Mathematical Tripos Part II
Michaelmas 2012
Professor A. Iserles

Integrable Systems – Lecture 4[1]

Poisson structures Hamiltonian formalism can be generalised by extending the concept of a Poisson bracket. This gives a framework which conveniently segues from ODEs to PDEs. Thereby a much simplified account. We cease distinguishing between positions and momenta and consider an m-dimensional phase space $\mathcal{M}$.

An $m \times m$ skew-symmetric matrix $\omega^{k,\ell} = \omega^{k,\ell}(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \mathcal{M}$, is a *Poisson structure* if the (generalised) Poisson bracket

$$\{f, g\} = \sum_{k,\ell=1}^{m} \omega^{k,\ell}(\boldsymbol{\xi}) \frac{\partial f}{\partial \xi^k} \frac{\partial g}{\partial \xi^\ell}$$

is skew-symmetric and obeys the Jacobi identity. It is trivial to verify that $\{\xi^k, \xi^\ell\} = \omega^{k,\ell}(\boldsymbol{\xi})$, hence the Jacobi identity can be reduced to a statement on the entries of ω.

A Poisson system Let $H : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ be a Hamiltonian: as before, we define the dynamics as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\}$$

(except that we are using generalised Poisson bracket), whereby the equations are

$$\dot{\xi}^k = \sum_{\ell=1}^{m} \omega^{k,\ell}(\boldsymbol{\xi}) \frac{\partial H}{\partial \xi^\ell} \qquad \text{or, in a vector form,} \qquad \boldsymbol{\xi}' = \omega(\boldsymbol{\xi}) \boldsymbol{\nabla} H(\boldsymbol{\xi}).$$

An example Let $\mathcal{M} = \mathbb{R}^3$ and $\omega^{k,\ell} = \sum_{j=1}^{3} \varepsilon^{k,\ell,j} \xi^j$, where $\{\varepsilon^{k,\ell,j}\}$ is the standard totally antisymmetric tensor:

$$\{\xi^1, \xi^2\} = \xi^3, \qquad \{\xi^3, \xi^1\} = \xi^2, \qquad \{\xi^2, \xi^3\} = \xi^1.$$

Given *any* smooth function f of $r = \|\boldsymbol{\xi}\| = \left({\xi^1}^2 + {\xi^2}^2 + {\xi^3}^2\right)^{1/2}$, it is possible to prove that $\{f(r), \xi^k\} = 0$, $k = 1, 2, 3$: such a function $f(r)$ is called a *Casimir*. In particular, let $I_1, I_2, I_3 > 0$ and choose

$$H(\boldsymbol{\xi}) = \tfrac{1}{2} \left(\frac{{\xi^1}^2}{I_1} + \frac{{\xi^2}^2}{I_2} + \frac{{\xi^3}^2}{I_3} \right).$$

The Hamiltonian equations are now

$$\dot{\xi}^1 = \frac{I_3 - I_2}{I_2 I_3} \xi^2 \xi^3, \qquad \dot{\xi}^2 = \frac{I_1 - I_3}{I_1 I_3} \xi^1 \xi^3, \qquad \dot{\xi}^3 = \frac{I_2 - I_1}{I_1 I_2} \xi^1 \xi^2$$

and they describe the motion of a rigid body fixed at its centre of gravity.

Poisson and Hamiltonian structures If m is even and the matrix ω is invertible then it is possible to prove (the Darboux Theorem) that there exists a coordinate transformation that reduces the generalised Poisson bracket to a 'standard' Poisson bracket, hence the underlying equations become Hamiltonian in a standard form.

Note that in the example ω is not invertible (no skew-symmetric odd-dimensional matrix is!), hence we obtain genuinely different creature to familiar Hamiltonian ODEs. Note further that the presence of (nontrivial) Casimirs is a feature of generalised Hamiltonian systems which is impossible for 'plain' Hamiltonian DEs.

[1] Please email all corrections and suggestions to these notes to A.Iserles@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html.

2 Integrability of PDEs

Our definition of integrability fails for *infinite-dimensional systems:* 'half of infinity' makes little sense! To consider a more appropriate approach, we need to look deeper into properties of solutions. We focus on two important examples *the Korteweg–de Vries (KdV) equation* and the *Sine–Gordon equation.*

The KdV equation is the standard model for nonlinear water waves in narrow channels. To derive it, recall the *wave equation* $\Psi_{tt} = v^2\Psi_{xx}$, which describes the propagation of (linear) waves travelling with constant velocity v. Its derivation rests upon three simplifying assumptions: (1) the equation is non-dissipative: the *Euclidean energy* $\|u(\,\cdot\,,t)\|_2$ is constant and the equation is invariant w.r.t. the time inversion $t \to -t$; (2) all oscillations are of small amplitude, hence nonlinear terms (e.g. Ψ^2) can be disregarded; and (3) there is no dispersion, i.e. group velocity is constant (i.e., all frequencies are transported by the same speed). In the derivation of KdV we will relax these assumptions.

The general solution of the wave equations is a superposition of two waves travelling in opposite directions, $\Psi(x,t) = f(x-vt) + g(x+vt)$, where $f,g \in \mathrm{C}^1$ are arbitrary. Each such wave is a solution of the 1st-order *advection equation,*

$$F_t - vF_x = 0 \quad \Rightarrow \quad F = f(x-vt), \qquad\qquad G_t + vG_x = 0 \quad \Rightarrow \quad G = g(x+vt).$$

To introduce dispersion, consider the complex wave $\Psi = \mathrm{e}^{\mathrm{i}[kx-\omega(k)t]}$: for the wave equation $\omega(k) = vk$, hence the group velocity $\omega'(k)$ equals phase velocity v. Instead, let $\omega(k) = v(k - \beta k^3 + \cdots)$ (the absence of even terms guarantees real dispersion relations). Actually, allowing just for small dispersion, we may disregard quintic terms and take $\omega(k) = vk - \beta vk^3$. Therefore $\Psi = \mathrm{e}^{\mathrm{i}[kx-v(kt-\beta k^3 t]}$ and is easily seen to obey the PDE

$$\Psi_t + v\Psi_x + \beta v\Psi_{xxx} = 0.$$

This we rewrite in a conservation form $\rho_t + j_x = 0$, where $\rho = v^{-1}\Psi$ is the *density* and $j = \Psi + \beta\Psi_{xx}$ is the *current.* The main idea is now to allow the current to be nonlinear by adding a quadratic term, $j = \Psi + \beta\Psi_{xx} + \frac{1}{2}\alpha\Psi^2$. Substituting into $\rho_t + j_x = 0$, we obtain

$$\Psi_t + v(\Psi_x + \beta\Psi_{xxx} + \alpha\Psi\Psi_x) = 0.$$

Changing variables $x \to x - vt$ and rescaling Ψ, we obtain the *standard-form KdV equation*

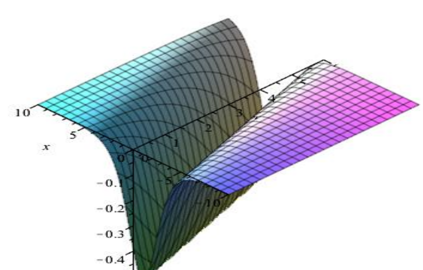
$$u_u - 6uu_x + u_{xxx} = 0. \tag{2.1}$$

The simplest (so-called 'one soliton') solution, found by Korteweg & de Vries, is

$$u(x,t) = \frac{-2\chi^2}{\cosh^2\chi(x - 4\chi^2 t - \phi_0)},$$

where χ and ϕ_0 are constants: ϕ_0 determines the location of the extremum at $t = 0$.

On the right: $u(x,t)$ *for* $\chi = \frac{1}{2}$.



Other solutions were found in 1965 by Gardner, Green, Kruskal & Miura, first numerically, and they are composed of two nonlinear waves of different amplitudes at $t = 0$. The initially-separated waves approached each other, initially distorting their shapes, but eventually the larger wave overtook the smaller and both re-emerged from this 'collision' with their shape and size intact: the only after-effect was a relative phase shift. This is behaviour typically associated with particles, rather than waves. Kruskal & Zabusky called these waves 'solitons'. We will later construct an N-soliton: N interacting 1-solitons.