Mathematical Tripos Part IB: Easter 2001

Numerical Analysis – Detailed Example 10[1]

Orthogonal polynomials

Let

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\mathrm{d}x.$$

We use the three-term recurrence relation to form the orthogonal polynomials:

$$\begin{array}{ll}
 & p_{-1}(x) \equiv 0, \qquad p_0(x) \equiv 1, \\
n=0: & \alpha_0 = \dfrac{\int_{-1}^{1} x\mathrm{d}x}{\int_{-1}^{1} \mathrm{d}x} = 0, \qquad \beta_0 = 0 \text{ (by definition)} \\
 & \quad \Rightarrow \qquad p_1(x) = x \cdot 1 - 0 = x, \\
n=1: & \alpha_1 = \dfrac{\int_{-1}^{1} x^3\mathrm{d}x}{\int_{-1}^{1} x^2\mathrm{d}x} = 0, \qquad \beta_1 = \dfrac{\int_{-1}^{1} x^2\mathrm{d}x}{\int_{-1}^{1} \mathrm{d}x} = \frac{1}{3} \\
 & \quad \Rightarrow \qquad p_2(x) = x \cdot x - \frac{1}{3} \cdot 1 = x^2 - \frac{1}{3}, \\
n=2: & \alpha_2 = \dfrac{\int_{-1}^{1} (x^2-\frac{1}{3})^2 x\mathrm{d}x}{\int_{-1}^{1} (x^2-\frac{1}{3})^2\mathrm{d}x} = 0, \qquad \beta_2 = \dfrac{\int_{-1}^{1} (x^2-\frac{1}{3})^2\mathrm{d}x}{\int_{-1}^{1} x^2\mathrm{d}x} = \frac{4}{15} \\
 & \quad \Rightarrow \qquad p_3(x) = x \cdot (x^2 - \frac{1}{3}) - \frac{4}{15}x = x^3 - \frac{3}{5}x, \\
n=3: & \alpha_3 = \dfrac{\int_{-1}^{1} (x^3-\frac{3}{5}x)^2 x\mathrm{d}x}{\int_{-1}^{1} (x^3-\frac{3}{5}x)^2\mathrm{d}x} = 0, \qquad \beta_3 = \dfrac{\int_{-1}^{1} (x^3-\frac{3}{5}x)^2\mathrm{d}x}{\int_{-1}^{1} (x^2-\frac{1}{3})^2\mathrm{d}x} = \frac{9}{35} \\
 & \quad \Rightarrow \qquad p_4(x) = x \cdot (x^3 - \frac{3}{5}x) - \frac{9}{35}(x^2 - \frac{1}{3}) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}.
\end{array}$$

Left to the reader:

1. Prove that $\alpha_n = 0$ for all $n \geq 0$;
2. Prove that each p_{2n} is an even polynomial and each p_{2n+1} is an odd polynomial.

[1] Corrections and suggestions to these notes should be emailed to `a.shadrin@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartIB/Handouts.html`.