Mathematical Tripos Part II
Michaelmas 2012
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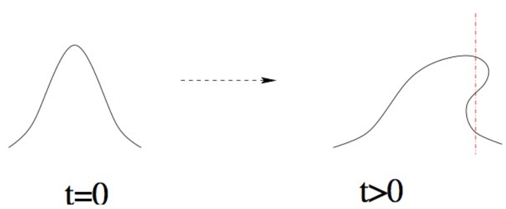
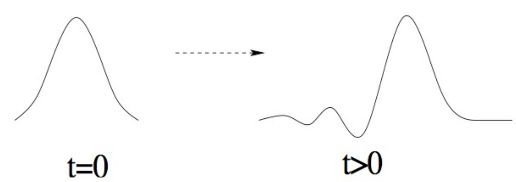
Integrable Systems – Lecture 5[1]

Note that had there been no dispersive term, we would have had a *Burgers' equation* $u_t = 6uu_x$, with a solution exhibiting shocks. On the other hand, had we had no nonlinear term, the equation $u_t + u_{xxx} = 0$ would have been dispersive (essentially, composed of terms of the form $e^{i(\mu x+\mu^3 t)}$ and the initial wave would have 'smeared out' after a while.

Solitons, shocks and dispersive waves It is important to distinguish between solitary waves and other linear or nonlinear waves. In the absence of a dispersive term we have the Burgers equation $u_t - 6uu_{xx}$, whose solutions exhibit shocks (naturally-arising discontinuities) – this is true because the characteristics of the equation are straight lines of (in general) different slope – and once characteristics clash, we are bound to have a discontinuity. In other words, once different waves 'clash', we have a discontinuity. On the other hand, once we excise the nonlinear term, we obtain a dispersive equation $u_t + u_{xxx}$ which can be solves easily by either separation of variables or Fourier analysis. Now we have linear waves which interact linearly.



Shock



Dispersive wave

Integrals of KdV Suppose infinitely smooth initial conditions, zero b.c. at $\pm\infty$ and sufficiently rapid decay of derivatives there. It is possible to prove that KdV admits an infinity of first integrals, which can be generated recursively. The first is the conservation of *momentum,*

$$\theta_1 = \int_{-\infty}^{\infty} \phi \, dx.$$

Next, we generate an infinite recursive sequence of functions:

$$p_1 = \phi, \qquad p_n = -p'_{n-1} + \sum_{j=1}^{n-2} p_j p_{n-1-j}, \quad n \geq 2.$$

E.g., $p_2 = -\phi'$ and $p_3 = \phi'' + \phi^2$. Then

$$\theta_n = \int_{-\infty}^{\infty} p_{2n-1} \, dx, \qquad n \geq 1,$$

are all first integrals. (As are such expressions for p_{2n}, except that they are not independent of other first integrals.) In particular, after trivial simplification,

$$\theta_2 = \int_{-\infty}^{\infty} \phi^2 \, dx \quad \text{(energy)}, \qquad \theta_3 = \int_{-\infty}^{\infty} [\tfrac{1}{3}\phi^3 - {\phi'}^2] \, dx$$

The sine–Gordon equation $\phi_{tt} = \phi_{xx} - \sin\phi$ is an example of a nonlinear wave equation. The equation arises in differential geometry and describes isometric embeddings of surfaces with

[1] Please email all corrections and suggestions to these notes to A.Iserles@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html.

constant negative Gaussian curvature in $\mathbf{R}^3$. Moving to the coordinates $\tau = \frac{1}{2}(x+t)$ and $\rho = \frac{1}{2}(x-t)$, the equation becomes $\phi_{\tau\rho} = \sin\phi$. Essentially, if the *first fundamental form* of a surface, parametrized by (ρ, τ), is

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + 2\cos\phi \,\mathrm{d}\rho \,\mathrm{d}\tau + \mathrm{d}\rho^2$$

then satisfying the sine-Gordon equation implies that the Gaussian quadrature is constant and equal to -1.

Bäcklund transformations We solve the Sine–Gordon equation, in the form $\phi_{\tau\rho} = \sin\phi$. The *Bäcklund relations* are

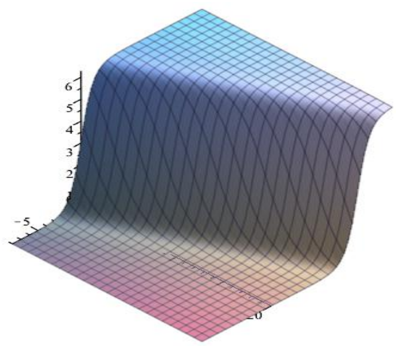
$$\partial_\rho(\psi - \phi) = 2b \sin\frac{\psi+\phi}{2}, \qquad \partial_\tau(\psi+\phi) = 2b^{-1}\sin\frac{\psi-\phi}{2},$$

where $b \neq 0$ is a constant. Therefore

$$\partial_\tau\partial_\rho(\psi-\phi) = 2b\partial_\tau \sin\frac{\psi+\phi}{2} = 2\sin\frac{\psi-\phi}{2}\cos\frac{\psi+\phi}{2} = \sin\psi - \sin\phi.$$

Supposing that ϕ is a solution of sine-Gordon, it follows that so is ψ. In particular, the trivial solution $\phi = 0$ generates the *kink* solution

$$\psi(x,t) = 4\arctan\left(\exp\left(\frac{x-\kappa t}{\sqrt{1-\kappa^2}} - x_0\right)\right), \quad |\kappa| < 1.$$



This is a *one-soliton solution.* A special case, a static kink, occurs for $\kappa = 0$.
It is possible to associate a *topological charge*

$$N = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\phi = \frac{1}{2\pi}[\lim_{x\to\infty}\phi(x,t) - \lim_{x\to-\infty}\phi(x,t)],$$

an integral that depends only on the boundary conditions, with the solution of the sG equation. It is conserved if

$$E(t) = \int_{-\infty}^{\infty}[\tfrac{1}{2}(\phi_t^2 + \phi_x^2) + (1-\cos\phi)]\,\mathrm{d}x < \infty.$$

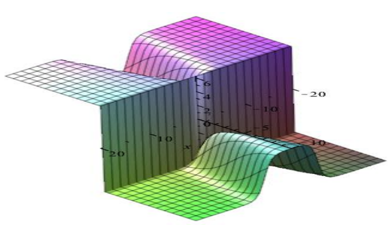
Note that

$$\begin{aligned} E' &= \int_{-\infty}^{\infty}(\phi_t\phi_{tt} + \phi_x\phi_{xt} + \phi_t\sin\phi)\,\mathrm{d}x = \int_{-\infty}^{\infty}[\phi_t(\phi_{xx} - \sin\phi) + \phi_x\phi_{xt} + \phi_t\sin\phi]\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty}(\phi_t\phi_{xx} + \phi_{tx}\phi_x)\,\mathrm{d}x = \int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\phi_t\phi_x)\,\mathrm{d}x = 0, \end{aligned}$$

provided that $\lim_{x\to\pm\infty}\phi_x(x,t) = 0$ and $\phi_t(\pm\infty, t)$ is bounded. Therefore the *energy* E is a first integral of the sG equation. Note however that a topological charge is not a first integral because, strictly speaking, we did not use the PDE in its definition! It is clear that for the kink solution the topological charge is $N = 1$.

An example of $N = 0$, yet nontrivial behaviour is provided by the *soliton–antisoliton pair*

$$\phi(x,t) = 4\arctan\left(\frac{\kappa\cosh\frac{x}{\sqrt{1-\kappa^2}}}{\sinh\frac{\kappa t}{\sqrt{1-\kappa^2}}}\right), \qquad |\kappa| < 1.$$



For $t \ll 0$ the solution is made out of a kink and an 'anti-kink', widely separated but approaching each other with velocity κ. They interact nonlinearly at $t = 0$ and then emerge unchanged. but having swapped places, and continue travelling at the same speed, one to the left and other to the right.