Mathematical Tripos Part IB: Easter 2001

Numerical Analysis – Lecture 8[1]

6.1 The error of polynomial interpolation

Let $[a, b]$ be a closed interval of $\mathbb{R}$. We denote by $C[a, b]$ the space of all continuous functions from $[a, b]$ to $\mathbb{R}$ and let $C^s[a, b]$, where s is a positive integer, stand for the linear space of all functions in $C[a, b]$ that possess s continuous derivatives.

Theorem Given $f \in C^{n+1}[a, b]$, let $p \in \mathbb{P}_n[x]$ interpolate the values $f(x_i)$, $i = 0, 1, \ldots, n$, where $x_0, \ldots, x_n \in [a, b]$ are pairwise distinct. Then for every $x \in [a, b]$ there exists $\xi \in [a, b]$ such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i). \tag{6.1}$$

Proof. The formula (6.1) is true when $x = x_j$ for $j \in \{0, 1, \ldots, n\}$, since both sides of the equation vanish. Let $x \in [a, b]$ be any other point and define

$$\phi(t) := [f(t) - p(t)] \prod_{i=0}^{n} (x - x_i) - [f(x) - p(x)] \prod_{i=0}^{n} (t - x_i), \qquad t \in [a, b].$$

[Note: The variable in ϕ is t, whereas x is a fixed parameter.] Note that $\phi(x_j) = 0$, $j = 0, 1, \ldots, n$, and $\phi(x) = 0$. Hence, ϕ has at least $n + 2$ distinct zeros in $[a, b]$. Moreover, $\phi \in C^{n+1}[a, b]$.
We now apply the *Rolle theorem:* if the function $g \in C^1[a, b]$ vanishes at two distinct points in $[a, b]$ then its derivative vanishes at an intermediate point. We deduce that ϕ' vanishes at (at least) $n + 1$ distinct points in $[a, b]$. Next, applying Rolle to ϕ', we conclude that ϕ'' vanishes at n points in $[a, b]$. In general, we prove by induction that $\phi^{(s)}$ vanishes at $n + 2 - s$ distinct points of $[a, b]$ for $s = 0, 1, \ldots, n + 1$. Letting $s = n + 1$, we have $\phi^{(n+1)}(\xi) = 0$ for some $\xi \in [a, b]$. Hence

$$0 = \phi^{(n+1)}(\xi) = [f^{(n+1)}(\xi) - p^{(n+1)}(\xi)] \prod_{i=0}^{n} (x - x_i) - [f(x) - p(x)] \frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}} \prod_{i=0}^{n} (\xi - x_i).$$

Since $p^{(n+1)} \equiv 0$ and $\mathrm{d}^{n+1} \prod_{i=0}^{n} (t - x_i) / \mathrm{d}t^{n+1} \equiv (n + 1)!$, we obtain (6.1). □

Runge's example We interpolate $f(x) = 1/(1 + x^2)$, $x \in [-5, 5]$, at the equally-spaced points $x_j = -5 + 10\frac{j}{n}$, $j = 0, 1, \ldots, n$. Some of the errors are displayed below

x	$f(x) - p(x)$	$\prod_{i=0}^{n}(x - x_i)$
0.75	3.2×10^{-3}	-2.5×10^{6}
1.75	7.7×10^{-3}	-6.6×10^{6}
2.75	3.6×10^{-2}	-4.1×10^{7}
3.75	5.1×10^{-1}	-7.6×10^{8}
4.75	$4.0 \times 10^{+2}$	-7.3×10^{10}

Table: Errors for $n = 20$

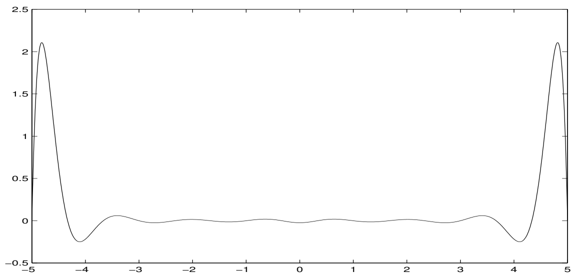

Figure: Errors for $n = 15$

The growth in the error is explained by the product term in (6.1) (the rightmost column of the table). Adding more interpolation points makes the largest error even worse. A remedy to this

[1]Corrections and suggestions to these notes should be emailed to `a.shadrin@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartIB/Handouts.html`.

state of affairs is to cluster points toward the end of the range. A considerably smaller error is attained for $x_j = 5\cos\frac{(n-j)\pi}{n}$, $j = 0, 1, \ldots, n$ (so-called *Chebyshev points*). It is possible to prove that this choice of points minimizes the magnitude of $\max_{x\in[-5,5]} |\prod_{i=0}^{n}(x - x_i)|$.

6.2 Divided differences: a definition

Given $f \in C[a, b]$ and pairwise-distinct points $x_0, \ldots, x_n \in [a, b]$, the *divided difference* $f[x_0, \ldots, x_n]$ of degree n is, by definition, the leading coefficient of $p \in \mathbb{P}_n[x]$ which interpolates f at these points (i.e., the coefficient of x^n in p).

We can derive the expression for $f[x_0, \ldots, x_n]$ from the Lagrange formula,

$$f[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} f(x_k) \prod_{\substack{\ell=0\\ \ell\neq k}}^{n} \frac{1}{x_k - x_\ell}. \tag{6.2}$$

Theorem Let $[\bar{a}, \bar{b}]$ be the smallest interval that contains $x_0, x_1, \ldots, x_n$ and let $f \in C^n[\bar{a}, \bar{b}]$. Then there exists $\xi \in [\bar{a}, \bar{b}]$ such that

$$f[x_0, x_1, \ldots, x_n] = \tfrac{1}{n!} f^{(n)}(\xi). \tag{6.3}$$

Proof. Let p be the interpolating polynomial. The error function $f - p$ has at least $n + 1$ zeros in $[\bar{a}, \bar{b}]$ and, applying Rolle's theorem n times, it follows that $f^{(n)} - p^{(n)}$ vanishes at some $\xi \in [\bar{a}, \bar{b}]$. But $p(x) = \frac{1}{n!}p^{(n)}(\zeta)x^n$ + lower order terms (for any $\zeta \in \mathbb{R}$), therefore, letting $\zeta = \xi$,

$$f[x_0, x_1, \ldots, x_n] = \tfrac{1}{n!}p^{(n)}(\xi) = \tfrac{1}{n!}f^{(n)}(\xi)$$

and we deduce (6.3). □

Application It is a consequence of the theorem that divided differences can be used to approximate derivatives.

6.3 Recurrence relations for divided differences

Our next topic is a useful way to calculate divided differences (and, ultimately, to deduce yet another means to construct an interpolating polynomial). We commence with the remark that $f[x_i]$ is the coefficient of x^0 in the polynomial of degree 0 (i.e., a constant) that interpolates $f(x_i)$, hence $f[x_i] = f(x_i)$.

Theorem Suppose that $x_0, x_1, \ldots, x_{k+1}$ are pairwise distinct, where $k \geq 0$. Then

$$f[x_0, x_1, \ldots, x_{k+1}] = \frac{f[x_1, x_2, \ldots, x_{k+1}] - f[x_0, x_1, \ldots, x_k]}{x_{k+1} - x_0}. \tag{6.4}$$

Proof. Let $p, q \in \mathbb{P}_k[x]$ be the polynomials that interpolate f at

$$\{x_0, x_1, \ldots, x_k\} \qquad \text{and} \qquad \{x_1, x_2, \ldots, x_{k+1}\}$$

respectively and define

$$r(x) := \frac{(x - x_0)q(x) + (x_{k+1} - x)p(x)}{x_{k+1} - x_0} \in \mathbb{P}_{k+1}[x].$$

We readily verify that $r(x_i) = f(x_i)$, $i = 0, 1, \ldots, k + 1$, hence, r is the $(k + 1)$-degree interpolating polynomial for f. Moreover, the leading coefficient of r is equal to the difference of those of q and p divided by $x_{k+1} - x_0$, what actually the recurrence (6.4) is. □