Mathematical Tripos Part IB: Easter 2001

Numerical Analysis – Detailed Example 12[1]

Gaussian quadrature

We wish to integrate

$$\int_0^\infty \mathrm{e}^{-x} \sin x \mathrm{d}x = \tfrac{1}{2}$$

numerically. To this end we set

$$\langle f, g \rangle = \int_0^\infty \mathrm{e}^{-x} f(x) g(x) \mathrm{d}x$$

and derive the underlying orthogonal polynomials (the are called *Laguerre polynomials*). The explicit formula is

$$L_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{x^k}{k!}, \qquad n \geq 0$$

(in this formulation they aren't monic – if you *must*, replace L_n by $n!L_n$). Hence, for example,

$$L_4(x) = \tfrac{1}{24}x^4 - \tfrac{2}{3}x^3 + 3x^2 - 4x + 1,$$

with the (approximate) zeros

$$c_1 = 0.32254768, \qquad c_2 = 1.74576110, \qquad c_3 = 4.53662029, \qquad c_4 = 9.39507091.$$

These will be the nodes of our quadrature. To derive the weights, we can use the explicit formula from the lecture,

$$b_k = \int_0^\infty \mathrm{e}^{-x} \prod_{\substack{j=1 \\ j \neq k}}^{4} \frac{x - c_j}{c_k - c_j} \mathrm{d}x, \qquad k = 1, 2, 3, 4$$

or, equivalently, solve the linear system

$$\sum_{l=1}^{4} c_l^{k-1} b_l = \int_0^\infty \mathrm{e}^{-x} x^{k-1} \mathrm{d}x = (k-1)!, \qquad k = 1, 2, 3, 4$$

(can you prove this?). The outcome is

$$b_1 = 0.60315410, \qquad b_2 = 0.35741869, \qquad b_3 = 0.03888791, \qquad b_4 = 0.00053929.$$

The quadrature formula is thus

$$\sum_{k=1}^{4} b_k \sin c_k \approx 0.50487928.$$

(As a matter of interest, choosing 8 quadrature points gives the answer 0.49998775373530.)

[1]Corrections and suggestions to these notes should be emailed to `a.shadrin@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartIB/Handouts.html`.