NUMERICAL ANALYSIS: EXAMPLES' SHEET 3

25. Find $\mathcal{D} \cap \mathbb{R}$, the intersection of the linear stability domain $\mathcal{D}$ with the real axis, for the following methods:

(1) $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)$ (2) $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{1}{2}h[\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})]$

(3) $\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + 2h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})$ (4) $\boldsymbol{y}_{n+2} = \boldsymbol{y}_{n+1} + \frac{1}{2}h[3\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}) - \boldsymbol{f}(t_n, \boldsymbol{y}_n)]$

(5) The RK method $\boldsymbol{k}_1 = \boldsymbol{f}(t_n, \boldsymbol{y}_n)$, $\boldsymbol{k}_2 = \boldsymbol{f}(t_n + h, \boldsymbol{y}_n + h\boldsymbol{k}_1)$, $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{1}{2}h(\boldsymbol{k}_1 + \boldsymbol{k}_2)$.

26. Show that, if z is a nonzero complex number that is on the boundary of the linear stability domain of the two-step BDF method

$$\boldsymbol{y}_{n+2} - \tfrac{4}{3}\boldsymbol{y}_{n+1} + \tfrac{1}{3}\boldsymbol{y}_n = \tfrac{2}{3}hf(t_{n+2}, \boldsymbol{y}_{n+2})$$

then the real part of z is positive. Thus deduce that this method is A-stable.

27. The (stiff) differential equation

$$y'(t) = -10^4(y - t^{-1}) - t^{-2}, \qquad t \ge 1, \qquad y(1) = 1,$$

has the analytic solution $y(t) = t^{-1}$, $t \ge 1$. Let it be solved numerically by Euler's method $y_{n+1} = y_n + h_n f(t_n, y_n)$ and the backward Euler method $y_{n+1} = y_n + h_n f(t_{n+1}, y_{n+1})$, where $h_n = t_{n+1} - t_n$ is allowed to depend on n and to be different in the two cases. Suppose that, for any $t_n \ge 1$, we have $|y_n - y(t_n)| \le 10^{-6}$, and that we require $|y_{n+1} - y(t_{n+1})| \le 10^{-6}$. Show that Euler's method can fail if $h_n = 2 \times 10^{-4}$, but that the backward Euler method always succeeds if $h_n \le 10^{-2} t_n t_{n+1}^2$.
Hint: Find relations between $y_{n+1} - y(t_{n+1})$ and $y_n - y(t_n)$ for general y_n and t_n.

28. This question concerns the predictor-corrector pair

$$\boldsymbol{y}^{\mathrm{P}}_{n+3} = -\tfrac{1}{2}\boldsymbol{y}_n + 3\boldsymbol{y}_{n+1} - \tfrac{3}{2}\boldsymbol{y}_{n+2} + 3h\boldsymbol{f}(t_{n+2}, \boldsymbol{y}_{n+2}),$$
$$\boldsymbol{y}^{\mathrm{C}}_{n+3} = \tfrac{1}{11}[2\boldsymbol{y}_n - 9\boldsymbol{y}_{n+1} + 18\boldsymbol{y}_{n+2} + 6h\boldsymbol{f}(t_{n+3}, \boldsymbol{y}_{n+3})].$$

Show that both methods are third order, and that the estimate of the error of the corrector formula by Milne's device has the value $\frac{6}{17}|\boldsymbol{y}^{\mathrm{P}}_{n+3} - \boldsymbol{y}^{\mathrm{C}}_{n+3}|$.

29. Let $\boldsymbol{p}$ be the cubic polynomial that is defined by $\boldsymbol{p}(t_j) = \boldsymbol{y}_j$, $j = n, n+1, n+2$, and by $\boldsymbol{p}'(t_{n+2}) = \boldsymbol{f}(t_{n+2}, \boldsymbol{y}_{n+2})$. Show that the predictor formula of the previous exercise is $\boldsymbol{y}^{\mathrm{P}}_{n+3} = \boldsymbol{p}(t_{n+2} + h)$. Further, show that the corrector formula is equivalent to the equation

$$\boldsymbol{y}^{\mathrm{C}}_{n+3} = \boldsymbol{p}(t_{n+2}) + \tfrac{5}{11}h\boldsymbol{p}'(t_{n+2}) - \tfrac{1}{22}h^2\boldsymbol{p}''(t_{n+2}) - \tfrac{7}{66}h^3\boldsymbol{p}'''(t_{n+2}) + \tfrac{6}{11}h\boldsymbol{f}(t_{n+2} + h, \boldsymbol{y}_{n+3}).$$

The point of these remarks is that $\boldsymbol{p}$ can be derived from available data, and then the above forms of the predictor and corrector can be applied for any choice of $h = t_{n+3} - t_{n+2}$.

30. Let $u(x)$, $0 \le x \le 1$, be a six-times differentiable function that satisfies the ODE $u''(x) = f(x)$, $0 \le x \le 1$, $u(0)$ and $u(1)$ being given. Further, we let $x_m = mh = m/M$, $m = 0, 1, \ldots, M$, for some positive integer M, and calculate the estimates $u_m \approx u(x_m)$, $m = 1, 2, \ldots, M-1$, by solving the difference equation

$$u_{m-1} - 2u_m + u_{m+1} = h^2 f(x_m) + \alpha h^2[f(x_{m-1}) - 2f(x_m) + f(x_{m+1})], \qquad m = 1, 2, \ldots, M-1,$$

where $u_0 = u(0)$, $u_M = u(1)$, and α is a positive parameter. Show that there exists a choice of α such that the local truncation error of the difference equation is $\mathcal{O}(h^6)$. In this case, deduce that the Euclidean norm of the vector of errors $u(x_m) - u_m$, $m = 0, 1, \ldots, M$, is bounded above by a constant multiple of $\|u^{(6)}\|_\infty h^{7/2}$, and provide an upper bound on this constant.

31. Let f be a smooth function from $\mathbb{R}$ to $\mathbb{R}$, and let $f^{(k)}$ denote its kth derivative. Further, let Δ_0 be the *central difference operator* $\Delta_0 f(mh) = f(mh+\frac{1}{2}h) - f(mh-\frac{1}{2}h)$ and Υ be the *averaging operator* $\Upsilon f(mh) = \frac{1}{2}[f(mh-\frac{1}{2}h) + f(mh+\frac{1}{2}h)]$. Deduce that the approximation

$$f^{(2q+1)}(mh) \approx h^{-2q-1}\Upsilon[\Delta_0^{2q+1} - \tfrac{1}{12}(q+2)\Delta_0^{2q+3}]f(mh)$$

has the form $f^{(2q+1)}(mh) \approx \sum_{j=-q-2}^{q+2} c_j f(mh+jh)$, where q is a nonnegative integer. We set $q=1$ for the rest of the question. In this case, find the values of the coefficients c_j, $j=-3,-2,\ldots,3$ (which are multiples of h^{-3}). Then show that the error of the approximation to $f'''(mh)$ is $\mathcal{O}(h^2)$.

32. The Laplace operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is approximated by the nine-point formula

$$\begin{aligned}(\Delta x)^2\nabla^2 u(l\Delta x, j\Delta x) \approx &-\tfrac{10}{3}u_{l,j} + \tfrac{2}{3}(u_{l+1,j} + u_{l-1,j} + u_{l,j+1} + u_{l,j-1}) \\ &+ \tfrac{1}{6}(u_{l+1,j+1} + u_{l+1,j-1} + u_{l-1,j+1} + u_{l-1,j-1}),\end{aligned}$$

where $u_{l,j} \approx u(l\Delta x, j\Delta x)$. Find the error of this approximation when u is any infinitely-differentiable function. Show that the error is smaller if u happens to satisfy Laplace's equation $\nabla^2 u = 0$.

33. Let $M \geq 2$ and $N \geq 2$ be integers and let $u \in \mathbb{R}^{(M-1)\times(N-1)}$ have the components $u_{m,n}$, $1 \leq m \leq M-1$, $1 \leq n \leq N-1$, where two subscripts occur because we associate the components with the interior points of a rectangular grid. Further, let $u_{m,n}$ be zero on the boundary of the grid, which means $u_{m,n} = 0$ if $0 \leq m \leq M$ and $0 \leq n \leq N$ and at least one of these four inequalities holds as an equation. Thus, for any real constants α, β and γ, we can define a linear transformation A from $\mathbb{R}^{(M-1)\times(N-1)}$ to $\mathbb{R}^{(M-1)\times(N-1)}$ by the equations

$$\begin{aligned}(A\boldsymbol{u})_{m,n} = \alpha u_{m,n} + \beta(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1}) + \gamma(u_{m-1,n-1} + u_{m+1,n-1} \\ + u_{m-1,n+1} + u_{m+1,n+1}), \qquad 1 \leq m \leq M-1,\ \ 1 \leq n \leq N-1.\end{aligned}$$

We now let the components of $\boldsymbol{u}$ have the special form $u_{m,n} = \sin(mk\pi/M)\sin(nl\pi/N)$, $1 \leq m \leq M-1$, $1 \leq n \leq N-1$, where k and l are integers. Prove that $\boldsymbol{u}$ is an eigenvector of A and find its eigenvalue. Hence deduce that, if α, β and γ provide the nine-point formula of Exercise 32, and if M and N are large, then the least modulus of an eigenvalue is approximately $4\sin^2(\frac{\pi}{2M}) + 4\sin^2(\frac{\pi}{2N})$.

34. The function $u(x) = x(x-1)$, $0 \leq x \leq 1$, is defined by the equations $u''(x) = 2$, $0 \leq x \leq 1$, and $u(0) = u(1) = 0$. A difference approximation to the differential equation provides the estimates $u_m \approx u(mh)$, $m = 1,2,\ldots,M-1$, through the system of equations $u_{m-1} - 2u_m + u_{m+1} = 2h^2$, $m = 1,2,\ldots,M-1$, where $u_0 = u_M = 0$, $h = 1/M$, and M is a large positive integer. Show that the exact solution of the system is just $u_m = u(mh)$, $m = 1,2,\ldots,M-1$.

We employ the notation $u_m^{(\infty)} = u(mh)$, because we let the system be solved by the Jacobi iteration, using the starting values $u_m^{(0)} = 0$, $m = 1,2,\ldots,M-1$. Prove that the iteration matrix has the spectral radius $\rho(H) = \cos(\pi/M)$. Further, by regarding the initial error vector $\boldsymbol{u}^{(0)} - \boldsymbol{u}^{(\infty)}$ as a linear combination of the eigenvectors of H, show that the largest component of $\boldsymbol{u}^{(k)} - \boldsymbol{u}^{(\infty)}$ for large k is approximately $(8/\pi^3)\cos^k(\pi/M)$. Hence deduce that the Jacobi method requires about $2.5M^2$ iterations to achieve $\|\boldsymbol{u}^{(k+1)} - \boldsymbol{u}^{(\infty)}\|_\infty \leq 10^{-6}$.

35. The function $u(x,y) = 18x(1-x)y(1-y)$, $0 \leq x,y \leq 1$, is the solution of the Poisson equation $u_{xx} + u_{yy} = 36(x^2+y^2-x-y) = f(x,y)$, say, subject to u being zero on the boundary of the unit square. We pick $\Delta x = 1/6$ and seek the solution of the five-point equations

$$u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - 4u_{m,n} = (\Delta x)^2 f(mh,nh), \qquad 1 \leq m \leq 5,\ 1 \leq n \leq 5,$$

where $u_{m,n}$ is zero if (mh,nh) is on the boundary of the square. Let the multigrid method be applied, using only this fine grid and a coarse grid of mesh size $1/3$, and let every $u_{m,n}$ be zero initially. Calculate the 25 residuals of the starting vector on the fine grid. Then, following the *restriction* procedure in the hand-outs, find the residuals for the initial calculation on the coarse grid. Further, show that if the equations on the coarse grid are solved exactly, then the resultant estimates of u at the four interior points of the coarse grid all have the value $5/6$. By applying the *prolongation operator* to these estimates, find the 25 starting values of $u_{m,n}$ for the subsequent iterations of Gauss–Seidel or Jacobi on the fine grid. Further, show that if one Jacobi iteration is performed, then $u_{3,3} = 23/24$ occurs, which is the estimate of $u(\frac{1}{2},\frac{1}{2}) = 9/8$.