Mathematical Tripos Part IB: Easter 2001

Numerical Analysis – Detailed Example 13[1]

The Peano Kernel Theorem

Given $f \in \mathrm{C}^3[0,\infty)$, we integrate

$$\int_0^\infty \mathrm{e}^{-x} f(x)\mathrm{d}x \approx b_1 f(c_1) + b_2 f(c_2).$$

The quadratic monic polynomial orthogonal w.r.t. $\langle g,h\rangle = \int_0^\infty \mathrm{e}^{-x} g(x)h(x)\mathrm{d}$ is x^2-4x+2, with zeros

$$c_1 = 2-\sqrt{2}, \qquad c_2 = 2+\sqrt{2},$$

which we choose as the quadrature nodes. The weights are

$$b_1 = \tfrac{1}{2}+\tfrac{\sqrt{2}}{4}, \qquad b_2 = \tfrac{1}{2}-\tfrac{\sqrt{2}}{4},$$

whence the quadrature is exact for all $f \in \mathbb{P}_3[x]$ (verify!). The error functional is

$$L[f] = \int_0^\infty \mathrm{e}^{-x} f(x)\mathrm{d}x - (\tfrac{1}{2}+\tfrac{\sqrt{2}}{4})f(2-\sqrt{2}) - (\tfrac{1}{2}-\tfrac{\sqrt{2}}{4})f(2+\sqrt{2}).$$

We evaluate the Peano kernel $K(\theta) = L[(x-\theta)_+^3]$:

1. $0 \le \theta \le 2-\sqrt{2}$:

$$\begin{aligned} K(\theta) &= \int_\theta^\infty \mathrm{e}^{-x}(x-\theta)^3\mathrm{d}x - (\tfrac{1}{2}+\tfrac{\sqrt{2}}{4})(2-\sqrt{2}-\theta)^3 - (\tfrac{1}{2}-\tfrac{\sqrt{2}}{4})(2+\sqrt{2}-\theta)^3 \\ &= \theta^3 - 3\theta^2 + 6\theta - 6 + 6\mathrm{e}^{-\theta}, \end{aligned}$$

2. $2-\sqrt{2} \le \theta \le 2+\sqrt{2}$:

$$\begin{aligned} K(\theta) &= \int_\theta^\infty \mathrm{e}^{-x}(x-\theta)^3\mathrm{d}x - (\tfrac{1}{2}-\tfrac{\sqrt{2}}{4})(2+\sqrt{2}-\theta)^3 \\ &= (\tfrac{1}{2}-\tfrac{\sqrt{2}}{4})\theta^3 - \tfrac{3}{2}\theta^2 + (3+\tfrac{3}{2}\sqrt{2})\theta - (3+2\sqrt{2}) + 6\mathrm{e}^{-\theta}, \end{aligned}$$

3. $2+\sqrt{2} \le \theta$:

$$K(\theta) = \int_\theta^\infty \mathrm{e}^{-x}(x-\theta)^3\mathrm{d}x = 6\mathrm{e}^{-\theta}.$$

It is possible to verify that $K(\theta) \ge 0$ (sketch the curve to persuade yourself!) and long calculation (or few seconds with symbolic algebra) show that

$$\|K\|_1 = \int_0^\infty K(\theta)\mathrm{d}\theta = 1 \qquad \Rightarrow \qquad |L[f]| \le \tfrac{1}{6}\|f^{(\mathrm{iv})}\|_\infty.$$

(Remember that we need to divide the bound by $3! = 6$.) Likewise,

$$\begin{aligned} \|K\|_2 = \left\{\int_0^\infty [K(\theta)]^2\mathrm{d}\theta\right\}^{1/2} &\approx 0.527664 \qquad \Rightarrow \qquad |L[f]| \le 0.087944\cdot\|f^{(\mathrm{iv})}\|_2, \\ \|K\|_\infty = \max_{\theta\ge 0}|K(\theta)| &\approx 0.403021 \qquad \Rightarrow \qquad |L[f]| \le 0.067170\cdot\|f^{(\mathrm{iv})}\|_1. \end{aligned}$$

[1]Corrections and suggestions to these notes should be emailed to `a.shadrin@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartIB/Handouts.html`.