NUMERICAL ANALYSIS: EXAMPLES' SHEET 4

36. Let $\mathcal{A}$ be the $m^2 \times m^2$ matrix that occurs in the five-point difference method for Laplace's equation on a square grid. By applying the orthogonal similarity transformation of Hockney's method, find a tridiagonal matrix, T say, that is similar to $\mathcal{A}$, and derive expressions for each element of T. Hence deduce the eigenvalues of T. Verify that they agree with the eigenvalues of Proposition 4.9.

37. Let $\beta_0 = 2$, $\beta_1 = 0$, $\beta_2 = 6$, $\beta_3 = -2$, $\beta_4 = 6$, $\beta_5 = 0$, $\beta_6 = 6$ and $\beta_7 = 2$. By applying the FFT algorithm, calculate $\sum_{j=0}^{7} \beta_j \mathrm{e}^{2\pi \mathrm{i} jk/8}$, $k = 0, 2, 4, 6$. Check your results by direct calculation.
Hint: Because all values of k are even, you can omit some parts of the usual FFT algorithm.

38. Let $u(x,t) : \mathbb{R}^2 \to \mathbb{R}$ be an infinitely-differentiable solution of the convection–diffusion equation $u_t = u_{xx} - bu_x$, where the subscripts denote partial derivatives and where b is a positive constant, and let $u(x,0)$, $0 \le x \le 1$, $u(0,t)$, $t > 0$, and $u(1,t)$, $t > 0$, be given. A difference method sets $\Delta x = 1/(M+1)$ and $\Delta t = T/N$, where M and N are positive integers and T is a fixed bound on t. Then it calculates the estimates $u_m^n \approx u(m\Delta x, n\Delta t)$, $1 \le m \le M$, $1 \le n \le N$, by applying the formula

$$u_m^{n+1} = u_m^n + \mu(u_{m-1}^n - 2u_m^n + u_{m+1}^n) - \tfrac{1}{2}(\Delta x) b\mu(u_{m+1}^n - u_{m-1}^n),$$

where $\mu = (\Delta t)/(\Delta x)^2$, the values of u_m^n being set to $u(m\Delta x, n\Delta t)$ when $(m\Delta x, n\Delta t)$ is on the boundary. Show that, subject to μ being constant, the local truncation error of the formula is $\mathcal{O}\big((\Delta x)^2\big)$.

Let $\varepsilon(\Delta x, \Delta t)$ be the greatest of the errors $|u(m\Delta x, n\Delta t) - u_m^n|$, $1 \le m \le M$, $1 \le n \le N$. Prove convergence from first principles: if $\Delta x \to 0$ and $\mu \le \frac{1}{2}$ is constant then $\varepsilon(\Delta x, \Delta t)$ also tends to zero.
Hint: Relate the maximum error at each time level to the maximum error at the previous time level.

39. Let $v(x,y)$ be a solution of Laplace's equation $v_{xx} + v_{yy} = 0$ on the unit square $0 \le x, y \le 1$, and let $u(x,y,t)$ solve the diffusion equation $u_t = u_{xx} + u_{yy}$, where the subscripts denote partial derivatives. Further, let u satisfy the boundary conditions $u(\xi,\eta,t) = v(\xi,\eta)$ at all points (ξ,η) on the boundary of the unit square for all $t \ge 0$. Prove that, if u and v are sufficiently differentiable, then the integral

$$\phi(t) = \int_0^1 \int_0^1 [u(x,y,t) - v(x,y)]^2 \mathrm{d}x\mathrm{d}y, \qquad t \ge 0,$$

has the property $\phi'(t) \le 0$. Then prove that $\phi(t)$ tends to zero as $t \to \infty$.
Hint: In the first part, try to replace u_{xx} and u_{yy} when they occur by $u_{xx} - v_{xx}$ and $u_{yy} - v_{yy}$ respectively.

40. Let $u(x,t) : \mathbb{R}^2 \to \mathbb{R}$ be a sufficiently differentiable function that satisfies the diffusion equation $u_t = u_{xx}$, and let θ be a positive constant. Using the notation $u_m^n \approx u(m\Delta x, n\Delta t)$, where $\mu = (\Delta t)/(\Delta x)^2$ is constant, we consider the implicit finite-difference scheme

$$u_m^{n+1} - \tfrac{1}{2}(\mu - \theta)(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) = u_m^n + \tfrac{1}{2}(\mu + \theta)(u_{m-1}^n - 2u_m^n + u_{m+1}^n).$$

Show that its local error is $\mathcal{O}\big((\Delta x)^4\big)$, unless $\theta = \frac{1}{6}$ (the Crandall method), which makes the local error of order $\mathcal{O}\big((\Delta x)^6\big)$. Is it possible for the order to be even higher?

41. The Crank–Nicolson formula is applied to the diffusion equation $u_t = u_{xx}$ on a rectangular mesh $(m\Delta x, n\Delta t)$, $m = 0, 1, \ldots, M+1$, $n = 0, 1, 2, \ldots$, where $\Delta x = 1/(M+1)$. We assume zero boundary conditions $u(0,t) = u(1,t) = 0$ for all $t \ge 0$. Prove that the estimates $u_m^n \approx u(m\Delta x, n\Delta t)$ satisfy the equation

$$\sum_{m=1}^{M} \left[(u_m^{n+1})^2 - (u_m^n)^2 \right] = -\tfrac{1}{2} \frac{\Delta t}{(\Delta x)^2} \sum_{m=1}^{M+1} (u_m^{n+1} + u_m^n - u_{m-1}^{n+1} - u_{m-1}^n)^2, \qquad n = 0, 1, 2, \ldots.$$

Because the right hand side is nonpositive, it follows that $\sum_{m=1}^{M} (u_m^n)^2$ is a monotonically decreasing function of n. We see that this property is analogous to part of Exercise 39 if $v \equiv 0$ there.

Hint: Substitute the value of $u_m^{n+1} - u_m^n$ that is given by the Crank–Nicolson formula into the elementary equation $\sum_{m=1}^{M}\left[(u_m^{n+1})^2 - (u_m^n)^2\right] = \sum_{m=1}^{M}(u_m^{n+1} - u_m^n)(u_m^{n+1} + u_m^n)$. It is also helpful occasionally to change the index m of the summation by one.

42. Let $a(x) > 0$, $x \in [0,1]$, be a given smooth function. We solve the diffusion equation with variable diffusion coefficient, $u_t = (au_x)_x$, given with an initial condition for $t = 0$ and boundary conditions at $x = 0$ and $x = 1$, $t \geq 0$, with the finite-difference method

$$u_m^{n+1} = u_m^n + \mu[a_{m-1/2}u_{m-1}^n - (a_{m-1/2} + a_{m+1/2})u_m^n + a_{m+1/2}u_{m+1}^n],$$

where $a_\alpha = a(\alpha\Delta x)$, $\mu = (\Delta t)/(\Delta x)^2$, $n \geq 0$, $m = 1, 2, \ldots, M$ and $\Delta x = 1/(M+1)$. Prove that the local error is $\mathcal{O}\big((\Delta x)^4\big)$.
Justifying carefully every step of your analysis, show (e.g. by using the eigenvalue technique) that the method is stable for all $0 < \mu < 1/(2a_{\max})$, where $a_{\max} = \max_{x\in[0,1]} a(x)$.
Hint: In the second part, you might prove first the following statement in linear algebra: Let A be an $n \times n$ tridiagonal matrix such that $A_{k,k} < |A_{k,k-1}| + |A_{k,k+1}|$, $k = 1, 2, \ldots, n$, where $A_{1,0} = A_{n,n+1} = 0$. Prove that all the eigenvalues of A are positive. Hence deduce that if a tridiagonal matrix B is such that $|B_{k,k} \pm (|B_{k,k-1}| + |B_{k,k+1}|)| < 1$ for all k then all its eigenvalues reside in the unit disc.
Hint to the hint: Let $A\boldsymbol{v} = \lambda\boldsymbol{v}$, where $\boldsymbol{v} \neq \boldsymbol{0}$. Choose k such that $|v_k| \geq |v_l|$, $l = 1, 2, \ldots, n$, and demonstrate that $A_{k,k-1}v_{k-1} + A_{k,k}v_k + A_{k,k+1}v_{k+1} = \lambda_k v_k$ implies $\lambda > 0$. This is similar to the proof from `Lecture 1` that the five-point equations yield a positive-definite system: both are a special case of a more general statement, the Gerschgorin theorem.

43. Apply the Fourier stability test to the difference equation

$$u_m^{n+1} = \tfrac{1}{2}(2 - 5\mu + 6\mu^2)u_m^n + \tfrac{2}{3}\mu(2 - 3\mu)(u_{m-1}^n + u_{m+1}^n) - \tfrac{1}{12}\mu(1 - 6\mu)(u_{m-2}^n + u_{m+2}^n), \qquad m \in \mathbb{Z}.$$

Deduce that the test is satisfied if and only if $0 \leq \mu \leq \frac{2}{3}$.

44. A square grid is drawn on the region $\{(x,t) : 0 \leq x \leq 1, t \geq 0\}$ in $\mathbb{R}^2$, the grid points being $(m\Delta x, n\Delta x)$, $0 \leq m \leq M+1$, $n = 0, 1, 2, \ldots$, where $\Delta x = 1/(M+1)$ and M is odd. Let $u(x,t)$ be an exact solution of the wave equation $u_{tt} = u_{xx}$ and let the boundary values $u(x,0)$, $0 \leq x \leq 1$, $u(0,t)$, $t > 0$, and $u(1,t)$, $t > 0$, be given. Further, an approximation to $\partial u/\partial t$ at $x = 0$ allows each of the function values $u(m\Delta x, \Delta x)$, $m = 1, 2, \ldots, M$, to be estimated to accuracy ε. Then the difference equation

$$u_m^{n+1} = u_{m+1}^n + u_{m-1}^n - u_m^{n-1}$$

is applied to estimate u at the remaining grid points. Prove that all of the moduli of the errors $|u_m^n - u(m\Delta x, n\Delta x)|$ are bounded above by $\frac{1}{2}\varepsilon M$, even when n is very large.
Hint: Let the error in $u(m\Delta x, \Delta x)$ be $\delta_{m,j}\varepsilon$, $m = 1, 2, \ldots, M$, where $\delta_{m,j}$ is the Kronecker delta and where j is an arbitrary integer in $\{1, 2, \ldots, M\}$. Draw a diagram that shows the contribution from this error to u_m^n for every m and n.

45. A rectangular grid is drawn on $\mathbb{R}^2$, with grid spacing Δx in the x-direction and Δt in the t-direction. Let the difference equation

$$\begin{aligned} u_m^{n+1} - 2u_m^n + u_m^{n-1} = \mu[&a(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) \\ &+ b(u_{m-1}^n - 2u_m^n + u_{m+1}^n) + c(u_{m-1}^{n-1} - 2u_m^{n-1} + u_{m+1}^{n-1})], \end{aligned}$$

where $\mu = (\Delta t)^2/(\Delta x)^2$, be used to approximate solutions of the wave equation $u_{tt} = u_{xx}$. Deduce that, with constant μ, the local error is $\mathcal{O}\big((\Delta x)^4\big)$ if and only if the parameters a, b and c satisfy $a = c$ and $a + b + c = 1$. Show also that, if these conditions hold, then the Fourier stability condition is achieved for all values of μ if and only if the parameters also satisfy $|b| \leq 2a$.