Mathematical Tripos Part IB: Easter Term 2003

Numerical Analysis – Lecture 6

6 Iterative methods for solving linear systems (cont.)

6.1 Norms of vectors and matrices

Definition 6.1 A norm on a vector space $\mathbb{X}$ (say, $\mathbb{X} = \mathbb{R}^n$) is a function $\|\cdot\| : \mathbb{X} \to \mathbb{R}$ which satisfies the following axioms:

$$
\begin{aligned}
&1) \quad \|x\| \ge 0 \ \forall x \in \mathbb{X}, \text{ and } \|x\| = 0 \text{ iff } x = 0;\\
&2) \quad \|\alpha x\| = |\alpha|\,\|x\| \quad \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{X};\\
&3) \quad \|x + y\| \le \|x\| + \|y\| \quad \forall x, y \in \mathbb{X}.
\end{aligned}
$$

A vector space with a norm is called a *normed* space.

Example 6.2 The following three vector norms in $\mathbb{R}^n$ are in common use

1)	$\|x\|_1 := \sum_{i=1}^n |x_i|$;	the ℓ_1-norm;
2)	$\|x\|_2 := \{\sum_{i=1}^n |x_i|^2\}^{1/2}$	the ℓ_2-norm or the Euclidean norm;
3)	$\|x\|_\infty := \max_i |x_i|$;	the ℓ_∞-norm or the max-norm.

They are particular cases of the ℓ_p-norm: $\|x\|_{\ell_p} := \|x\|_p := \{\sum_{i=1}^n |x_i|^p\}^{1/p}$, $1 \le p \le \infty$.

Definition 6.3 A matrix norm is a function $\|\cdot\| : \mathbb{R}^{n\times n} \to \mathbb{R}$ such that

$$
\begin{aligned}
&1)\text{-}3) \quad \|A\| \text{ is a norm on the space of matrices;}\\
&4) \quad \|AB\| \le \|A\|\,\|B\|.
\end{aligned}
$$

Example 6.4 Given a vector norm $\|\cdot\|_* : \mathbb{R}^n \to \mathbb{R}$, the function

$$\|A\| := \|A\|_* := \sup_{x \ne 0} \frac{\|Ax\|_*}{\|x\|_*} \tag{6.1}$$

is a matrix norm. (Prove it.) It is called the *induced* matrix norm. By definition,

$$\|Ax\|_* \le \|A\|_* \|x\|_* \,.$$

It is not too difficult to determine the matrix norms induced by three basic vector norms.

1)	$\|A\|_1 = \max_j \sum_i |a_{ij}|$,	i.e., max absolute column sum;
2)	$\|A\|_2 = \sqrt{\rho(A^T A)}$,	i.e., [the spectarl radius of $A^T A]^{1/2}$;
3)	$\|A\|_\infty = \max_i \sum_j |a_{ij}|$,	i.e., max absolute row sum.

Lemma 6.5 *For any induced matrix norm $\|\cdot\|$, and for any $A \in \mathbb{R}^{n\times n}$, we have*

$$\rho(A) \le \|A\| \,.$$

If A is symmetric, then $\rho(A) = \|A\|_2$.

Proof. If λ is an eigenvalue of A and w is the corresponding eigenvector, then

$$|\lambda|\,\|w\| = \|\lambda w\| = \|Aw\| \le \|A\|\,\|w\|.$$

Thus, $|\lambda| \le \|A\|$ for any λ, hence $\rho(A) \le \|A\|$. If $A = A^T$, then $\|A\|_2 = \sqrt{\rho(A^2)} = \sqrt{\rho^2(A)} = \rho(A)$. ■

Definition 6.6 A sequence (x_k) in a normed space $\mathbb{X}$ is called *convergent* if there is an $x \in \mathbb{X}$ s.t.

$$\lim_{k\to\infty} \|x - x_k\| = 0 \,.$$

6.2 Sufficient condition for convergence

Theorem 6.7 *Suppose that x^*, a solution of $Ax = b$, satisfies $x^* = Hx^* + v$ and we are given the scheme*

$$x^{k+1} = Hx^k + v, \qquad x^0, v \in \mathbb{R}^n.$$

If $\|SHS^{-1}\| < 1$ for some induced matrix norm $\|\cdot\|$ and for some nonsingular S, then $x^k \to x^*$.

Proof. Set $\widetilde{H} := SHS^{-1}$, $\gamma := \|\widetilde{H}\| < 1$, and $\alpha := \|S\|\,\|S^{-1}\|$. Then the error $e^k := x^k - x^*$ satisfies

$$\|e^k\| = \|H^k e^0\| = \|S^{-1}\widetilde{H}^k S e^0\| \le \|S^{-1}\|\,\|\widetilde{H}^k\|\,\|S\|\,\|e^0\| \le \alpha\|\widetilde{H}\|^k\,\|e^0\| = \alpha\gamma^k\|e^0\| \to 0. \qquad \blacksquare$$

As an application of this result, consider the following (partially) familiar class of matrices.

Definition 6.8 A matrix A is called strictly diagonally dominant by rows or by columns, if

$$|a_{ii}| > \textstyle\sum_{j\ne i} |a_{ij}|, \quad i = \overline{1,n}\,, \qquad \text{or} \qquad |a_{jj}| > \textstyle\sum_{i\ne j} |a_{ij}|, \quad j = \overline{1,n}\,,$$

respectively.

Theorem 6.9 *Let A be strictly diagonally dominant by rows or by columns. Then the Jacobi method converges.*

Proof. The iteration matrix for Jacobi is

$$H_{\mathrm{J}} = D^{-1}(L_0 + U_0) = \left[\begin{array}{cccc} 0 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ \hline \frac{a_{21}}{a_{22}} & 0 & \cdots & \frac{a_{2n}}{a_{22}} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \cdots & 0 \end{array}\right].$$

If A is diagonally dominant by rows, then taking the matrix ℓ_∞-norm of H_{J} we obtain

$$\|H_{\mathrm{J}}\|_\infty = \max_i \textstyle\sum_j |h_{ij}| = \max_i \sum_{i\ne j} |\frac{a_{ij}}{a_{ii}}| \le \gamma < 1\,.$$

If A is diagonally dominant by columns, then with $S = D^{-1}$ we form the matrix

$$\widetilde{H}_{\mathrm{J}} := SH_{\mathrm{J}}S^{-1} = (L_0 + U_0)D^{-1} = \left[\begin{array}{c|c|c|c} 0 & \frac{a_{12}}{a_{22}} & \cdots & \frac{a_{1n}}{a_{nn}} \\ \frac{a_{21}}{a_{11}} & 0 & \cdots & \frac{a_{2n}}{a_{nn}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{a_{11}} & \frac{a_{n2}}{a_{22}} & \cdots & 0 \end{array}\right],$$

and take its ℓ_1-matrix-norm

$$\|\widetilde{H}_{\mathrm{J}}\|_1 := \max_j \textstyle\sum_i |\widetilde{h}_{ij}| = \max_j \sum_{i\ne j} |\frac{a_{ij}}{a_{jj}}| \le \gamma < 1.$$

Application of Theorem 6.7 yields the result. $\blacksquare$

Remark 6.10 The sufficient condition of Theorem 6.7 looks stronger than that of Theorem 5.10 because, for any nonsingular S and for any induced matrix norm, we have

$$\rho(H) = \rho(SHS^{-1}) \le \|SHS^{-1}\|.$$

(The first equality holds for all eigenvalues, the second one is due to Lemma 6.5.) However, they are equivalent: for any matrix H, and for any $\epsilon > 0$ there exists an induced matrix norm $\|\cdot\|_*$ such that $\|H\|_* \le \rho(H) + \epsilon$ (but one cannot put $\epsilon = 0$ here).

6.3 Examples

Example 6.11 Prove from the first principles that, if A is strictly diagonally dominant by rows, then the Gauss–Seidel method converges.

Proof. Set $e^{(k)} = x^{(k)} - x^*$. Then the Gauss–Seidel method reads

$$e_i^{(k+1)} = -\sum_{j<i} \frac{a_{ij}}{a_{ii}} e_j^{(k+1)} - \sum_{j>i} \frac{a_{ij}}{a_{ii}} e_j^{(k)}, \qquad j = \overline{1,n}\,.$$

Notice that the strict diagonal dominance by rows implies

$$\sum_{j\neq i} \left|\frac{a_{ij}}{a_{ii}}\right| \le \gamma < 1, \qquad j = \overline{1,n}\,,$$

For the 1-st component of $e^{(k+1)}$ we have the inequality

$$|e_1^{(k+1)}| = \left|\sum_{j\neq 1} \frac{a_{1j}}{a_{11}} e_j^{(k)}\right| \le \max_j |e_j^{(k)}| \sum_{j\neq 1} \left|\frac{a_{1j}}{a_{11}}\right| \le \gamma \|e^{(k)}\|_\infty\,.$$

Assuming that the similar estimate is valid for the first $(i-1)$ components of $e^{(k+1)}$, i.e.,

$$|e_j^{(k+1)}| \le \gamma\, \|e^{(k)}\|_\infty, \qquad j < i,$$

we derive the same estimate for the i-th component

$$|e_i^{(k+1)}| = \left|\sum_{j<i} \frac{a_{ij}}{a_{ii}} e_j^{(k+1)} + \sum_{j>i} \frac{a_{ij}}{a_{ii}} e_j^{(k)}\right| \le \|e^{(k)}\|_\infty \sum_{j\neq i} \left|\frac{a_{ij}}{a_{ii}}\right| \le \gamma \|e^{(k)}\|_\infty.$$

Thus, $\|e^{(k+1)}\|_\infty := \max_i |e_i^{(k+1)}| \le \gamma\, \|e^{(k)}\|_\infty$, hence $\|e^{(k)}\|_\infty \le \gamma^k\, \|e^0\|_\infty \to 0$. ∎

Remark 6.12 We did not use Theorem 6.7 explicitly, however what we have proved is that, for a strictly diagonally dominant A, and without any assumtions on e^k, the Gauss–Seidel method implies $\|e^{k+1}\|_\infty \le \gamma \|e^k\|_\infty$. Since $e^{k+1} = H_{\rm GS} e^k$, where $H_{\rm GS}$ is the iteration matrix, this means the following: for a strictly diagonally dominant A,

$$\|H_{\rm GS} z\|_\infty \le \gamma\, \|z\|_\infty \quad \forall z \in \mathbb{R}^n \quad \Rightarrow \quad \|H_{\rm GS}\|_\infty \le \gamma < 1,$$

i.e., the condition of Theorem 6.7 is fulfilled.

6.4 Exercises

Exercises with a $*$-mark are *not* subject to supervision! They are just for fun!

6.1.* Prove that if A is strictly diagonally dominant by columns, then the Gauss–Seidel method converges.
[*Hint*: Prove that, with $S = (L_0 + D)$ and the ℓ_1-norm, we have $\|SH_{\rm GS}S^{-1}\|_1 < 1$.]
[*Hint to Hint*: Transpose the matrix $\widetilde{H}_{\rm GS}$ and use the equality $\|B\|_1 = \|B^T\|_\infty$ together with some consequences of Example 6.11.]

6.2.* Prove that, in the method of simple iteration, if we know the bound not only for $\rho(A) = \lambda_{\max} =: M$ but also for $\lambda_{\min} =: m$, then the choice

$$\tau_{\rm opt} := \frac{2}{M+m}$$

is optimal in the sense that it provides the best rate of convergence in ℓ_2 (of the method of simple iteration) on the class of SPD-matrices A with $\sigma(A) \in [m, M]$, namely

$$\|e^k\|_2 \le \gamma_{\rm opt}^k \, \|e^0\|_2, \qquad \gamma_{\rm opt} = \frac{M-m}{M+m} < 1\,.$$

6.3.* For a given vector norm $\|\cdot\|$, the value

$$\kappa(A) := \|A\|\,\|A^{-1}\|$$

is called the *condition number* of the matrix A. Prove that, if

$$A(x + \delta x) = y + \delta y,$$

then

$$\frac{\|\delta x\|}{\|x\|} \le \kappa(A) \frac{\|\delta y\|}{\|y\|}\,.$$

Thus, the condition number measures *stability* of solution of equation $Ax = y$, i.e. the extent to which the relative error in the right-hand side influences the relative error of solution. (Notice that it is not related to the closeness to zero of the determinant of A.)

6.4.* Prove that, if $\|A\|_*$ is an induced matrix norm, then $\|A\|_{**} := \|SAS^{-1}\|_*$ is also an induced matrix norm, that is find a vector norm $\|\cdot\|_{**}$ such that

$$\|A\|_{**} := \sup_{x \ne 0} \frac{\|Ax\|_{**}}{\|x\|_{**}}\,.$$