Mathematical Tripos Part IB: Easter 2001

Numerical Analysis – Lecture 9[1]

6.6 The Newton interpolation formula

The recursive formula allows for fast evaluation of the *divided difference table,* in the following manner:

x_i	$f[\,] = f(\,)$	$f[,]$	$f[,,]$	$f[,,,]$
x_0	$\to f[x_0] \searrow$			
		$f[x_0, x_1] \searrow$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2] \searrow$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

This can be done in $\mathcal{O}(n^2)$ operations and the outcome are the numbers $\{f[x_0, \ldots, x_k]\}_{k=0}^n$.

We now provide an alternative representation of the interpolating polynomial.

Theorem. Suppose that $x_0, \ldots, x_n$ are pairwise distinct and the polynomial $p_n \in \mathbb{P}_n$ interpolates f at these points. Then it may be written in the (*Newton*) form

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \ldots, x_n] \prod_{i=0}^{n-1} (x - x_i).$$

Proof. By induction on n. The statement is obvious for $n = 0$ and we suppose that it is true for n. To prove its validity for $n + 1$, we have to prove that

$$p_{n+1}(x) - p_n(x) = f[x_0, \ldots, x_{n+1}] \prod_{i=0}^{n} (x - x_i).$$

Set $r := p_{n+1} - p_n$. Then $r \in \mathbb{P}_{n+1}[x]$ and $r(x_i) = 0$, $i = 0, \ldots, n$. Thus, $r(x) = \mathrm{const} \times \prod_{i=0}^{n} (x - x_i)$, the constant being the leading coefficient of r. But the leading coefficient of r is equal to the leading coefficient of p_{n+1} i.e., to $f[x_0, \ldots, x_{n+1}]$ and we are done. $\square$

We have derived the *Newton interpolation formula,* which requires only the top row of the divided difference table. It has several advantages over Lagrange's. In particular, its evaluation at a given point x (provided that divided differences are known) requires just $\mathcal{O}(n)$ operations, as long as we do it by the *Horner scheme*

$$\begin{aligned} p_n(x) = \{\cdots\{\{&f[x_0, \ldots, x_n](x - x_{n-1}) + f[x_0, \ldots, x_{n-1}]\} \times (x - x_{n-2}) + f[x_0, \ldots, x_{n-2}]\} \\ &\times (x - x_{n-3}) + \cdots\} + f[x_0]. \end{aligned}$$

On the other hand, the Lagrange formula is often better when we wish to manipulate the interpolation polynomial as part of a larger mathematical expression. We'll see an example in the section on *Gaussian quadrature.*

7 Orthogonal polynomials

7.1 Orthogonality in general linear spaces

We have already seen the scalar product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^n x_i y_i$, acting on $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Likewise, given arbitrary *weights* $w_1, w_2, \ldots, w_n > 0$, we may define $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^n w_i x_i y_i$.

[1]Corrections and suggestions to these notes should be emailed to `a.shadrin@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartIB/Handouts.html`.

In general, a *scalar* (or *inner*) *product* is any function $\mathbb{V} \times \mathbb{V} \to \mathbb{R}$, where $\mathbb{V}$ is a vector space over the reals, subject to the following three axioms:

Symmetry: $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle \; \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$;

Nonnegativity: $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0 \; \forall \boldsymbol{x} \in \mathbb{V}$ and $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ iff $\boldsymbol{x} = \boldsymbol{0}$; and

Linearity: $\langle a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z} \rangle = a\langle \boldsymbol{x}, \boldsymbol{z} \rangle + b\langle \boldsymbol{y}, \boldsymbol{z} \rangle \; \forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{V}$, $a, b \in \mathbb{R}$.

Given a scalar product, we may define *orthogonality:* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$ are orthogonal if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$.

Let $\mathbb{V} = C[a, b]$, $w \in \mathbb{V}$ be a fixed *positive* function and define $\langle f, g \rangle := \int_a^b w(x) f(x) g(x)\, \mathrm{d}x$ for all $f, g \in \mathbb{V}$. It is easy to verify all three axioms of the scalar product.

7.2 Orthogonal polynomials – definition, existence, uniqueness

Given a scalar product in $\mathbb{V} = \mathbb{P}_n[x]$, we say that $p_n \in \mathbb{P}_n[x]$ is the nth *orthogonal polynomial* if $\langle p_n, p \rangle = 0$ for all $p \in \mathbb{P}_{n-1}[x]$. *[Note: different inner products lead to different orthogonal polynomials.]* A polynomial in $\mathbb{P}_n[x]$ is *monic* if the coefficient of x^n therein equals one.

Theorem. For every $n \geq 0$ there exists a unique monic orthogonal polynomial of degree n. Moreover, any $p \in \mathbb{P}_n[x]$ can be expanded as a linear combination of $p_0, p_1, \ldots, p_n$,

Proof. We let $p_0(x) \equiv 1$ and prove the theorem by induction on n. Thus, suppose that $p_0, \ldots, p_n$ have been already derived consistently with both assertions of the theorem and let $q(x) := x^{n+1} \in \mathbb{P}_{n+1}[x]$. Motivated by the *Gram–Schmidt algorithm,* we choose

$$p_{n+1}(x) = q(x) - \sum_{k=0}^{n} \frac{\langle q, p_k \rangle}{\langle p_k, p_k \rangle} p_k(x), \qquad x \in \mathbb{R}. \tag{7.1}$$

Clearly, $p_{n+1} \in \mathbb{P}_{n+1}[x]$ and it is monic (since all the terms in the sum are of degree $\leq n$). Let $m \in \{0, 1, \ldots, n\}$. It follows from (7.1) and the induction hypothesis that

$$\langle p_{n+1}, p_m \rangle = \langle q, p_m \rangle - \sum_{k=0}^{n} \frac{\langle q, p_k \rangle}{\langle p_k, p_k \rangle} \langle p_k, p_m \rangle = \langle q, p_m \rangle - \frac{\langle q, p_m \rangle}{\langle p_m, p_m \rangle} \langle p_m, p_m \rangle = 0.$$

Hence, p_{n+1} is orthogonal to $p_0, \ldots, p_n$. Consequently, according to the second inductive assertion, it is orthogonal to all $p \in \mathbb{P}_n[x]$.

To prove uniqueness, we suppose the existence of two monic orthogonal polynomials $p_{n+1}, \tilde{p}_{n+1} \in \mathbb{P}_{n+1}[x]$. Let $p := p_{n+1} - \tilde{p}_{n+1} \in \mathbb{P}_n[x]$, hence $\langle p_{n+1}, p \rangle = \langle \tilde{p}_{n+1}, p \rangle = 0$, and this implies

$$0 = \langle p_{n+1}, p \rangle - \langle \tilde{p}_{n+1}, p \rangle = \langle p_{n+1} - \tilde{p}_{n+1}, p \rangle = \langle p, p \rangle,$$

and we deduce $p \equiv 0$.

Finally, in order to prove that each $p \in \mathbb{P}_{n+1}[x]$ is a linear combination of $p_0, \ldots, p_{n+1}$, we note that we can always write it in the form $p = cp_{n+1} + q$, where c is the coefficient of x^{n+1} in p and where $q \in \mathbb{P}_n[x]$. According to the induction hypothesis, q can be expanded as a linear combination of $p_0, p_1, \ldots, p_n$, hence our assertion is true. $\square$

Well-known examples of orthogonal polynomials include

Name	Notation	Interval	Weight function
Legendre	P_n	$[-1, 1]$	$w(x) \equiv 1$
Chebyshev	T_n	$[-1, 1]$	$w(x) = (1 - x^2)^{-1/2}$
Laguerre	L_n	$[0, \infty)$	$w(x) = \mathrm{e}^{-x}$
Hermite	H_n	$(-\infty, \infty)$	$w(x) = \mathrm{e}^{-x^2}$