Mathematical Tripos Part IB: Easter 2003

Numerical Analysis – Exercise Sheet 1

1.1. Calculate *all* LU factorizations of the matrix

$$A = \begin{bmatrix} 10 & 6 & -2 & 1 \\ 10 & 10 & -5 & 0 \\ -2 & 2 & -2 & 1 \\ 1 & 3 & -2 & 3 \end{bmatrix},$$

where all diagonal elements of L are one. By using one of these factorizations, find *all* solutions of the equation $Ax = b$ where $b^T = [-2, 0, 2, 1]$.

1.2. By using column pivoting if necessary to exchange rows of A, an LU factorization of a real $n \times n$ matrix A is calculated, where L has ones on its diagonal, and where the moduli of the off-diagonal elements of L do not exceed one. Let α be the largest of the moduli of the elements of A. Prove by induction on i that elements of U satisfy the condition $|u_{ij}| \leq 2^{i-1}\alpha$. Then construct 2×2 and 3×3 nonzero matrices A that yield $|u_{22}| = 2\alpha$ and $|u_{33}| = 4\alpha$ respectively.

2.1. Let A be a real $n \times n$ matrix that has the factorization $A = LU$, where L is lower triangular with ones on its diagonal and U is upper triangular. Prove that, for every integer $k \in \{1, 2, \ldots, n\}$, the first k rows of U span the same space as the first k rows of A. Prove also that the first k columns of A are in the k-dimensional subspace that is spanned by the first k columns of L. Hence deduce that no LU factorization of the given form exists if we have $\operatorname{rank} H_k < \operatorname{rank} B_k$, where H_k is the leading $k \times k$ submatrix of A and where B_k is the $n \times k$ matrix whose columns are the first k columns of A.

2.2. Calculate the Cholesky factorization of the matrix

$$\begin{bmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & 1 & 4 & 1 & \\ & & & 1 & 5 & 1 \\ & & & & 1 & \lambda \end{bmatrix}.$$

Deduce from the factorization the value of λ that makes the matrix singular. Also find this value of λ by seeking the vector in the null-space of the matrix whose first component is one.

2.3. Let A be an $n \times n$ nonsingular band matrix that satisfies the condition $a_{ij} = 0$ if $|i - j| > r$, where r is small, and let Gaussian elimination *with column pivoting* be used to solve $Ax = b$. Identify all the coefficients of the intermediate equations that can become nonzero. Hence deduce that the total number of additions and multiplications of the complete calculation can be bounded by a constant multiple of nr^2.

3.1. Let a_1, a_2 and a_3 denote the columns of the matrix

$$A = \begin{bmatrix} 6 & 6 & 1 \\ 3 & 6 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Apply the Gram–Schmidt procedure to A, which generates orthonormal vectors q_1, q_2 and q_3. Note that this calculation provides real numbers r_{jk} such that $a_k = \sum_{j=1}^{k} r_{jk} q_j$, $k = 1, 2, 3$. Hence express A as the product $A = QR$, where Q and R are orthogonal and upper-triangular matrices respectively.

4.1. Calculate the QR factorization of the matrix of Exercise 3.1 by using three Givens rotations. Explain why the initial rotation can be any one of the three types $\Omega^{(1,2)}$, $\Omega^{(1,3)}$ and $\Omega^{(2,3)}$. Prove that the final factorization is independent of this initial choice in exact arithmetic, provided that we satisfy the condition that in each row of R the leading nonzero element is positive.

4.2. Let A be an $n \times n$ matrix, and for $i = 1, 2, \ldots, n$ let $k(i)$ be the number of zero elements in the i-th row of A that come before all nonzero elements in this row and before the diagonal element a_{ii}. Show that the QR factorization of A can be calculated by using at most $\frac{1}{2}n(n-1) - \sum k(i)$ Givens rotations. Hence show that, if A is an upper triangular matrix except that there are nonzero elements in its first column, i.e. $a_{ij} = 0$ when $2 \leq j < i \leq n$, then its QR factorization can be calculated by using only $2n - 3$ Givens rotations. [*Hint*: Your should find the order of the first $(n-2)$ rotations that brings your matrix to the form considered above.]

4.3. Calculate the QR factorization of the matrix of Exercise 3.1 by using two Householder reflections. Show that, if this technique is used to generate the QR factorization of a general $n \times n$ matrix A, then the computation can be organised so that the total number of additions and multiplications is bounded above by a constant multiple of n^3.

5.1. The iteration $x^{k+1} = Hx^k + b$ is applied for $k = 0, 1, \ldots$, where H is the real 2×2 matrix

$$H = \begin{bmatrix} \alpha & \gamma \\ 0 & \beta \end{bmatrix},$$

with γ large and $|\alpha| < 1$, $|\beta| < 1$. Calculate the elements of H^k and show that they tend to zero as $k \to \infty$. Further, establish the equation $x^k - x^* = H^k(x^0 - x^*)$, where x^* is defined by $x^* = Hx^* + b$. Thus deduce that the sequence $(x^k)_{k=0}^{\infty}$ converges to x^*.

5.2. For some choice of x^0 the iterative method

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x^{k+1} + \begin{bmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ \eta & \zeta & 0 \end{bmatrix} x^k = b$$

is applied for $k = 0, 1, \ldots$, in order to solve the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ \xi & 1 & 1 \\ \eta & \zeta & 1 \end{bmatrix} x = b,$$

where ξ, η and ζ are constants. Find all values of the constants such that the sequence $(x^k)_{k=0}^{\infty}$ converges for every x^0 and b. Give an example of nonconvergence when $\xi = \eta = \zeta = -1$. Is the solution always found in at most two iterations when $\xi = \zeta = 0$?

7.1. Let

$$A = \begin{bmatrix} 3 & 4 & 7 & -2 \\ 5 & 4 & 9 & 3 \\ 1 & -1 & 0 & 3 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 11 \\ 29 \\ 16 \\ 10 \end{bmatrix}.$$

Calculate the QR factorization of A by using Householder reflections. In this case A is singular and you should choose Q so that the last row of R is zero. Hence identify all the least squares solutions of the inconsistent system $Ax = b$, where we require x to minimize $\|Ax - b\|_2$. Verify that all the solutions give the same vector of residuals $Ax - b$, and that this vector is orthogonal to the columns of A. There is no need to calculate the elements of Q explicitly.