{b}_1\|}_{r_{11}} = 3,$$

$$\boldsymbol{q}_1 = \frac{\boldsymbol{b}_1}{\|\boldsymbol{b}_1\|} = \tfrac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix},$$

$$\boldsymbol{b}_2 = \boldsymbol{a}_2 - \underbrace{(\boldsymbol{q}_1^{\mathrm{T}} \boldsymbol{a}_2)}_{r_{12}} \boldsymbol{q}_1 = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} - 3 \cdot \tfrac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}, \quad \underbrace{\|\boldsymbol{b}_2\|}_{r_{22}} = 3,$$

$$\boldsymbol{q}_2 = \frac{\boldsymbol{b}_2}{\|\boldsymbol{b}_2\|} = \tfrac{1}{3} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix},$$

$$\boldsymbol{b}_3 = \boldsymbol{a}_3 - \underbrace{(\boldsymbol{q}_1^{\mathrm{T}} \boldsymbol{a}_3)}_{r_{13}} \boldsymbol{q}_1 - \underbrace{(\boldsymbol{q}_2^{\mathrm{T}} \boldsymbol{a}_3)}_{r_{23}} \boldsymbol{q}_2 = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} - 3 \cdot \tfrac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - 3 \cdot \tfrac{1}{3} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \quad \underbrace{\|\boldsymbol{b}_3\|}_{r_{33}} = 3,$$

$$\boldsymbol{q}_3 = \frac{\boldsymbol{b}_3}{\|\boldsymbol{b}_3\|} = \tfrac{1}{3} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

Therefore

$$A = QR, \quad Q = [\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3] = \frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{bmatrix}, \quad R = \begin{bmatrix} 3 & 3 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{bmatrix}.$$

[1] Corrections and suggestions to these notes should be emailed to a.shadrin@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartIB/Handouts.html.