Mathematical Tripos Part IB: Easter 2001

Numerical Analysis – Lecture 10[1]

7.3 The three-term recurrence relation

How to construct orthogonal polynomials? (7.1) might help, but it suffers from the same problem as the Gram–Schmidt algorithm in Euclidean spaces: loss of accuracy due to imprecisions in the calculation of scalar products. A considerably better procedure follows from our next theorem.

Lemma. Let $(p_k)_{k=0}^n$ be the sequence of orthogonal polynomials. Then, for all $p \in \mathbb{P}_n$,

$$p = \sum_{k=0}^{n} c_k p_k, \qquad c_k = \frac{\langle p, p_k\rangle}{\langle p_k, p_k\rangle} \tag{7.2}$$

Proof. By the previous theorem, an expansion exists. The scalar product of both sides with p_k gives $\langle p, p_k\rangle = c_k\langle p_k, p_k\rangle$, hence the expression for a_k □

Theorem. Monic orthogonal polynomials are given by the formula

$$\begin{aligned} p_{-1}(x) &\equiv 0, \qquad p_0(x) \equiv 1, \\ p_{n+1}(x) &= (x-\alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad n = 0, 1, \ldots \end{aligned} \tag{7.3}$$

where

$$\alpha_n := \frac{\langle xp_n, p_n\rangle}{\langle p_n, p_n\rangle}, \qquad \beta_n := \frac{\langle p_n, p_n\rangle}{\langle p_{n-1}, p_{n-1}\rangle} > 0.$$

Proof. Based on (7.2), let us look at the coefficients of the expansion

$$xp_n(x) = \sum_{k=0}^{n+1} c_k p_k(x), \qquad c_k = \frac{\langle xp_n, p_k\rangle}{\langle p_k, p_k\rangle} = \frac{\langle p_n, xp_k\rangle}{\langle p_k, p_k\rangle}$$

$k = n+1 \rightsquigarrow$ $c_{n+1} = 1$, since both $xp_n(x)$ and $p_{n+1}(x)$ are monic polynomials of degree $n+1$.
$k = n \rightsquigarrow$ By definition $c_n = \alpha_n$.
$k = n-1 \rightsquigarrow$ Because of monicity, we have the equality $xp_{n-1} = p_n + q$ with some $q \in \mathbb{P}_{n-1}$, so that $\langle p_n, xp_{n-1}\rangle = \langle p_n, p_n\rangle$, hence $c_{n-1} = \beta_n$.
$k < n-1 \rightsquigarrow$ Since $xp_k \in \mathbb{P}_{n-1}$, we obtain $\langle p_n, xp_k\rangle = 0$, thus $c_k = 0$.

It folows that $xp_n(x) = p_{n+1} + \alpha_n p_n + \beta_n p_{n-1}$, and that is equivalent to (7.3). □

Example (*Chebyshev polynomials*). We choose the scalar product

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}}, \qquad f, g \in C[-1, 1]$$

and define $T_n \in \mathbb{P}_n$ by the relation $T_n(x) = \cos(n\theta)$ where $x = \cos\theta$. Hence $T_0(x) \equiv 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$ etc. Changing the integration variable,

$$\begin{aligned} \langle T_n, T_m\rangle &= \int_{-1}^{1} T_n(x)T_m(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int_0^{\pi} \cos n\theta \cos m\theta \,\mathrm{d}\theta \\ &= \tfrac{1}{2}\int_0^{\pi} [\cos(n+m)\theta + \cos(n-m)\theta]\,\mathrm{d}\theta = 0 \end{aligned}$$

whenever $n \neq m$. The recurrence relation for Chebyshev polynomials is particularly simple,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

[1] Corrections and suggestions to these notes should be emailed to `a.shadrin@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartIB/Handouts.html`.

as can be verified at once from the identity

$$\cos[(n+1)\theta] + \cos[(n-1)\theta] = 2\cos(\theta)\cos(n\theta).$$

Note that the T_ns aren't monic, hence the inconsistency with (7.3). To obtain monic polynomials take $T_n(x)/2^{n-1}$, $n \geq 1$.

7.4 Least-squares polynomial fitting

Consider the space $C[a,b]$ with the scalar product $\langle g, h\rangle = \int_a^b g(x)h(x)w(x)\,\mathrm{d}x$, $w(x) > 0$, and the underlying norm $\|f\| := \langle f, f\rangle^{1/2}$.

Given $f \in C[a,b]$, we wish to pick $p \in \mathbb{P}_n$ so as to minimise $\|f - p\|$. Intuitively speaking, p approximates f and is an alternative to an interpolating polynomial. (The situation is similar to the one that we have already encountered in numerical linear algebra, least-squares' fitting *vs* solving linear equations.)

Let $p_0, p_1, \ldots, p_n$ be ortho*normal* polynomials w.r.t. the underlying inner product, i.e., $p_k \in \mathbb{P}_k$, $\langle p_k, p_\ell\rangle = \delta_{k\ell}$. They form a basis of $\mathbb{P}_n$, therefore every $p \in \mathbb{P}_n$ can be written as $p = \sum_{k=0}^n c_k p_k$. Because of orthonormality,

$$\langle f - p, f - p\rangle = \left\langle f - \sum_{k=0}^n c_k p_k, f - \sum_{k=0}^n c_k p_k \right\rangle = \langle f, f\rangle - 2\sum_{k=0}^n c_k\langle p_k, f\rangle + \sum_{k=0}^n c_k^2,$$

and if we set $a_k := \langle p_k, f\rangle$, then we can rewrite the right-hand side as

$$\langle f - p, f - p\rangle = \langle f, f\rangle - \sum_{k=0}^n a_k^2 + \sum_{k=0}^n (a_k - c_k)^2$$

Since a_k's are independent of p, it is clear that the minimum occurs iff $c_k = a_k$. Thus, the optimal $p^* \in \mathbb{P}_n$ is given by

$$p^* = \sum_{k=0}^n a_k p_k = \sum_{k=0}^n \langle p_k, f\rangle p_k, \qquad \|p^*\|^2 = \sum_{k=0}^n \langle p_k, f\rangle^2, \tag{7.4}$$

and

$$\|f - p^*\|^2 = \|f\|^2 - \sum_{k=0}^n \langle p_k, f\rangle^2 = \|f\|^2 - \|p^*\|^2. \tag{7.5}$$

The latter identity is a reminiscent of the Pythagoras theorem.

How to choose n? Note that $a_k = \langle p_k, f\rangle$ is independent of n. Thus, we can continue to add terms to (7.4) until $\|f - p^*\|^2$ is below specified *tolerance* ε. Because of (7.5), we need to pick n so that $\sum_{k=0}^n \langle p_k, f\rangle^2 > \|f\|^2 - \varepsilon$.

Theorem (*The Parseval identity*). Let $[a,b]$ be finite. Then

$$\sum_{k=0}^\infty \langle p_k, f\rangle^2 = \|f\|^2 \tag{7.6}$$

Incomplete proof. Let

$$\sigma_n := \|p^*\|^2 = \sum_{k=0}^n \langle p_k, f\rangle^2,$$

hence $\inf\limits_{p\in P_n} \|f - p\|^2 = \|f - p^*\|^2 = \|f\|^2 - \sigma_n$. According to the *Weierstrass theorem*, any function in $C[a,b]$ can be approximated arbitrarily close by a polynomial, hence $\lim\limits_{n\to\infty} \inf\limits_{p\in P_n} \|f - p\|^2 = 0$ and we deduce that $\sigma_n \overset{n\to\infty}{\longrightarrow} \|f\|^2$ and (7.6) is true. □