Mathematical Tripos Part IB: Easter Term 2002

Numerical Analysis – Lecture 12

12 Error of approximation

12.1 Introduction

Given a linear functional λ and an approximation scheme $\lambda(f) \approx \sum_{i=0}^{N} a_i f(x_i)$, we are interested in the error (which is a linear functional as well)

$$e_\lambda(f) := \lambda(f) - \sum_{i=0}^{N} a_i f(x_i),$$

If λ acts on $f \in C^{n+1}[a,b]$, then it is natural to seek an estimate in terms of $\|f^{(n+1)}\|_\infty$, i.e.,

$$|e_\lambda(f)| \le c_\lambda \, \|f^{(n+1)}\|_\infty, \quad \forall f \in C^{n+1}[a,b]\,. \tag{12.1}$$

Since $f^{(n+1)} \equiv 0$ on $\mathcal{P}_n$, such an estimate can exist only if

$$e_\lambda(f) = 0 \quad \text{on} \quad \mathcal{P}_n,$$

i.e., the approximation formula must be *exact on* $\mathcal{P}_n$ (e.g., it can be interpolating of degree n).

If (12.1) holds with some c_λ and moreover, for any $\epsilon > 0$, there is an $f_\epsilon \in C^{n+1}[a,b]$ such that

$$|e_\lambda(f_\epsilon)| > (c_\lambda - \epsilon)\, \|f_\epsilon^{(n+1)}\|_\infty\,,$$

then the constant c_λ is called *least* or *sharp*. The next section gives a general approach to obtaining *sharp* estimates for the functionals e_λ that vanish on $\mathcal{P}_n$.

12.2 The Peano kernel theorem

Our point of departure is the *Taylor formula with an integral remainder term*,

$$f(x) = \sum_{i=0}^{n} \frac{1}{i!}(x-a)^i f^{(i)}(a) + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt, \tag{12.2}$$

where the sum is the *Taylor polynomial* q_n to f (at the point a). One can can verify (12.2) by integration by parts. It is standard to write the remainder in the slightly different form, which makes the range of integration independent of x:

$$R(x) = \frac{1}{n!}\int_a^b (x-t)_+^n \, f^{(n+1)}(t)\,dt, \qquad \text{with} \quad (x-t)_+^n := \big(\max\{x-t, 0\}\big)^n\,.$$

Let λ be a linear functional on C^{n+1}. Then

$$\lambda(f) = \lambda(q_n) + \lambda(R) = \lambda(q_n) + \frac{1}{n!}\,\lambda\Big(\int_a^b (x-t)_+^n \, f^{(n+1)}(t)\,dt\Big).$$

Let us put formally λ under the integration sign, so that, for any *fixed* value of t, it will act on the function

$$g_t(\bullet) := (\bullet - t)_+^n\,,$$

a function, say, of x which (for a given t) is defined as $g_t(x) := (x-t)_+^n$, $x \in [a,b]$. To each value $t \in \mathbb{R}$ there corresponds a value $\lambda(g_t) \in \mathbb{R}$, thus we have a function

$$K_\lambda(t) := \lambda(g_t) := \lambda\big((\bullet - t)_+^n\big), \qquad t \in \mathbb{R}\,,$$

called the *Peano kernel* of the functional λ. So, formally, we may write

$$\lambda(f) = \lambda(q_n) + \frac{1}{n!}\int_a^b K_\lambda(t) f^{(n+1)}(t)dt\,. \tag{12.3}$$

Definition 12.1 Denote by Λ_0 the set of linear functionals which are linear combinations of the functionals of two types

$$1)\ \lambda(f) = f^{(k)}(\bar{x}) \quad (0 \le k \le n), \quad \bar{x} \in [a,b], \qquad 2)\ \lambda(f) = \int_a^{\bar{x}} f(x)\, w(x) dx, \quad \bar{x} \in [a,b].$$

Lemma 12.2 (*without proof*) *If $\lambda \in \Lambda_0$, then equality* (12.3) *is valid.*

Theorem 12.3 (The Peano kernel theorem) *Let $\lambda \in \Lambda_0$ and let $\lambda(f) \approx \sum_{i=0}^N a_i f(x_i)$ be an approximation formula which is exact on $\mathcal{P}_n$. Then the error functional e_λ (and any other functional that vanishes on $\mathcal{P}_n$) has the integral representation*

$$e_\lambda(f) = \frac{1}{n!} \int_a^b K_e(t) f^{(n+1)}(t) dt\,. \tag{12.4}$$

Proof. The formula follows from (12.3) because $e \in \Lambda_0$ and $e(q_n) = 0$. ∎

12.3 Error bounds

Theorem 12.4 *Under assumptions of the previous theorem, we have the sharp inequality*

$$|e_\lambda(f)| \le \frac{1}{n!}\, \|K_{e_\lambda}\|_1\, \|f^{(n+1)}\|_\infty\,. \tag{12.5}$$

Proof. Apply the inequality $\Big| \int\limits_a^b g(t)h(t)dt \Big| \le \max\limits_{t\in[a,b]} |h(t)| \int\limits_a^b |g(t)|\, dt =: \|g\|_1 \|h\|_\infty$ to the right-hand side of (12.4). ∎

Theorem 12.5 *Let $\lambda(f) \approx \sum_{i=0}^n a_i f(x_i)$ be an interpolating formula. If the Peano kernel K_{e_λ} does not change its sign, then*

$$|e_\lambda(f)| \le \frac{1}{(n+1)!}\, |\lambda(\omega)|\, \|f^{(n+1)}\|_\infty\,, \qquad \omega(x) = \prod_{i=0}^n (x - x_i)\,. \tag{12.6}$$

Proof. Put $f(x) = \omega(x)$ into (12.4). Since $\omega^{(n+1)}(t) \equiv (n+1)!$, we obtain $e_\lambda(\omega) = (n+1) \int_a^b K(t)dt$, and since $\omega(x_i) = 0$, we have $e_\lambda(\omega) = \lambda(\omega)$. So, for *any* K,

$$\lambda(\omega) = (n+1) \int_a^b K(t)\, dt\,.$$

If K does not change, then $| \int_a^b K(t)dt| = \|K\|_1$, hence $\frac{1}{n!}\|K\|_1 = \frac{1}{(n+1)!}\, |\lambda(\omega)|$. ∎

Example 12.6 For example, we have an analogue of (10.4) for the k-th derivative

$$|f^{(k)}(\bar{x}) - p_n^{(k)}(\bar{x})| \le \frac{1}{(n+1)!} |\omega^{(k)}(\bar{x})|\, \|f^{(n+1)}\|_\infty\,,$$

provided that K does not change its sign.

Finally, there is a simple (though not very accurate) error bound for numerical integration.

Theorem 12.7 *Let $I(f)$ be a quadrature with $n+1$ nodes (x_i) that is exact on $\mathcal{P}_n$. Then*

$$|e_I(f)| \le \frac{1}{(n+1)!}\, \|\omega\|_1\, \|f^{(n+1)}\|_\infty\,. \tag{12.7}$$

Proof. We have

$$\begin{aligned} |e(f)| \quad &:= \quad |I(f-p)| = \Big| \int\limits_a^b [f(x) - p(x)]dx \Big| \le \int\limits_a^b |f(x) - p(x)| dx \\ &\le \quad \Big\{ \tfrac{1}{(n+1)!} \int\limits_a^b |\omega(x)| dx \Big\} \cdot \|f^{(n+1)}\|_\infty = \tfrac{1}{(n+1)!} \|\omega\|_1 \|f^{(n+1)}\|_\infty \end{aligned}$$

12.4 Examples

Example 12.8 Take formula (11.3):

$$f'(0) \approx p_2'(0) = -\tfrac{3}{2}f(0) + 2f(1) - \tfrac{1}{2}f(2)\,.$$

The error can be expressed as

$$e(f) \;=\; f'(0) - [-\tfrac{3}{2}f(0) + 2f(1) - \tfrac{1}{2}f(2)] \;=\; \frac{1}{2!}\int_0^2 K(t)f'''(t)dt$$

where with $g_t(x) := (x-t)_+^2$

$$\begin{aligned} K(t) = e(g_t) \quad &= \quad 2(0-t)_+^1 + \tfrac{3}{2}(0-t)_+^2 - 2(1-t)_+^2 + \tfrac{1}{2}(2-t)_+^2 \\ &= \left\{ \begin{array}{r} -2(1-t)^2 + \frac{1}{2}(2-t)^2 \\ \frac{1}{2}(2-t)^2 \end{array} \right. = \left\{ \begin{array}{ll} 2t - \frac{3}{2}t^2, & t \in [0,1], \\ \frac{1}{2}(2-t)^2, & t \in [1,2]. \end{array} \right. \end{aligned}$$

So, $K(t) \ge 0$, and integration gives the value

$$\|K\|_1 := \int_0^2 |K(t)|\,dt = (\int_0^1 + \int_1^2)\,K(t)\,dt = \tfrac{1}{2} + \tfrac{1}{6} = \tfrac{2}{3}\,,$$

therefore

$$|e_\lambda(f)| \le \frac{1}{2!}\,\frac{2}{3}\,\|f'''\|_\infty = \frac{1}{3}\,\|f'''\|_\infty\,.$$

Since $K(t) \ge 0$, we could also apply Theorem 12.5: with $\omega(x) = x(x-1)(x-2)$, it follows that

$$|e(f)| \le \tfrac{1}{3!}|\omega'(0)|\,\|f'''\|_\infty = \tfrac{1}{3}\,\|f'''\|_\infty,$$

in accordance with the previous calculations.

Example 12.9 For the Simpson rule,

$$\int_{-1}^1 f(t)\,dt \approx \int_{-1}^1 p_2(t)\,dt = \tfrac{1}{3}f(-1) + \tfrac{4}{3}f(0) + \tfrac{1}{3}f(1)$$

the error is

$$e(f) = \int_{-1}^1 f(t)\,dt - \tfrac{1}{3}f(-1) - \tfrac{4}{3}f(0) - \tfrac{1}{3}f(1) = \int_{-1}^1 K(t)f'''(t)\,dt$$

where with $g_t(x) := (x-t)_+^2$

$$\begin{aligned} K(t) = e(g_t) \quad &= \quad \int_{-1}^1 (x-t)_+^2 dx - \tfrac{1}{3}(-1-t)_+^2 - \tfrac{4}{3}(0-t)_+^2 + \tfrac{1}{3}(1-t)_+^2 \\ &= \left\{ \begin{array}{ll} \frac{1}{3}(1-t)^3 - \frac{4}{3}t^2 & - \frac{1}{3}(1-t)^2 \\ \frac{1}{3}(1-t)^3 & - \frac{1}{3}(1-t)^2 \end{array} \right. = \left\{ \begin{array}{ll} -\frac{1}{3}\,t\,(1+t)^2, & t \in [-1,0] \\ -\frac{1}{3}\,t\,(1-t)^2, & t \in [0,1] \end{array} \right. \end{aligned}$$

Now, $K(t)$ changes its sign at $t=0$, so its L_1-norm has the value

$$\|K\|_1 := \int_0^2 |K(t)|\,dt = (\int_{-1}^0 - \int_0^1)\,K(t)\,dt = 2 \cdot \tfrac{1}{3}\Big(\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4}\Big) = \tfrac{1}{18}$$

so that

$$e_\lambda(f) \le \frac{1}{2!}\frac{1}{18}\|f'''\|_\infty = \frac{1}{36}\,\|f'''\|_\infty\,.$$

From the simpler estimate (12.7), we find

$$\omega(x) = x(x^2-1), \quad \|\omega\|_1 = 2 \cdot \Big|\tfrac{1}{4} - \tfrac{1}{2}\Big| = \tfrac{1}{2},$$

whence

$$e(f) \le \frac{1}{3!}\|\omega\|_1\|f'''\|_\infty = \frac{1}{12}\,\|f'''\|_\infty\,,$$

just to compare with the sharp bound above.