Mathematical Tripos Part IB: Easter Term 2002

Numerical Analysis – Lecture 10

10 Error bounds for polynomial interpolation

Here we study the *interpolation error*

$$e_n(x) = f(x) - p_n(x), \quad p_n \in \mathcal{P}_n,$$

restricted to the class of differentiable functions f that possess, say, k continuous derivatives on the interval $[a,b]$; we denote this class by $C^k[a,b]$. We start with an estimate provided by the basic relation (9.3) between two consecutive interpolating polynomials.

Theorem 10.1 *Let $p_n \in \mathcal{P}_n$ interpolate $f \in C[a,b]$ at $n+1$ distinct points $x_0, ..., x_n$. Then for any $x \notin (x_i)$*

$$\boxed{f(x) - p_n(x) = f[x_0, ..., x_n, x]\,\omega(x)} \tag{10.1}$$

Proof. Given $x_0, .., x_n$, let $\bar{x} := x_{n+1}$ be any other point. Then, by (9.3), the corresponding polynomials p_n and p_{n+1} are related by

$$p_{n+1}(x) = p_n(x) + f[x_0, ..., x_n, \bar{x}]\,\omega(x), \qquad \omega(x) := \prod_{i=0}^{n}(x - x_i)\,.$$

In particular, putting $x = \bar{x}$, and noticing that $p_{n+1}(\bar{x}) = f(\bar{x})$, we obtain

$$f(\bar{x}) = p_n(\bar{x}) + f[x_0, ..., x_n, \bar{x}]\,\omega(\bar{x}),$$

the latter equality being the same as (10.1). ∎

This theorem shows the error to be "like the next term" in the Newton form. However, we cannot evaluate the right-hand side of (10.1) without knowing the number $f(x)$. But as we now show we can relate it to the $(n+1)$-st derivative of f. For this we need a version of the Rolle's theorem:

Lemma 10.2 *If $g \in C^k[a,b]$ is zero at $k+\ell$ distinct points, then $g^{(k)}$ has at least ℓ distinct zeros in $[a,b]$.*

Proof. By Rolle's theorem, if $\phi \in C^1$ is zero at two point, then ϕ' is zero at an intermediate point. So, we deduce that g' vanishes at least at $(k-1)+\ell$ distinct points. Next, applying Rolle to g', we conclude that g'' vanishes at $(k-2)+\ell$ points, and so on. ∎

Theorem 10.3 *Let $[\bar{a},\bar{b}]$ be the smallest interval that contains $x_0, ..., x_k$ and let $f \in C^k[\bar{a},\bar{b}]$. Then there exists $\xi \in [\bar{a},\bar{b}]$ such that*

$$\boxed{f[x_0, ..., x_k] = \frac{1}{k!}\,f^{(k)}(\xi)} \tag{10.2}$$

Proof. Let $p \in \mathcal{P}_k$ be the interpolating polynomial to f on (x_i). The error function $f - p$ has at least $k+1$ zeros in $[\bar{a},\bar{b}]$ so, by Rolle's theorem, $f^{(k)} - p^{(k)}$ must vanish at some $\xi \in [\bar{a},\bar{b}]$, i.e.,

$$p^{(k)}(\xi) = f^{(k)}(\xi)$$

On the other hand, if $p(x) = a_k x^k +$ (lower order terms), then (for any ξ)

$$p^{(k)}(\xi) = k!\,a_n =: k!\,f[x_0, ..., x_n]\,.$$ ∎

So, combining (10.1) with (10.2) where we put $k = n+1$, we obtain

Theorem 10.4 *Let $f \in C^{n+1}[a,b]$, and let $p_n \in \mathcal{P}_n$ interpolate f at $n+1$ points $(x_i)_{i=0}^n \in [a,b]$. Then for every $x \in [a,b]$ there exists $\xi \in [a,b]$ such that*

$$\boxed{f(x) - p_n(x) = \frac{1}{(n+1)!}\,\omega(x) f^{(n+1)}(\xi)} \tag{10.3}$$

The equality (10.3) with the value $f^{(n+1)}(\xi)$ for *some* ξ is of hardly any use. Usually one has a bound for $f^{(n+1)}$ in terms of some *norm*, e.g., the L_∞-norm (the *max*-norm)

$$\|g\|_\infty := \|g\|_{L_\infty[a,b]} := \max_{t\in[a,b]} |g(t)|\,.$$

The estimate (10.3) takes then the form

$$\boxed{|f(x) - p_n(x)| \le \frac{1}{(n+1)!}|\omega(x)|\,\|f^{(n+1)}\|_\infty} \tag{10.4}$$

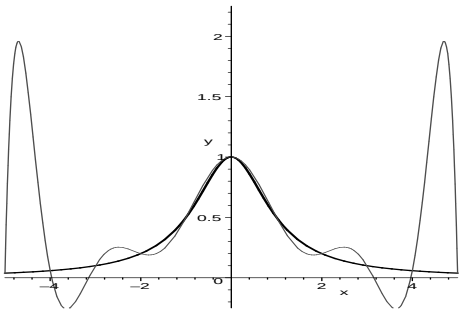
If we want to find the maximal error over the interval, then maximizing first the right- and then the left-hand side over $x \in [a,b]$ we get yet one more error bound for polynomial interpolation

$$\boxed{\|f - p_\Delta\|_\infty \le \frac{1}{(n+1)!}\|\omega_\Delta\|_\infty\,\|f^{(n+1)}\|_\infty} \tag{10.5}$$

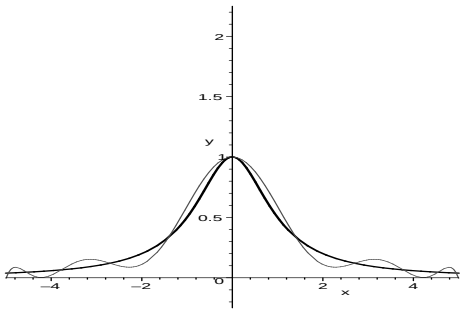
Here we put the lower index in ω_Δ in order to emphasize the dependence of $\omega(x) := \prod_{i=0}^n (x - x_i)$ on the sequence of interpolating points $\Delta := (x_i)_{i=0}^n$. The choice of Δ makes a big difference!

Example 10.5

$$f(x) = \frac{1}{1+x^2}, \quad x \in [-5,5].$$



a) Interpolation at uniform knots $\{-5+k\}_{k=0}^{10}$



b) Interpolation at Chebyshev knots $\{-5\cos\frac{2k+1}{22}\pi\}_{k=0}^{10}$

Definition 10.6 The *Chebyshev* polynomial of degree n on $[-1,1]$ is defined by

$$T_n(x) = \cos n\theta, \quad x = \cos\theta, \quad \theta \in [0,\pi].$$

One sees at once that, on $[-1,1]$,

1) T_n takes its maximal absolute value 1 with alternating signs $n+1$ times:

$$\|T_n\|_\infty = 1, \quad T_n(t_k) = (-1)^k, \quad t_k = \cos\frac{\pi k}{n}, \quad k = \overline{0,n};$$

2) T_n has n distinct zeros:

$$T_n(x_k) = 0, \quad x_k^* = \cos\frac{2k+1}{2n}\pi, \quad k = \overline{0,n-1}.$$

Lemma 10.7 *The Chebyshev polynomials T_n satisfy the recurrence relation*

$$T_0(x) \equiv 1, \quad T_1(x) = x, \tag{10.6}$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \ge 1. \tag{10.7}$$

In particular, T_n is really an algebraic polynomial of degree n with the leading coefficient 2^{n-1}.

Proof. Expressions (10.6) are straightforward, the recurrence follows from the equality

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$$

via the substitution $x = \cos\theta$. ∎

Theorem 10.8 *On the interval* $[-1,1]$*, among all polynomials of degree* n *with leading coefficient equal to one, the Chebyshev polynomial* γT_n *deviates least from zero, i.e.,*

$$\inf_{(a_i)} \left\| x^n + a_{n-1}x^{n-1} + \cdots + a_0 \right\|_\infty = \gamma \, \|T_n\|_\infty, \quad \gamma := 1/2^{n-1} \, .$$

Proof. Suppose there is a polynomial $q_n(x) = x^n + \cdots + a_0$ such that $\|q\|_\infty < \gamma$, and set

$$r := \gamma \, T_n - q_n .$$

1) The leading coefficient of both q_n and γT_n is one, thus r is of degree at most $n-1$.

2) On the other hand, at the points $t_k := \cos \frac{\pi k}{n}$, the Chebyshev polynomial γT_n takes the values $\pm\gamma$ altenatively, while $|q_n(t_k)| < \gamma$, hence r alternates in sign at these points, therefore it has a zero in each of n intervals (t_k, t_{k+1}), i.e. at least n zeros in the interval $[-1,1]$, a contradiction to $r \in \mathcal{P}_{n-1}$. ∎

Corollary 10.9 *For* $\Delta = (x_i)_{i=0}^n \subset [-1,1]$*, let* $\omega_\Delta(x) = \prod_{i=0}^n (x - x_i)$*. Then, for all* n*, we have*

$$\inf_\Delta \|\omega_\Delta\|_\infty = \|\omega_{\Delta_*}\|_\infty = 1/2^n \, .$$

Theorem 10.10 *For* $f \in C^{n+1}[-1,1]$*, the best choice of interpolating points is* $\Delta_* = (x_k^*) = \big(\cos \frac{2k+1}{2n+2}\pi \big)_{k=0}^n$*,*

$$\boxed{\|f - p_{\Delta_*}\|_\infty \le \frac{1}{2^n} \frac{1}{(n+1)!} \, \|f^{(n+1)}\|_\infty}$$

10.1 Additional issues

Theorem 10.4 can be proved without use of the Newton formula.

Theorem 10.4 *Let* $f \in C^{n+1}[a,b]$*, and let* $p_n \in \mathcal{P}_n$ *interpolate* f *at* $n+1$ *points* $(x_i)_{i=0}^n \in [a,b]$*. Then for every* $x \in [a,b]$ *there exists* $\xi \in [a,b]$ *such that*

$$\boxed{f(x) - p_n(x) = \frac{1}{(n+1)!} \, \omega(x) f^{(n+1)}(\xi)} \tag{10.3}$$

Proof. If x coincides with any x_i from the interpolating set, then both sides of (10.3) vanish, hence the formula trivially holds. So, we let $\bar{x}$ be any other point, and consider the function

$$\phi(x) := \big[f(x) - p(x) \big] - c_{\bar{x}} \omega(x)$$

with *some* constant $c_{\bar{x}}$. For *any* $c_{\bar{x}}$, $\phi(x) = 0$ at $n+1$ points $x = x_i$, and we choose *particular* $c_{\bar{x}}$ so that $\phi(x) = 0$ at $x = \bar{x}$ as well, i.e.,

$$c_{\bar{x}} := \frac{f(\bar{x}) - p(\bar{x})}{\omega(\bar{x})} \, .$$

Then ϕ has $n+2$ distinct zeros, and, by Rolle's theorem, $\phi^{(n+1)}(\xi) = 0$ for some $\xi \in [a,b]$. So,

$$0 = \phi^{(n+1)}(\xi) = [f^{(n+1)}(\xi) - p^{(n+1)}(\xi)] - c_{\bar{x}} \omega^{(n+1)}(\xi) = f^{(n+1)}(\xi) - c_{\bar{x}} \, (n+1)!$$

whence

$$c_{\bar{x}} := \frac{f(\bar{x}) - p(\bar{x})}{\omega(\bar{x})} = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \, ,$$

and that is the same as (10.3).

10.2 Exercises

10.1. [*Not easy!*] Given f and g in $C[a,b]$, let $h := fg$. Prove by induction that the divided differences of h satisfy the equation

$$h[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} f[x_0, x_1, \ldots, x_k]\, g[x_k, x_{k+1}, \ldots, x_n].$$

By expressing the differences in terms of derivatives and by letting the points $x_0, x_1, \ldots, x_n$ become coincident, deduce the Leibnitz formula for the nth derivative of a product of two functions.

10.2. Let f be a function in $C^4[0,1]$ and let p be a cubic polynomial that interpolates $f(0), f'(0), f(1)$ and $f'(1)$. Deduce from the Rolle theorem that for every $x \in [0,1]$ there exists $\xi \in [0,1]$ such that the equation

$$f(x) - p(x) = \tfrac{1}{24} x^2 (x-1)^2 f^{(4)}(\xi)$$

is satisfied.

10.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function and let p be the polynomial of degree at most n that interpolates f at the pairwise distinct points $x_0, x_1, \ldots, x_n$. Further, let x be any real number that is not an interpolation point. Deduce the identity

$$f(x) - p(x) = f[x_0, x_1, \ldots, x_n, x] \prod_{i=0}^{n} (x - x_i)$$

from the definition of the divided difference $f[x_0, x_1, \ldots, x_n, x]$.