Mathematical Tripos Part IB: Easter 2002

Numerical Analysis – Exercise Sheet 1

1. Calculate *all* LU factorizations of the matrix

$$A = \begin{bmatrix} 10 & 6 & -2 & 1 \\ 10 & 10 & -5 & 0 \\ -2 & 2 & -2 & 1 \\ 1 & 3 & -2 & 3 \end{bmatrix},$$

where all diagonal elements of L are one. By using one of these factorizations, find *all* solutions of the equation $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b}^{\mathrm{T}} = [\ -2 \quad 0 \quad 2 \quad 1\ ]$.

2. Let A be a real $n \times n$ matrix that has the factorization $A = LU$, where L is lower triangular with ones on its diagonal and U is upper triangular. Prove that, for every integer $k \in \{1, 2, \ldots, n\}$, the first k rows of U span the same space as the first k rows of A. Prove also that the first k columns of A are in the k-dimensional subspace that is spanned by the first k columns of L. Hence deduce that no LU factorization of the given form exists if we have $\operatorname{rank} H_k < \operatorname{rank} B_k$, where H_k is the leading $k \times k$ submatrix of A and where B_k is the $n \times k$ matrix whose columns are the first k columns of A.

3. By using column pivoting if necessary to exchange rows of A, an LU factorization of a real $n \times n$ matrix A is calculated, where L has ones on its diagonal, and where the moduli of the off-diagonal elements of L do not exceed one. Let α be the largest of the moduli of the elements of A. Prove by induction on i that elements of U satisfy the condition $|U_{i,j}| \le 2^{i-1}\alpha$. Then construct 2×2 and 3×3 nonzero matrices A that yield $|U_{2,2}| = 2\alpha$ and $|U_{3,3}| = 4\alpha$ respectively.

4. Calculate the Cholesky factorization of the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 & \lambda \end{bmatrix}.$$

Deduce from the factorization the value of λ that makes the matrix singular. Also find this value of λ by seeking the vector in the null-space of the matrix whose first component is one.

5. Let A be an $n \times n$ nonsingular band matrix that satisfies the condition $A_{i,j} = 0$ if $|i - j| > r$, where r is small, and let Gaussian elimination *with column pivoting* be used to solve $A\boldsymbol{x} = \boldsymbol{b}$. Identify all the coefficients of the intermediate equations that can become nonzero. Hence deduce that the total number of additions and multiplications of the complete calculation can be bounded by a constant multiple of nr^2.

6. The iteration $\boldsymbol{x}_{k+1} = H\boldsymbol{x}_k + \boldsymbol{b}$ is applied for $k = 0, 1, \ldots$, where H is the real 2×2 matrix

$$H = \begin{bmatrix} \alpha & \gamma \\ 0 & \beta \end{bmatrix},$$

with γ large and $|\alpha| < 1$, $|\beta| < 1$. Calculate the elements of H^k and show that they tend to zero as $k \to \infty$. Further, establish the equation $\boldsymbol{x}_k - \boldsymbol{x}^* = H^k(\boldsymbol{x}_0 - \boldsymbol{x}^*)$, where $\boldsymbol{x}^*$ is defined by $\boldsymbol{x}^* = H\boldsymbol{x}^* + \boldsymbol{b}$. Thus deduce that the sequence $\{\boldsymbol{x}_k\}_{k=0}^{\infty}$ converges to $\boldsymbol{x}^*$.

7. For some choice of $\boldsymbol{x}_0$ the iterative method

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_{k+1} + \begin{bmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ \eta & \zeta & 0 \end{bmatrix} \boldsymbol{x}_k = \boldsymbol{b}$$

is applied for $k = 0, 1, \ldots$, in order to solve the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ \xi & 1 & 1 \\ \eta & \zeta & 1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{b},$$

where ξ, η and ζ are constants. Find all values of the constants such that the sequence $\{\boldsymbol{x}_k\}_{k=0}^{\infty}$ converges for every $\boldsymbol{x}_0$ and $\boldsymbol{b}$. Give an example of nonconvergence when $\xi = \eta = \zeta = -1$. Is the solution always found in at most two iterations when $\xi = \zeta = 0$?

8. Let $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ denote the columns of the matrix

$$A = \begin{bmatrix} 6 & 6 & 1 \\ 3 & 6 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Apply the Gram–Schmidt procedure to A, which generates orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2$ and $\boldsymbol{q}_3$. Note that this calculation provides real numbers $R_{k,\ell}$ such that $\boldsymbol{a}_k = \sum_{\ell=1}^{k} R_{k,\ell}\boldsymbol{q}_\ell$, $k = 1, 2, 3$. Hence express A as the product $A = QR$, where Q and R are orthogonal and upper-triangular matrices respectively.

9. Calculate the QR factorization of the matrix of Exercise 8 by using three Givens rotations. Explain why the initial rotation can be any one of the three types $\Omega^{(1,2)}$, $\Omega^{(1,3)}$ and $\Omega^{(2,3)}$. Prove that the final factorization is independent of this initial choice in exact arithmetic, provided that we satisfy the condition that in each row of R the leading nonzero element is positive.

10. Let A be an $n \times n$ matrix, and for $i = 1, 2, \ldots, n$ let $k(i)$ be the number of zero elements in the ith row of A that come before all nonzero elements in this row and before the diagonal element $A_{i,i}$. Show that the QR factorization of A can be calculated by using at most $\frac{1}{2}n(n-1) - \sum k(i)$ Givens rotations. Hence show that, if A is an upper triangular matrix except that there are nonzero elements in its first column, i.e. $A_{i,j} = 0$ when $2 \le j < i \le n$, then its QR factorization can be calculated by using only $2n - 3$ Givens rotations.

11. Calculate the QR factorization of the matrix of Exercise 8 by using two Householder reflections. Show that, if this technique is used to generate the QR factorization of a general $n \times n$ matrix A, then the computation can be organised so that the total number of additions and multiplications is bounded above by a constant multiple of n^3.

12. Let

$$A = \begin{bmatrix} 3 & 4 & 7 & -2 \\ 5 & 4 & 9 & 3 \\ 1 & -1 & 0 & 3 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 11 \\ 29 \\ 16 \\ 10 \end{bmatrix}.$$

Calculate the QR factorization of A by using Householder reflections. In this case A is singular and you should choose Q so that the last row of R is zero. Hence identify all the least squares solutions of the inconsistent system $A\boldsymbol{x} = \boldsymbol{b}$, where we require $\boldsymbol{x}$ to minimize $\|A\boldsymbol{x} - \boldsymbol{b}\|_2$. Verify that all the solutions give the same vector of residuals $A\boldsymbol{x} - \boldsymbol{b}$, and that this vector is orthogonal to the columns of A. There is no need to calculate the elements of Q explicitly.