Mathematical Tripos Part IB: Easter 2001

Numerical Analysis – Detailed Example 11[1]

Least squares polynomial fitting

Let

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\mathrm{d}x$$

and recall that

$$\begin{aligned}
p_0(x) &\equiv 1,\\
p_1(x) &= x,\\
p_2(x) &= x^2 - \tfrac{1}{3},\\
p_3(x) &= x^3 - \tfrac{3}{5}x,\\
p_4(x) &= x^4 - \tfrac{6}{7}x^2 + \tfrac{3}{35}.
\end{aligned}$$

We wish to approximate $f(x) = \mathrm{e}^x$. Letting

$$c_k = \frac{\langle p_k, f\rangle}{\langle p_k, k_k\rangle} = \frac{\int_{-1}^{1} p_k(x)\mathrm{e}^{-x}\mathrm{d}x}{\int_{-1}^{1}[p_k(x)]^2\mathrm{d}x}$$

we have

$$\begin{aligned}
c_0 &= \tfrac{1}{2}(\mathrm{e}-\mathrm{e}^{-1}) \approx 1.175201,\\
c_1 &= 3\mathrm{e}^{-1} \approx 1.103638,\\
c_2 &= \tfrac{15}{4}\mathrm{e} - \tfrac{105}{4}\mathrm{e}^{-1} \approx 0.536722,\\
c_3 &= -\tfrac{175}{4}\mathrm{e} + \tfrac{1295}{4}\mathrm{e}^{-1} \approx 0.176139,\\
c_4 &= \tfrac{50085}{4384}\mathrm{e} - \tfrac{368235}{4384}\mathrm{e}^{-1} \approx 0.154895.
\end{aligned}$$

Therefore the least-squares approximation by a quartic is (up to four significant digits)

$$p(x) = \sum_{k=0}^{4} c_k p_k(x) = 0.1549x^4 + 0.1761x^3 + 0.4040x^2 + 0.9980x + 1.0096.$$

Plot the pointwise error $|\mathrm{e}^x - p(x)|$ to persude yourself that it is smaller than ≈ 0.026 for $x \in [-1,1]$.

[1]Corrections and suggestions to these notes should be emailed to `a.shadrin@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartIB/Handouts.html`.