Mathematical Tripos Part IB: Easter Term 2002

Numerical Analysis – Lecture 10

10 Error bounds for polynomial interpolation

Here we study the *interpolation error*

$$e_n(x) = f(x) - p_n(x), \quad p_n \in \mathcal{P}_n,$$

restricted to the class of differentiable functions f that possess, say, k continuous derivatives on the interval $[a, b]$; we denote this class by $C^k[a, b]$. We start with an estimate provided by the basic relation (9.3) between two consecutive interpolating polynomials.

Theorem 10.1 *Let $p_n \in \mathcal{P}_n$ interpolate $f \in C[a, b]$ at $n+1$ distinct points $x_0, ..., x_n$. Then for any $x \notin (x_i)$*

$$\boxed{f(x) - p_n(x) = f[x_0, ..., x_n, x]\, \omega(x)} \tag{10.1}$$

Proof. Given $x_0, .., x_n$, let $\bar{x} := x_{n+1}$ be any other point. Then, by (9.3), the corresponding polynomials p_n and p_{n+1} are related by

$$p_{n+1}(x) = p_n(x) + f[x_0, ..., x_n, \bar{x}]\, \omega(x), \qquad \omega(x) := \prod_{i=0}^{n} (x - x_i)\,.$$

In particular, putting $x = \bar{x}$, and noticing that $p_{n+1}(\bar{x}) = f(\bar{x})$, we obtain

$$f(\bar{x}) = p_n(\bar{x}) + f[x_0, ..., x_n, \bar{x}]\, \omega(\bar{x}),$$

the latter equality being the same as (10.1). ∎

This theorem shows the error to be "like the next term" in the Newton form. However, we cannot evaluate the right-hand side of (10.1) without knowing the number $f(x)$. But as we now show we can relate it to the $(n+1)$-st derivative of f. For this we need a version of the Rolle's theorem:

Lemma 10.2 *If $g \in C^k[a, b]$ is zero at $k + \ell$ distinct points, then $g^{(k)}$ has at least ℓ distinct zeros in $[a, b]$.*

Proof. By Rolle's theorem, if $\phi \in C^1$ is zero at two point, then ϕ' is zero at an intermediate point. So, we deduce that g' vanishes at least at $(k-1)+\ell$ distinct points. Next, applying Rolle to g', we conclude that g'' vanishes at $(k-2)+\ell$ points, and so on. ∎

Theorem 10.3 *Let $[\bar{a}, \bar{b}]$ be the smallest interval that contains $x_0, ..., x_k$ and let $f \in C^k[\bar{a}, \bar{b}]$. Then there exists $\xi \in [\bar{a}, \bar{b}]$ such that*

$$\boxed{f[x_0, ..., x_k] = \frac{1}{k!}\, f^{(k)}(\xi)} \tag{10.2}$$

Proof. Let $p \in \mathcal{P}_k$ be the interpolating polynomial to f on (x_i). The error function $f - p$ has at least $k + 1$ zeros in $[\bar{a}, \bar{b}]$ so, by Rolle's theorem, $f^{(k)} - p^{(k)}$ must vanish at some $\xi \in [\bar{a}, \bar{b}]$, i.e.,

$$p^{(k)}(\xi) = f^{(k)}(\xi)$$

On the other hand, if $p(x) = a_k x^k +$ (lower order terms), then (for any ξ)

$$p^{(k)}(\xi) = k!\, a_n =: k!\, f[x_0, ..., x_n]\,.$$ ∎

So, combining (10.1) with (10.2) where we put $k = n + 1$, we obtain

Theorem 10.4 *Let $f \in C^{n+1}[a, b]$, and let $p_n \in \mathcal{P}_n$ interpolate f at $n + 1$ points $(x_i)_{i=0}^n \in [a, b]$. Then for every $x \in [a, b]$ there exists $\xi \in [a, b]$ such that*

$$\boxed{f(x) - p_n(x) = \frac{1}{(n+1)!}\, \omega(x) f^{(n+1)}(\xi)} \tag{10.3}$$

The equality (10.3) with the value $f^{(n+1)}(\xi)$ for *some* ξ is of hardly any use. Usually one has a bound for $f^{(n+1)}$ in terms of some *norm*, e.g., the L_∞-norm (the *max*-norm)

$$\|g\|_\infty := \|g\|_{L_\infty[a,b]} := \max_{t\in[a,b]} |g(t)| .$$

The estimate (10.3) takes then the form

$$\boxed{|f(x) - p_n(x)| \le \frac{1}{(n+1)!}|\omega(x)|\,\|f^{(n+1)}\|_\infty} \tag{10.4}$$

If we want to find the maximal error over the interval, then maximizing first the right- and then the left-hand side over $x \in [a,b]$ we get yet one more error bound for polynomial interpolation

$$\boxed{\|f - p_\Delta\|_\infty \le \frac{1}{(n+1)!}\|\omega_\Delta\|_\infty\,\|f^{(n+1)}\|_\infty} \tag{10.5}$$

Here we put the lower index in ω_Δ in order to emphasize the dependence of $\omega(x) := \prod_{i=0}^n (x - x_i)$ on the sequence of interpolating points $\Delta := (x_i)_{i=0}^n$. The choice of Δ makes a big difference!

Definition 10.5 The *Chebyshev* polynomial of degree n on $[-1,1]$ is defined by

$$T_n(x) = \cos n\theta, \quad x = \cos\theta, \quad \theta \in [0,\pi].$$

One sees at once that, on $[-1,1]$,

1) T_n takes its maximal absolute value 1 with alternating signs $n+1$ times:

$$\|T_n\|_\infty = 1, \quad T_n(t_k) = (-1)^k, \quad t_k = \cos\tfrac{\pi k}{n}, \quad k = 0,...,n,$$

2) T_n has n distinct zeros:

$$T_n(x_k) = 0, \quad x_k^* = \cos\tfrac{(2k+1)\pi}{2n}, \quad k = 0,...,n-1.$$

Lemma 10.6 *The Chebyshev polynomials T_n satisfy the recurrence relation*

$$T_0(x) \equiv 1, \quad T_1(x) = x, \tag{10.6}$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \ge 1. \tag{10.7}$$

In particular, T_n is really an algebraic polynomial of degree n with the leading coefficient 2^{n-1}.

Proof. Expressions (10.6) are straightforward, the recurrence follows from the equality

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$$

via the substitution $x = \cos\theta$. ■

Theorem 10.7 *On the interval $[-1,1]$, among all polynomials of degree n with leading coefficient equal to one, the Chebyshev polynomial γT_n deviates least from zero, i.e.,*

$$\inf_{(a_i)} \left\|x^n + a_{n-1}x^{n-1} + \cdots + a_0\right\|_\infty = \gamma\,\|T_n\|_\infty, \quad \gamma := 1/2^{n-1} .$$

Proof. Suppose there is a polynomial $q_n(x) = x^n + \cdots + a_0$ such that $\|q\|_\infty < \gamma$, and set

$$r := \gamma T_n - q_n.$$

1) The leading coefficient of both q_n and γT_n is one, thus r is of degree at most $n-1$.

2) On the other hand, at the points $t_k := \cos\frac{\pi k}{n}$, the Chebyshev polynomial γT_n takes the values $\pm\gamma$ altenatively, while $|q_n(t_k)| < \gamma$, hence r alternates in sign at these points, therefore it has a zero in each of n intervals (t_k, t_{k+1}), i.e. at least n zeros in the interval $[-1,1]$, a contradiction to $r \in \mathcal{P}_{n-1}$. ■

Corollary 10.8 *For $\Delta = (x_i)_{i=0}^n \subset [-1,1]$, let $\omega_\Delta(x) = \prod_{i=0}^n (x - x_i)$. Then, for all n, we have*

$$\inf_\Delta \|\omega_\Delta\|_\infty = \|\omega_{\Delta_*}\|_\infty = 1/2^n .$$

Theorem 10.9 *For $f \in C^{n+1}[-1,1]$, the best choice of interpolating points is $\Delta_* = (x_k^*) = \left(\cos\frac{(2k+1)\pi}{2n+2}\right)_{k=0}^n$,*

$$\boxed{\|f - p_{\Delta_*}\|_\infty \le \frac{1}{2^n}\frac{1}{(n+1)!}\,\|f^{(n+1)}\|}$$

10.1 Additional issues

Theorem 10.4 can be proved without use of the Newton formula.

Theorem 10.4 *Let $f \in C^{n+1}[a,b]$, and let $p_n \in \mathcal{P}_n$ interpolate f at $n+1$ points $(x_i)_{i=0}^n \in [a,b]$. Then for every $x \in [a,b]$ there exists $\xi \in [a,b]$ such that*

$$\boxed{f(x) - p_n(x) = \frac{1}{(n+1)!}\,\omega(x) f^{(n+1)}(\xi)} \tag{10.3}$$

Proof. If x coincides with any x_i from the interpolating set, then both sides of (10.3) vanish, hence formula trivially holds. So, we let $\bar{x}$ be any other point, and consider the function

$$\phi(x) := \big[f(x) - p(x)\big] - c_{\bar{x}}\omega(x)$$

with *some* constant $c_{\bar{x}}$. For *any* $c_{\bar{x}}$, $\phi(x) = 0$ at $n+1$ points $x = x_i$, and we choose *particular* $c_{\bar{x}}$ so that $\phi(x) = 0$ at $x = \bar{x}$ as well, i.e.,

$$c_{\bar{x}} := \frac{f(\bar{x}) - p(\bar{x})}{\omega(\bar{x})}\,.$$

Then ϕ has $n+2$ distinct zeros, and, by Rolle's theorem, $\phi^{(n+1)}(\xi) = 0$ for some $\xi \in [a,b]$. So,

$$0 = \phi^{(n+1)}(\xi) = [f^{(n+1)}(\xi) - p^{(n+1)}(\xi)] - c_{\bar{x}}\omega^{(n+1)}(\xi) = f^{(n+1)}(\xi) - c_{\bar{x}}(n+1)!$$

whence

$$c_{\bar{x}} := \frac{f(\bar{x}) - p(\bar{x})}{\omega(\bar{x})} = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\,,$$

and that is the same as (10.3).