Mathematical Tripos Part II
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Integrable Systems – Lecture 2[1]

Action-angle variables

Given the Hamiltonian system (1.2), it is often sufficient to identify just n (rather than $2n-1$) first integrals, since each reduces the order of the system by two. This motivates a (non-unique) definition of an integrable system.

An integrable Hamiltonian system consists of a *phase-space* $\mathcal{M} \subseteq \mathbb{R}^{2n}$, together with n smoothly differentiable first integrals $f_1, \ldots, f_n : \mathcal{M} \to \mathbb{R}$, mutually independent in the following sense: at any $\boldsymbol{x} \in \mathcal{M}$ the gradients $\nabla f_1(\boldsymbol{x}), \ldots, \nabla f_n(\boldsymbol{x})$ are linearly independent, and which are in involution (i.e. $\{f_k, f_\ell\} = 0$, $k, \ell = 1, \ldots, n$. We will demonstrate that an integrable Hamiltonian system is completely solvable.

Consider coordinate transformation $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{p}, \boldsymbol{q})$, $\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{q})$ in the Hamiltonian system (1.2). It is said to be *canonical* if it preserves the Poisson bracket,

$$\sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial Q_j} \frac{\partial g}{\partial P_j} - \frac{\partial f}{\partial P_j} \frac{\partial g}{\partial Q_j} \right)$$

for all smoothly differentiable $f, g : \mathcal{M} \to \mathbb{R}$.

Lemma 2 *The new equation in variables $(\boldsymbol{P}, \boldsymbol{Q})$ is Hamiltonian with the same Hamiltonian energy H.*

Proof We will prove that $P_k' = -\partial H(\boldsymbol{P}, \boldsymbol{Q})/\partial Q_k$, $k = 1, \ldots, n$ – the proof for Q_k follows in an identical manner. We have

$$\begin{aligned} P_k' &= \frac{\mathrm{d}P_k(\boldsymbol{p}, \boldsymbol{q})}{\mathrm{d}t} = \sum_{j=1}^{n} \left(\frac{\partial P_k}{\partial p_j} p_j' + \frac{\partial P_k}{\partial q_j} q_j' \right) = \sum_{j=1}^{n} \left(-\frac{\partial P_k}{\partial p_j} \frac{\partial H(\boldsymbol{p}, \boldsymbol{q})}{\partial q_j} + \frac{\partial P_k}{\partial q_j} \frac{\partial H(\boldsymbol{p}, \boldsymbol{q})}{\partial p_j} \right) \\ &= \{P_k, H(\boldsymbol{p}, \boldsymbol{q})\} \stackrel{\text{canonicity}}{=} \sum_{j=1}^{n} \left(\frac{\partial P_k}{\partial Q_j} \frac{\partial H(\boldsymbol{P}, \boldsymbol{Q})}{\partial P_j} - \frac{\partial P_k}{\partial P_j} \frac{\partial H(\boldsymbol{P}, \boldsymbol{Q})}{\partial Q_j} \right) = -\frac{\partial H(\boldsymbol{P}, \boldsymbol{Q})}{\partial Q_k}, \end{aligned}$$

because $\partial P_k / \partial Q_j = 0$ and $\partial P_k / \partial P_j = \delta_{k,j}$. □

Let $S : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ be smoothly differentiable and such that

$$\det \left(\frac{\partial^2 S(\boldsymbol{q}, \boldsymbol{P}, t)}{\partial q_j \partial P_k} \right) \neq 0.$$

Then (exercise!) we can construct a canonical transformation by setting $p_k = \partial S / \partial q_k$, $Q_k = \partial S / \partial P_k$, except that the new Hamiltonian energy will be time-dependent, of the form $H + \partial S / \partial t$. This is an example of a *generating function* of (1.2). Now, the idea is to seek a canonical transformation such that in the new variables $H = H(\boldsymbol{P})$, in other words

$$P_k(t) \equiv P_k(0), \qquad Q_k(t) = Q_k(0) + t \frac{\partial H(\boldsymbol{P})}{\partial P_k}, \qquad k = 1, \ldots, n.$$

Finding such transformation in practice is very difficult, often impossible, but it is possible to explore the subject from a qualitative standpoint.

[1]Please email all corrections and suggestions to these notes to `A.Iserles@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartII/IntegrableSystems/Handouts.html`.

Theorem 3 (Arnold–Liouville) *Given an integrable Hamiltonian system (1.2) with first integrals $f_1, \ldots, f_n$, of which $H = f_1$, let*

$$\mathcal{M}_{\boldsymbol{c}} = \{(\boldsymbol{p}, \boldsymbol{q}) \in \mathcal{M} \,:\, f_k(\boldsymbol{p}, \boldsymbol{q}) = c_k,\ k = 1, \ldots, n\},$$

where $c_1, \ldots, c_n$ are constants (dependent on the initial conditions), be an n-dimensional level surface. Then

(a) *If $\mathcal{M}_{\boldsymbol{c}}$ is compact and connected then it is diffeomorphic to a torus $\mathbb{T}^n = \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \mathbb{S}^1$. It is possible to introduce in the neighbourhood of that torus the* action-angle *coordinates $I_1, \ldots, I_n, \varphi_1, \ldots, \varphi_n$, where $\varphi_k \in [0, 2\pi]$, s.t. the* angles *φ_k are coordinates on $\mathcal{M}_f$ and the* actions *I_k are first integrals.*

(b) *The transformed system is*

$$I_k' = 0, \qquad \varphi_k' = \omega_k(I_1, \ldots, I_n), \qquad k = 1, \ldots, n$$

and can be solved by quadratures (a finite number of algebraic operations and explicit integrations).

Proof (Outline.) The motion takes place on the surface (a sub-manifold of $\mathcal{M}$)

$$f_k(\boldsymbol{p}, \boldsymbol{q}) = c_k, \quad k = 1, \ldots, n \qquad \text{where } c_k = f_k(\boldsymbol{p}(0), \boldsymbol{q}(0)),$$

of dimension n. For every $\boldsymbol{x} \in \mathcal{M}$ we wish to prove that there is exactly one torus passing through $\boldsymbol{x}$ – in other words, $\mathcal{M}$ admits a *foliation* by n-dimensional *leaves:* Each leaf is a torus corresponding to a different choice of constants $\boldsymbol{c}$.

Assume that $\det(\partial f_j / \partial p_k) \neq 0$, so that, by the implicit function theorem, the system $\boldsymbol{f}(\boldsymbol{p}, \boldsymbol{q}) = \boldsymbol{c}$ can be inverted – specifically, solved for $p_1, \ldots, p_n$. In other words $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q}, \boldsymbol{c})$. Differentiate the identity $f_i(\boldsymbol{p}(\boldsymbol{q}, \boldsymbol{c}), \boldsymbol{q}) = c_i$ w.r.t. q_j:

$$\sum_{k=1}^{n} \frac{\partial f_i}{\partial p_k} \frac{\partial p_k}{\partial q_j} + \frac{\partial f_i}{\partial q_j} = 0.$$

Multiply by $\partial f_m / \partial p_j$ and sum up in j,

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial f_m}{\partial p_j} \frac{\partial f_i}{\partial p_k} \frac{\partial p_k}{\partial q_j} + \sum_{j=1}^{n} \frac{\partial f_m}{\partial p_j} \frac{\partial f_i}{\partial q_j} = 0.$$

Swapping the indices $i \leftrightarrow m$ and subtracting the two expressions, we thus obtain

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial f_m}{\partial p_j} \frac{\partial f_i}{\partial p_k} \frac{\partial p_k}{\partial q_j} - \frac{\partial f_i}{\partial p_j} \frac{\partial f_m}{\partial p_k} \frac{\partial p_k}{\partial q_j} \right) + \{f_i, f_m\} = 0.$$

The Poisson bracket vanishes because the f_ks are in involution and, rearranging indices in the double sum,

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial f_i}{\partial p_k} \frac{\partial f_m}{\partial p_j} \left(\frac{\partial p_k}{\partial q_j} - \frac{\partial p_j}{\partial q_k} \right) = 0.$$

As we have assumed, the matrices $(\partial f_i / \partial p_k)$ are invertible and we thus deduce that

$$\frac{\partial p_k}{\partial q_j} - \frac{\partial p_j}{\partial q_k} = 0, \qquad k, j = 1, \ldots, n. \tag{1.3}$$

This implies that $\oint \sum_{j=1}^{n} p_j \, \mathrm{d}q_j = 0$ for any closed contractible curve on the torus $\mathbb{T}^n$, as a consequence of the *Stokes theorem:* e.g. for $n = 3$ Stokes reads

$$\int_{\mathcal{D}} (\nabla \times \boldsymbol{p}) \cdot \mathrm{d}\boldsymbol{q} = \oint_{\partial \mathcal{D}} \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{q},$$

where $\mathcal{D}$ is a surface and

$$(\nabla \times \boldsymbol{p})_m = \tfrac{1}{2} \sum_{j,k} \varepsilon_{j,k,m} \left(\frac{\partial p_k}{\partial q_j} - \frac{\partial p_j}{\partial q_k} \right).$$

To be continued... □