Part II Integrable Systems, Sheet One

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1. **A Lax pair.** Consider a one-parameter family of self–adjoint operators $L(t)$ in some complex inner product space such that

$$L(t) = U(t)L(0)U(t)^{-1}$$

where $U(t)$ is a unitary operator, i.e. $U(t)U(t)^* = 1$ where U^* is the adjoint of U.

Show that $L(t)$ and $L(0)$ have the same eigenvalues. Show that there exist a skew-symmetric operator A such that $U_t = -AU$ and

$$L_t = [L, A].$$

2. **Lax pair for KdV.** Show that the KdV equation is equivalent to

$$L_t = [L, A]$$

where the Lax operators are

$$L = -\frac{d^2}{dx^2} + u, \qquad A = 4\frac{d^3}{dx^3} - 3\Big(u\frac{d}{dx} + \frac{d}{dx}u\Big), \qquad u = u(x,t).$$

3. **Dispersionless KdV and shocks.** Use a chain rule to verify that the implicit solution to the dispersionless KdV

$$u_t = 6uu_x$$

is given by

$$u(x,t) = F(x + 6tu(x,t)),$$

where F is an arbitrary differentiable function of one variable. [This solution is obtained by the method of characteristics. Read about it if you want to, or see if your supervisor is willing to explain it to you.]

Assume that

$$u(x,0) = -\frac{2}{\cosh^2(x)}$$

and show that u_x is unbounded, i.e. that for any $M > 0$ there exists $t > 0$ so that $u_x(x_0, t) > M$ for some x_0. Deduce that u_x becomes infinite after finite time. This is called a shock. Draw a graph of $u(x,t)$ illustrating this situation. Compare it with the one–soliton solution to the KdV equation with the same initial condition.

4. **Lax representation of ODEs**. Let $L(t), A(t)$ be complex valued n by n matrices such that

$$\dot{L} = [L, A].$$

Deduce that Trace$(L^p), p \in \mathbb{Z}$ does not depend on t.

[It is possible to show that systems integrable in a sense of Arnold–Liouville's theorem can be put in this form, with the Poisson commuting first integrals given by traces of powers of L].

Assume that

$$\begin{aligned} L &= (\Phi_1 + i\Phi_2) + 2\Phi_3\lambda - (\Phi_1 - i\Phi_2)\lambda^2, \\ A &= -i\Phi_3 + i(\Phi_1 - i\Phi_2)\lambda \end{aligned}$$

where λ is a parameter and find the system of ODEs satisfied by matrices $\Phi_j(t), j = 1, 2, 3$.

[The Lax relations should hold for any value of the parameter λ. The system you are asked to find known as Nahm's equations. It underlies the construction of non–abelian magnetic monopoles.]

Now take $\Phi_j(t) = -i\sigma_j w_j(t)$ (no summation) where σ_j are matrices

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which satisfy $[\sigma_j, \sigma_k] = i\sum_{k=1}^{3} \varepsilon_{jkl}\sigma_l$. Show that the system reduces to the Euler equations

$$\dot{w_1} = w_2 w_3, \qquad \dot{w_2} = w_1 w_3, \qquad \dot{w_3} = w_1 w_2.$$

Use Trace(L^p) to construct first integrals of this system.

5. **2–soliton solution.** Assume that the scattering data consists of two energy levels $E_1 = -\chi_1^2, E_2 = -\chi_2^2$ where $\chi_1 > \chi_2$ and a vanishing reflection coefficient. Solve the Gelfand–Levitan–Marchenko equation to find the 2-soliton solution.

   [Follow the derivation of the 1–soliton in the Notes but try not to look at the N–soliton unless you really get stuck.]

6. **Integral equation.** Let $L\psi = k^2\psi$ where $L = -\partial_x^2 + u$. Consider ψ of the form

$$\psi(x) = e^{ikx} + \int_x^\infty K(x,z)e^{ikz}dz$$

   where $K(x,z), \partial_z K(x,z) \to 0$ as $z \to \infty$ for any fixed x. Use integration by parts to show

$$\psi = e^{ikx}\Big(1 + \frac{i\hat{K}}{k} - \frac{\hat{K}_z}{k^2}\Big) - \frac{1}{k^2}\int_x^\infty K_{zz}e^{ikz}dz,$$

   where $\hat{K} = K(x,x)$ and $\hat{K}_z = (\partial_z K)|_{z=x}$. Deduce that the Schrödinger equation is satisfied if

$$u(x) = -2(\hat{K}_x + \hat{K}_z), \qquad \text{and}$$
$$K_{xx} - K_{zz} - uK = 0 \quad \text{for} \quad z > x.$$

7. **First integrals for KdV.** Consider the Riccati equation

$$\frac{dS}{dx} - 2ikS + S^2 = u.$$

   for the first integrals of KdV. Assume that

$$S = \sum_{n=1}^{\infty} \frac{S_n(x)}{(2ik)^n}$$

   and find the recursion relations

$$S_1(x,t) = -u(x,t), \quad S_{n+1} = \frac{dS_n}{dx} + \sum_{m=1}^{n-1} S_m S_{n-m}.$$

   Solve the first few relations to show that

$$S_2 = -\frac{\partial u}{\partial x}, \quad S_3 = -\frac{\partial^2 u}{\partial x^2} + u^2, \quad S_4 = -\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial}{\partial x}u^2.$$

   and find S_5. Use the KdV equation to verify directly that

$$\frac{d}{dt}\int_{\mathbb{R}} S_3 dx = 0, \qquad \frac{d}{dt}\int_{\mathbb{R}} S_5 dx = 0.$$