NUMERICAL ANALYSIS: EXAMPLES' SHEET 2

13. Let A be an $n \times n$ symmetric tridiagonal matrix that is not deflatable (i.e., all the elements of A that are adjacent to the diagonal are nonzero). Prove that A has n distinct eigenvalues. Prove also that, if A has a zero eigenvalue and a single iteration of the QR algorithm is applied to A, then the resultant tridiagonal matrix is deflatable.
Hint: In the second part deduce that a diagonal element of R is zero.

14. Let A be a 2×2 symmetric matrix whose trace does not vanish, let $A_0 = A$, and let the sequence of matrices $\{A_k : k = 1, 2, \ldots\}$ be calculated by applying the QR algorithm to A_0 (without any origin shifts). Express the matrix element $(A_{k+1})_{1,1}$ in terms of the elements of A_k. Show that, except in the special case when A is already diagonal, the sequence $\{(A_k)_{1,1} : k = 0, 1, \ldots\}$ converges monotonically to the eigenvalue of A of larger modulus.
Hint: The sign of this eigenvalue is same as the sign of the trace of A.

15. Apply a single step of the QR method to the matrix

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 1 & \varepsilon \\ 0 & \varepsilon & 0 \end{bmatrix}.$$

You should find that the $(2,3)$ element of the new matrix is $\mathcal{O}(\varepsilon^3)$ and that the new matrix has exactly the same trace as A.

16. (For those who like analysis). Let A be a real 4×4 upper Hessenberg matrix whose eigenvalues all have nonzero imaginary parts, where the moduli of the two complex pairs of eigenvalues are different. Prove that, if the matrices A_k, $k = 0, 1, 2, \ldots$, are calculated from A by the QR algorithm, then the subdiagonal elements $(A_k)_{2,1}$ and $(A_k)_{4,3}$ stay bounded away from zero, but $(A_k)_{3,2}$ converges to zero as $k \to \infty$.

17. Apply a single iteration of the QR algorithm with double shifts to the matrix

$$A = \begin{bmatrix} 0 & 2 & -1 & -1 \\ -1 & 1 & 0 & 2 \\ 0 & \varepsilon & -1 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

assuming $|\varepsilon|$ is so small that $\mathcal{O}(\varepsilon^2)$ terms are negligible and can be disregarded. You should find that the first column of $(A - s_0 I)(A - s_1 I)$, where s_0 and s_1 are the shifts, has the elements 1, 0, $-\varepsilon$ and 0. Further, you should find that the iteration provides a matrix that is deflatable, because its $(3,2)$ element is of magnitude ε^2.

18. Let $h = 1/M$, where $M \geq 1$ is an integer, and let Euler's method be applied to calculate the estimates $\{\boldsymbol{y}_n\}_{n=1,2,\ldots,M}$ of $\boldsymbol{y}(nh)$ for each of the differential equations

$$y' = -\frac{y}{1+t} \quad \text{and} \quad y' = 2\frac{y}{1+t}, \qquad 0 \leq t \leq 1,$$

starting with $y_0 = y(0) = 1$ in both cases. By using induction and by cancelling as many terms as possible in the resultant products, deduce simple explicit expressions for y_n, $n = 1, 2, \ldots, M$, which should be free from summations and products of n terms. Hence deduce the exact solutions of the equations from the limit $h \to 0$. Verify that the magnitude of the errors $y_n - y(nh)$, $n = 1, 2, \ldots, M$, is at most $\mathcal{O}(h)$.

19. Assuming that $\boldsymbol{f}$ satisfies the Lipschitz condition and possesses a bounded third derivative in $[0, t^*]$, apply the method of analysis of the Euler method, given in the lectures, to prove that the trapezoidal rule

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \tfrac{1}{2}h[\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})]$$

converges and that $\|\boldsymbol{y}_n - \boldsymbol{y}(t_n)\| \le ch^2$ for some $c > 0$ and all n such that $0 \le nh \le t^*$.

20. The s-step Adams–Bashforth method is of order s and has the form

$$\boldsymbol{y}_{n+s} = \boldsymbol{y}_{n+s-1} + h\sum_{j=0}^{s-1} \sigma_j \boldsymbol{f}(t_{n+j}, \boldsymbol{y}_{n+j}).$$

Calculate the actual values of the coefficients in the case $s = 3$

Denoting the polynomials generating the s-step Adams–Bashforth by $\{\rho_s, \sigma_s\}$, prove that

$$\sigma_s(z) = z\sigma_{s-1}(z) + \alpha_{s-1}(z-1)^{s-1},$$

where $\alpha_s \neq 0$ is a constant s.t. $\rho_s(z) - \sigma_s(z)\log z = \alpha_s(z-1)^{s+1} + \mathcal{O}\big(|z-1|^{s+2}\big)$, $z \to 1$.
[Hint: Use induction, the order conditions and the fact that the degree of each σ_s is $s-1$.]

21. By solving a three-term recurrence relation, calculate analytically the sequence of values $\{\boldsymbol{y}_n : n = 2, 3, 4, \ldots\}$ that is generated by the *explicit midpoint rule*

$$\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + 2h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}),$$

when it is applied to the ODE $y' = -y$, $t \ge 0$. Starting from the values $y_0 = 1$ and $y_1 = 1 - h$, show that the sequence diverges as $n \to \infty$ for *all* $h > 0$. Recall, however, that order ≥ 1, the root condition and suitable starting conditions imply convergence in a *finite* interval. Prove that the above implementation of the explicit midpoint rule is consistent with this theorem.
Hint: In the last part, relate the roots of the recurrence relation to $\pm e^{\mp h} + \mathcal{O}\big(h^3\big)$.

22. Show that the multistep method

$$\sum_{j=0}^{3} \rho_j \boldsymbol{y}_{n+j} = h\sum_{j=0}^{2} \sigma_j \boldsymbol{f}(t_{n+j}, \boldsymbol{y}_{n+j})$$

is fourth order only if the conditions $\rho_0 + \rho_2 = 8$ and $\rho_1 = -9$ are satisfied. Hence deduce that this method cannot be both fourth order and satisfy the root condition

23. An s-stage explicit Runge–Kutta method of order s with constant step size $h > 0$ is applied to the differential equation $y' = \lambda y$, $t \ge 0$. Prove the identity

$$y_n = \left[\sum_{l=0}^{s} \frac{1}{l!}(h\lambda)^l\right]^n y_0, \qquad n = 0, 1, 2, \ldots.$$

24. The following four-stage Runge–Kutta method has order four,

$$\begin{aligned}
\boldsymbol{k}_1 &= \boldsymbol{f}(t_n, \boldsymbol{y}_n)\\
\boldsymbol{k}_2 &= \boldsymbol{f}(t_n + \tfrac{1}{3}h, \boldsymbol{y}_n + \tfrac{1}{3}h\boldsymbol{k}_1)\\
\boldsymbol{k}_3 &= \boldsymbol{f}(t_n + \tfrac{2}{3}h, \boldsymbol{y}_n - \tfrac{1}{3}h\boldsymbol{k}_1 + h\boldsymbol{k}_2)\\
\boldsymbol{k}_4 &= \boldsymbol{f}(t_n + h, \boldsymbol{y}_n + h\boldsymbol{k}_1 - h\boldsymbol{k}_2 + h\boldsymbol{k}_3)\\
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + h(\tfrac{1}{8}\boldsymbol{k}_1 + \tfrac{3}{8}\boldsymbol{k}_2 + \tfrac{3}{8}\boldsymbol{k}_3 + \tfrac{1}{8}\boldsymbol{k}_4).
\end{aligned}$$

By considering the equation $y' = y$, show that the order is at most four. Then, *for scalar functions,* prove that the order is at least four in the easy case when f is independent of y, and that the order is at least three in the relatively easy case when f is independent of t.
[You are not expected to derive all of the (gory) details when $f(t, y)$ depends on both t and $\boldsymbol{y}$.]