Mathematical Tripos Part II
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Integrable Systems – Lecture 1[1]

1 Integrability of ODEs

What and why?

A mathematician has a huge bag of tricks to solve *linear* DEs: Fourier analysis, Green functions, integral transforms, semigroup theory, operator theory… But the world is nonlinear. The theory of integrable systems aims to illuminate the world of nonlinear DEs.

What is an integrable system? Different definitions abound but they have all to do with the existence of *invariants:* functionals that remain constant as the solution evolves in time. Such invariants have threefold significance: *for a physicist* they represent quantities that often have profound physical significance, e.g. the energy of the system or phase-space volume; *for a computational scientist* they provide crucial guideline how to discretise and compute it efficiently and *for a mathematician* they provide a powerful mechanism to explore qualitative properties of the solution.

In a deep sense, integrable systems provide a probe into the *geometry* of DEs. For example, the simplest invariant is a *first integral:* a function that remains constant as a solution evolves. More formally, if the DE in question is $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$, $\boldsymbol{y}(0) = \boldsymbol{y}_0 \in \mathbb{R}^d$, a conservation law is a function $g : \mathbb{R}^d \to \mathbb{R}$ such that $g(\boldsymbol{y}(t)) \equiv g(\boldsymbol{y}_0)$. In other words, the solution *foliates* $\mathbb{R}^d$ (or a part thereof) and each particular solution $\boldsymbol{y}(t)$ evolves on the *manifold* of constant g.

Hamiltonian formalism

Consider a system with n degrees of freedom: its motion is described by a trajectory in a $(2n)$-dimensional *phase space* $\mathcal{M}$, an open, nonempty set of $\mathbb{R}^{2n}$ with the local coordinates (p_k, q_k), $k = 1, 2, \ldots, n$ (generalised momenta and positions, respectively).

Dynamical variables are smoothly differentiable functions $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ s.t. $f = f(\boldsymbol{p}, \boldsymbol{q}, t)$ (think of t as time). Given two dynamical variables, f and g, we define their *Poisson bracket* as

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right). \tag{1.1}$$

It is easy to confirm that the bracket (1.1) satisfies the following two conditions:

Skew-symmetry: $\{f, g\} = -\{g, f\}$ (hence $\{f, f\} = 0$);

The Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

Exercise: Prove that $\{p_k, p_\ell\} = 0$, $\{q_k, q_\ell\} = 0$ and $\{p_k, q_\ell\} = \delta_{k,\ell}$ for all $k, \ell = 1, \ldots, n$.

Our starting point is a smoothly differentiable function $H : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$, the *Hamiltonian.* (In many cases of interest H is independent of t.) Given a dynamical variable f, we consider

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\}.$$

Requiring that $f = p_k$ and $f = q_k$ results in the *Hamiltonian equations of motion,*

$$p_k' = -\frac{\partial H}{\partial q_k}, \quad q_k' = \frac{\partial H}{\partial p_k}, \qquad k = 1, \ldots, n. \tag{1.2}$$

[1]Please email all corrections and suggestions to these notes to A.Iserles@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartII/IntegrableSystems/Handouts.html.

For example,

$$p_k' = \sum_{j=1}^{n} \left(\frac{\partial p_k}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial p_k}{\partial p_j} \frac{\partial H}{\partial q_j} \right) = -\frac{\partial H}{\partial q_k}.$$

Alternatively to (1.2), we can write a Hamiltonian system in the form

$$\boldsymbol{y}' = \begin{bmatrix} O & -I \\ I & O \end{bmatrix} \nabla H(\boldsymbol{y}), \qquad \text{where} \qquad \boldsymbol{y} = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix}.$$

A dynamic variable f s.t. $\mathrm{d}f/\mathrm{d}t = 0$ (equivalently, $f(\boldsymbol{p}(t), \boldsymbol{q}(t), t) \equiv \text{const.}$) is called a *first integral* (or a *constant of motion*) of (1.1). The system is solvable once it admits sufficiently many first integrals because each such integral can be used to eliminate one equation – hence, in principle, we need $2n-1$ first integrals to restrict the solution to a curve – in effect, solve the equation. (Later we'll see that we can often do with much fewer first integrals.)

Trivially, the Hamiltonian itself is always a first integral, because $\partial H/\partial t = 0$ and $\{H, H\} = 0$.

Example Let $n = 1$, $\mathcal{M} = \mathbb{R}^2$ and $H(p, q) = \frac{1}{2}p^2 + V(q)$ – in other words, the Hamiltonian equations are

$$p' = -\frac{\mathrm{d}V}{\mathrm{d}q}, \qquad q' = p.$$

Since H is a first integral, we have $H(p(t), q(t)) = \frac{1}{2}p^2(t) + V(q(t)) \equiv E$, say, where $E = H(p(0), q(0))$ is the *Hamiltonian energy.* Therefore

$$p = \pm\sqrt{2[E - V(q)]}$$

and integrating $\mathrm{d}t/\mathrm{d}q = 1/p$ yields a solution in an implicit form,

$$t = c \pm \int \frac{\mathrm{d}q}{\sqrt{2[E - V(q)]}}.$$

Sometimes we can find explicitly the integral *and* invert the relation $t = t(q)$: this yields an explicit solution. Although this is usually impossible, nonetheless the system is considered integrable.

Geometric interpretation Each first integral confines the solution to some $(2n-1)$-dimensional manifold. If there are m different first integrals which are independent (in a well defined sense – essentially, we require that their Poisson brackets vanish, i.e. that they are *in involution*), the intersection of 'their' manifolds is itself a manifold, of dimension $2n - m$. In particular, in the example, $n = 2$, $m = 1$ and a one-dimensional manifold is a curve along which the solution evolves.

Proposition 1 *Suppose that f and g are first integrals which are not in involution. Then $\{f, g\}$ is also a first integral.*

Proof The main part of the proof is to demonstrate that $h = \{f, g\}$ is a dynamical variable. By the chain rule,

$$\begin{aligned} h' = \{f', g\} + \{f, g'\} &= \left\{ \frac{\partial f}{\partial t} + \{f, H\}, g \right\} + \left\{ f, \frac{\partial g}{\partial t} + \{g, H\} \right\} \\ &= \left\{ \frac{\partial f}{\partial t}, g \right\} + \left\{ f, \frac{\partial g}{\partial t} \right\} + \{\{f, H\}, g\} + \{f, \{g, H\}\}. \end{aligned}$$

But, using chain rule again, $\{\partial f/\partial t, g\} + \{f, \partial g/\partial t\} = \partial\{f, g\}/\partial t = \partial h/\partial t$ and, by skew-symmetry and the Jacobi identity, $\{\{f, H\}, g\} + \{f, \{g, H\}\} = -\{g, \{f, H\}\} - \{f, \{H, g\}\} = \{\{f, g\}, H\} = \{h, H\}$. Therefore $h' = \partial h/\partial t + \{h, H\}$ and $\{f, g\}$ is a dynamical variable. Now, being first integrals, f and g are independent of t, hence $\{f, H\}, \{g, H\} = 0$ and we use again the Jacobi identity to deduce that $\{h, H\} = 0$, thus $h' = 0$. □

More first integrals can be obtained by means of the *Noether theorem.* The Hamiltonian energy $H(\boldsymbol{p}, \boldsymbol{q})$ is always a first integral and in general it might be the only first integral of a Hamiltonian system.