Integrable Systems – Lecture 3[1]

Proof   (... continuation) There are n curves passing through $\boldsymbol{x} \in \mathcal{M}$ that cannot be contracted to a point, so that the corresponding integrals don't vanish. We thus define the action coordinates

$$I_k = \frac{1}{2\pi} \oint_{\Gamma_k} \sum_j p_j \, dq_j, \tag{1.4}$$

where Γ_k is a closed curve which is the kth *basic cycle* of the torus $\mathbb{T}^n$: 'cycle' in general is a sub-manifold without boundary.

Thus,

$$\Gamma_k = \{\rho_1, \ldots, \rho_n) \in \mathbb{T}^n \; : \; \rho_k \in [0, 2\pi], \; \rho_j = \text{const for } j \neq k\},$$

where $\boldsymbol{\rho}$ are coordinates on $\mathbb{T}^n$. (This is a non-trivial point!) The Stokes theorem implies that the above actions are independent of the choice of the cycle Γ_k: Given two such cycles Γ_k and $\tilde{\Gamma}_k$, of opposite orientation,
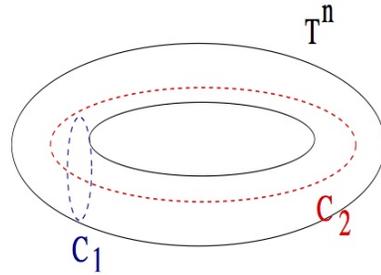
$$\oint_{\Gamma_k} \sum_j p_j \, dq_j + \oint_{\tilde{\Gamma}_k} \sum_j p_j \, dq_j = \int \sum_{i,j} \left(\frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i} \right) dq_j \wedge dq_i = 0.$$

The actions (1.4) are also first integrals, because $\oint p(q, c) \, dq$ depends only on $c_k = f_k$, while f_k is a first integral. Moreover, they are in involution,

$$\{I_i, I_j\} = \sum_{r,s,k} \left(\frac{\partial I_i}{\partial f_r} \frac{\partial f_r}{\partial q_k} \frac{\partial I_j}{\partial f_s} \frac{\partial f_s}{\partial p_k} - \frac{\partial I_i}{\partial f_r} \frac{\partial f_r}{\partial p_k} \frac{\partial I_j}{\partial f_s} \frac{\partial f_s}{\partial q_k} \right) = \sum_{r,s} \frac{\partial I_i}{\partial f_r} \frac{\partial I_j}{\partial f_s} \{f_r, f_s\} = 0.$$

In particular, $\{I_k, H\} = 0$.
The torus $\mathcal{M}_f$ can be equivalently represented by $\{I_k = \tilde{c}_k, \; k = 1, \ldots, n\}$ for some constants $\tilde{\boldsymbol{c}}$, but resist the temptation just to define $I_k = f_k$, because then the transformation $(p, q) \to (I, \phi)$ need not be canonical.

Next we construct the angle coordinates $\phi_1, \ldots, \phi_n$, canonically conjugate to the actions, using the generating function $S(\boldsymbol{q}, \boldsymbol{I}) = \int_{\boldsymbol{q}_0}^{\boldsymbol{q}} \sum_j p_j \, dq_j$, where $\boldsymbol{q}_0 \in \mathbb{T}^n$ – this definition is independent of the path joining $\boldsymbol{q}_0$ and $\boldsymbol{q}$ because of $\partial p_k / \partial q_j - \partial p_j / \partial q_k = 0$ and the Stokes theorem.

Thus, the angles are $\phi_i = \partial S / \partial I_i$, $i = 1, \ldots, n$. They are periodic with period 2π: thus, consider the paths C and $C \cup C_k$ (where C_k represents the kth cycle) joining $\boldsymbol{q}_0$ and $\boldsymbol{q}$:

$$S(\boldsymbol{q}, \boldsymbol{I}) = \int_{C \cup C_k} \sum_j p_j \, dq_j = \int_C \sum_j p_j \, dq_j + \int_{C_k} \sum_j p_j \, dq_j = S(\boldsymbol{q}, \boldsymbol{I}) + 2\pi I_k$$

and, taking derivatives, $\phi_k = \partial S / \partial I_k = \phi_k + 2\pi$.

The transformations $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{\phi}, \boldsymbol{I})$, $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{\phi}, \boldsymbol{I})$ and their inverse, $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{p}, \boldsymbol{q})$, $\boldsymbol{I} = \boldsymbol{I}(\boldsymbol{p}, \boldsymbol{q})$, are canonical (being defined by a generating function!) and invertible. Therefore $\{I_j, I_k\} = 0$, $\{\phi_j, \phi_k\} = 0$, $\{\phi_j, I_k\} = \delta_{j,k}$ and

$$\phi_k' = \{\phi_k, \tilde{H}\}, \quad I_k' = \{I_k, \tilde{H}\}, \qquad k = 1, \ldots, n,$$

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where $\tilde{H}(\boldsymbol{\phi}, \boldsymbol{I}) = H(\boldsymbol{q}(\boldsymbol{\phi}, \boldsymbol{I}), \boldsymbol{p}(\boldsymbol{\phi}, \boldsymbol{I}))$. But the actions are first integrals, therefore $I_k = -\partial \tilde{H}/\partial \phi_k = 0$, and we deduce that $\tilde{H} = \tilde{H}(\boldsymbol{I})$ and $\phi'_k = \partial \tilde{H}/\partial I_k = \omega_k(\boldsymbol{I})$, where the ω_ks are also first integrals. This proves (b). Finally, integrating these canonical equations results in

$$\phi_k(t) = \phi_k(0) + \omega_k(\boldsymbol{I})t, \quad I_k(t) \equiv I_k(0), \qquad k = 1, \dots, n,$$

n circular motions with constant angular velocities. The proof is complete. $\square$

The KAM theory In some situation the main consequences of the Arnold–Liouville Theorem, in particular periodic motion on a (deformed) torus, remains true for Hamiltonians of the form $\tilde{H}(\boldsymbol{I}) + \varepsilon K(\boldsymbol{I}, \boldsymbol{\phi})$: this is the KAM (Kolmogorov–Arnold–Moser) theory.

Example I: Harmonic oscillator $H(p, q) = \frac{1}{2}(p^2 + \omega^2 q^2)$. The solutons foliate $\mathcal{M}_f$ into ellipses $\frac{1}{2}(p^2 + \omega^2 q^2) = \frac{1}{2}(p_0^2 + \omega q_0^2) := E$. For a fixed value of *energy* E we can take $\Gamma = \mathcal{M}_f$, therefore

$$I = \frac{1}{2\pi} \oint_{\mathcal{M}_f} p \, \mathrm{d}q = \frac{1}{2\pi} \iint_S \mathrm{d}p \, \mathrm{d}q = \frac{E}{\omega},$$

where S is the area enclosed by $\mathcal{M}_f$ and we have used the Stokes theorem to move from an integral on the boundary to an area integral inside. Therefore the new Hamiltonian is $H(I) = \omega I$ and $\phi' = \partial H/\partial I = \omega$ implies $\phi = \omega t + \phi_0$. Finally, $I = \frac{1}{2}(p^2/\omega + \omega q^2)$, hence the generating function is

$$S(q, I) = \int p \, \mathrm{d}q = \pm \int (2I\omega - \omega^2 q^2)^{1/2} \, \mathrm{d}q.$$

Taking the $+$ sign, we thus have

$$\phi = \frac{\partial S}{\partial I} = \int \frac{\omega \, \mathrm{d}q}{(2I\omega - \omega^2 q^2)^{1/2}} = \arcsin\left(q\sqrt{\frac{\omega}{2I}}\right) - \phi_0 \quad \Rightarrow \quad q = \sqrt{\frac{2I}{\omega}} \sin(\phi_0 + \phi).$$

Thus, we recover the familiar solution of the harmonic oscillator,

$$p = \sqrt{2E} \cos(\phi_0 + \omega t), \qquad q = \sqrt{2E}/\omega \sin(\phi_0 + \omega t).$$

Example II: The Kepler problem $\boldsymbol{p} = [p_\phi, p_r]^\top$, $\boldsymbol{q} = [\phi, r]^\top$, $H = p_\phi^2/(2r^2) + p_r^2/2 - \alpha/r$, $\alpha > -0$ a constant. It is easy to see that $\{H, p_\phi\} = 0$, hence H and p_ϕ are two first integrals in involution and the system is integrable. Let $\mathcal{M}_f = \{H = E, \ p_\phi = \mu\}$, therefore

$$p_\phi = \mu, \qquad p_r = \pm \left(2E - \frac{\mu^2}{r^2} + \frac{2\alpha}{r}\right)^{1/2},$$

with arbitrary ϕ and a single constraint on (p_r, r). Thus, ϕ and either r or p_r parametrise $\mathcal{M}_f$. Varying ϕ and fixing the other coordinate gives a cycle $\Gamma_\phi \subset \mathcal{M}_f$ and

$$I_\phi = \frac{1}{2\pi} \oint_{\Gamma_\phi} (p_\phi \, \mathrm{d}\phi + p_r \, \mathrm{d}r) = \frac{1}{2\pi} \int_0^{2\pi} p_\phi \, \mathrm{d}\phi = p_\phi.$$

Likewise, to find I_r, we fix ϕ (in addition to H and p_ϕ), whereby we have a cycle Γ_r and

$$I_r = \frac{1}{2\pi} \oint_{\Gamma_r} p_r \, \mathrm{d}p_r = \frac{1}{2\pi} \int_{r_-}^{r_+} \left(2E - \frac{\mu^2}{r^2} + \frac{2\alpha}{r}\right)^{1/2} \mathrm{d}r = \frac{\sqrt{-2E}}{\pi} \int_{r_-}^{r_+} \frac{(r - r_-)^{1/2}(r_+ - r)^{1/2}}{r} \, \mathrm{d}r.$$

To calculate the integral we choose a circular contour of radius $R > r_+$ with a branch cut at $[r_-, r_+]$. This leads to a lengthy calculation and, finally,

$$I_r = \frac{\alpha}{\sqrt{2|E|}} - \mu \quad \Rightarrow \quad H = -\frac{\alpha^2}{2(I_r + I_\phi)^2}.$$

Therefore the absolute values of the frequencies are *equal,* given by

$$\frac{\partial H}{\partial I_r} = \frac{\partial H}{\partial I_\phi} = \frac{\alpha^2}{(I_r + I_\phi)^3} = \left(\frac{2}{r_+ + r_-}\right)^{3/2} \sqrt{\alpha}.$$