Mathematical Tripos Part IB: Easter Term 2002

Numerical Analysis – Lecture 5

5 Iterative methods for linear systems

5.1 Norms of vectors and matrices

Definition 5.1 The norm on a vector space $\mathbb{X}$ (say, $\mathbb{X} = \mathbb{R}^n$) is a functional $\|\cdot\| : \mathbb{X} \to \mathbb{R}$ which satisfies the following axioms:

$$\begin{aligned}
&1) \quad \|x\| \ge 0, \text{ and } \|x\| = 0 \quad \Leftrightarrow \quad x = 0;\\
&2) \quad \|\alpha x\| = |\alpha|\,\|x\|, \quad \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{X};\\
&3) \quad \|x+y\| \le \|x\| + \|y\|, \quad \forall x, y \in \mathbb{X}.
\end{aligned}$$

A vector space with a norm is called the *normed* space.

Example 5.2 The following three vector norms in $\mathbb{R}^n$ are in common use

1) $\|x\|_1 := \sum_{i=1}^n |x_i|$; the ℓ_1-norm;
2) $\|x\|_2 := \{\sum_{i=1}^n |x_i|^2\}^{1/2}$ the ℓ_2-norm or the Euclidean norm;
3) $\|x\|_\infty := \max_i |x_i|$; the ℓ_∞-norm or the max-norm.

They are particular cases of the ℓ_p-norm: $\|x\|_{\ell_p} := \|x\|_p := \{\sum_{i=1}^n |x_i|^p\}^{1/p}$, $1 \le p \le \infty$

Definition 5.3 The matrix norm is a functional $\|\cdot\| : \mathbb{R}^{n\times n} \to \mathbb{R}$ such that

1)-3) $\|A\|$ is a norm on the space of matrices;
4) $\|AB\| \le \|A\|\,\|B\|$.

Example 5.4 Given a vector norm $\|\cdot\|_* : \mathbb{R}^n \to \mathbb{R}$, the functional

$$\|A\| := \|A\|_* := \sup_{x\ne 0} \frac{\|Ax\|_*}{\|x\|_*} \tag{5.1}$$

is a matrix norm. (Prove it.) It is called the *induced* matrix norm. By definition,

$$\|Ax\|_* \le \|A\|_* \|x\|_* .$$

It is not too difficult to determine the matrix norms induced by three basic vector norms.

1) $\|A\|_1 = \max_j \sum_i |a_{ij}|$, i.e., max absolute column sum;
2) $\|A\|_2 = \sqrt{\rho(A^T A)}$, i.e., [the dominant eigenvalue of $A^T A]^{1/2}$;
3) $\|A\|_\infty = \max_i \sum_j |a_{ij}|$, i.e., max absolute row sum.

Definition 5.5 Let $A \in \mathbb{R}^{n\times n}$. The set $\sigma(A) := (\lambda_1, \ldots, \lambda_n)$ of the eigenvalues of A is called *spectrum* of A, and the value

$$\rho(A) = \max_i |\lambda_i|$$

is called the *spectral radius* of A.

Lemma 5.6 *For any induced matrix norm $\|\cdot\|$, and for any $A \in \mathbb{R}^{n\times n}$, we have*

$$\rho(A) \le \|A\| .$$

If A is symmetric, then $\rho(A) = \|A\|_2$.

Proof. If λ is an eigenvalue of A and w is the corresponding eigenvector, then

$$|\lambda|\,\|w\| = \|\lambda w\| = \|Aw\| \leq \|A\|\,\|w\|.$$

Thus, $|\lambda| \leq \|A\|$ for any λ, hence $\rho(A) \leq \|A\|$. If $A = A^T$, then $\|A\|_2 = \sqrt{\rho(A^2)} = \sqrt{\rho^2(A)} = \rho(A)$. ∎

Definition 5.7 A sequence (x_k) in a normed space $\mathbb{X}$ is called *convergent* if there is an $x \in \mathbb{X}$ s.t.

$$\lim_{k\to\infty} \|x - x_k\| \to 0\,.$$

5.2 Basic iterative schemes

The general *iterative* method for solving $Ax = b$ is a rule $x^{k+1} = f_k(x^0, x^1, \ldots, x^k)$. We will consider the simplest ones: *linear, one-step, stationary* iterative schemes

$$x^{k+1} = Hx^k + v, \qquad x^0, v \in \mathbb{R}^n\,. \tag{5.2}$$

Here one chooses H and v so that $x^* = Hx^* + v$, the fix point of the iteration, satisfies $Ax^* = b$. Standard terminology:

the *iteration matrix* H, the *error* $e^k := x^k - x^*$, the *residual* $r^k := Ae^k = Ax^k - b$.

For a given class of matrices A, we are interested in the *convergent* methods, i.e. the methods such that $x^k \to x^* = A^{-1}b$ for every starting value x^0. Substracting $x^* = Hx^* + v$ from (5.2) we obtain

$$e^{k+1} = He^k = \cdots = H^{k+1}e^0,$$

i.e., a method is convergent if $\|e^k\| = \|H^k e^0\| \to 0$ for any $e^{(0)} \in \mathbb{R}^n$.

Scheme 5.8 (Regular splitting) This is the scheme

$$Mx^{k+1} = Nx^k + b, \qquad A =: M - N, \qquad M \text{ is non-singular}.$$

The simplest splitting is $A = L_0 + D + U_0$, where L_0, D and U_0 are portions of A: subdiagonal (strictly lower triangular), diagonal and superdiagonal (strictly upper triangular), respectively. We assume that no diagonal element of A is zero.

Method 5.9 (Jacobi)

$$Dx^{k+1} = -(L_0 + U_0)x^k + b.$$

Method 5.10 (Gauss–Seidel)

$$(L_0 + D)x^{k+1} = -U_0x^k + b.$$

Remark 5.11 There is no need to invert any matrix. Writing the original system $Ax = b$ as

$$\textstyle\sum_{j=1}^n a_{ij}x_j = b_i, \qquad i = \overline{1,n}\,,$$

the Jacobi method reads

$$x_i^{k+1} = -\tfrac{1}{a_{ii}}\left(\textstyle\sum_{j\neq i} a_{ij}x_j^k - b_i\right), \qquad i = \overline{1,n}\,,$$

while the Gauss–Seidel becomes

$$x_i^{k+1} = -\tfrac{1}{a_{ii}}\left(\textstyle\sum_{j<i} a_{ij}x_j^{k+1} + \sum_{j>i} a_{ij}x_j^k - b_i\right), \qquad i = \overline{1,n}\,.$$

Example 5.12 Jacobi and Gauss–Seidel methods for an approximate solution of

$$\begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix} \quad \text{with the starting vector } x^{(0)} = (1,1,1)^T.$$

1) The Jacobi method:

$$\begin{aligned} x_1^{(k+1)} &= \tfrac{1}{4}\,(5 \qquad\quad -2x_2^{(k)} \quad -x_3^{(k)}) \\ x_2^{(k+1)} &= \tfrac{1}{5}\,(4 - 2x_1^{(k)} \qquad\quad -2x_3^{(k)}) \\ x_3^{(k+1)} &= \tfrac{1}{6}\,(7 \;\; -x_1^{(k)} - 2x_2^{(k)}) \end{aligned}$$

With $x^{(0)} = (1,1,1)^T$, the first two iterations are

$$\begin{aligned} x_1^{(1)} &= \tfrac{1}{4}\,(5 - 2\cdot 1 - 1\cdot 1) &&= \tfrac{1}{2} && & x_1^{(2)} &= \tfrac{1}{4}\,(5 - 2\cdot 0 - 1\cdot \tfrac{2}{3}) &&= \tfrac{13}{12} \\ x_2^{(1)} &= \tfrac{1}{5}\,(4 - 2\cdot 1 - 2\cdot 1) &&= 0 && \Rightarrow & x_2^{(2)} &= \tfrac{1}{5}\,(4 - 2\cdot \tfrac{1}{2} - 2\cdot \tfrac{2}{3}) &&= \tfrac{1}{3} \\ x_3^{(1)} &= \tfrac{1}{6}\,(7 - 1\cdot 1 - 2\cdot 1) &&= \tfrac{2}{3} && & x_3^{(2)} &= \tfrac{1}{6}\,(7 - 1\cdot \tfrac{1}{2} - 2\cdot 0) &&= \tfrac{13}{12} \end{aligned}$$

2) The Gauss–Seidel method:

$$\begin{aligned} x_1^{(k+1)} &= \tfrac{1}{4}\,(5 \qquad\quad -2x_2^{(k)} \quad -x_3^{(k)}) \\ x_2^{(k+1)} &= \tfrac{1}{5}\,(4 - 2x_1^{(k+1)} \qquad\quad -2x_3^{(k)}) \\ x_3^{(k+1)} &= \tfrac{1}{6}\,(7 \;\; -x_1^{(k+1)} - 2x_2^{(k+1)}) \end{aligned}$$

With $x^{(0)} = (1,1,1)^T$, the first two iterations are

$$\begin{aligned} x_1^{(1)} &= \tfrac{1}{4}\,(5 - 2\cdot 1 - 1\cdot 1) &&= \tfrac{1}{2} && & x_1^{(2)} &= \tfrac{1}{4}\,(5 - 2\cdot \tfrac{1}{5} - 1\cdot \tfrac{61}{60}) &&= \tfrac{43}{48} &&= 0.8958 \\ x_2^{(1)} &= \tfrac{1}{5}\,(4 - 2\cdot \tfrac{1}{2} - 2\cdot 1) &&= \tfrac{1}{5} && \Rightarrow & x_2^{(2)} &= \tfrac{1}{5}\,(4 - 2\cdot \tfrac{43}{48} - 2\cdot \tfrac{61}{60}) &&= \tfrac{7}{200} &&= 0.035 \\ x_3^{(1)} &= \tfrac{1}{6}\,(7 - 1\cdot \tfrac{1}{2} - 2\cdot \tfrac{1}{5}) &&= \tfrac{61}{60} && & x_3^{(2)} &= \tfrac{1}{6}\,(7 - 1\cdot \tfrac{43}{48} - 2\cdot \tfrac{7}{200}) &&= \tfrac{7241}{7200} &&= 1.0057 \end{aligned}$$

3) The exact solution is $x^* = (1,0,1)^T$.

4) Both methods converge (because A is diagonally dominant), the Gauss–Seidel method much faster.

Scheme 5.13 (Iterative refinement) This is the scheme

$$x^{k+1} = x^k - S(Ax^k - b).$$

If $S = A^{-1}$, then $x^{k+1} = A^{-1}b = x^*$, so that one should choose S as an approximation to A^{-1}. The iteration matrix of this scheme is

$$H = I - SA.$$

Remark 5.14 Any regular splitting can be viewed as an iterative refinement, e.g.,

$$\text{Jacobi:} \quad S = D^{-1}, \qquad \text{Gauss-Seidel:} \quad S = (L_0 + D)^{-1}.$$

Method 5.15 (Simple iteration)

$$x^{k+1} = x^k - \tau(Ax^k - b), \quad \tau \in \mathbb{R}.$$