Mathematical Tripos Part II
Michaelmas 2012
Professor A. Iserles

Integrable Systems – Lecture 6[1]

The Liouville equation Bäcklund transformation is valid in a much broader context and they can link solutions of different PDEs. For example, suppose that

$$\partial_\rho(\psi - \phi) = 2a \exp\frac{\phi+\psi}{2}, \qquad \partial_\tau(\psi+\phi) = -a^{-1}\exp\frac{\phi-\psi}{2}, \qquad a \neq 0.$$

Then (proof left as an exercise) ϕ is the solution of the Liuoville equation $\phi_{\rho\tau} = \mathrm{e}^{\phi}$ iff ψ is the solution of $\psi_{\rho\tau} = 0$ – the latter equation can, of course, be solved trivially.

The Camassa–Holm equation An interesting example of an integrable PDE is the *Camassa–Holm equation* which (in a simplified form) reads

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

It is used to model waves in shallow water. We will return to the CH equation once we deal with Hamiltonian formulation of soliton equations (this greedy equation has *two* such formulations!), now we just mention that solutions consist of a very strange version of non-smooth, crested solitons: *peakons,* of the form

$$u(x,t) = \sum_j m_j(t)\mathrm{e}^{-|x-x_j(t)|}.$$

3 The inverse scattering transform

The Schrödinger equation and scattering Consider an infinite-dimensional complex linear space $\mathcal{H}$ of functions, referred to as *wave functions* or *state vectors.* We equip it with an inner product

$$\langle\Psi,\Phi\rangle = \int_{-\infty}^{\infty} \bar{\Psi}(x)\Phi(x)\,\mathrm{d}x.$$

Functions Ψ s.t. $\langle\Psi,\Psi\rangle < \infty$ (think e^{-x^2}) are called *bound states,* while if $\langle\Psi,\Psi\rangle = \infty$ (e.g. $\mathrm{e}^{\mathrm{i}x}$) then ψ is a *scattering state.* (The terminology originates in quantum mechanics: physicists are more careless than mathematicians when it comes to infinities…)

The time-independent (univariate) Schrödinger equation is the spectral problem

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + u\Psi = E\Psi, \tag{3.1}$$

where $u = u(x)$ is the *interaction potential.* $\hbar$ (the Planck constant) and m (the mass) are constants, which need not concern us, while E, the spectral point, is the energy of the underlying quantum system. The set of all E is in general discrete (point spectrum, a.k.a. eigenvalues) for bound states, continuous (continuous spectrum) for scattering states Ψ, and this depends on the potential u.

The quantum-mechanical interpretation is that the probability density of a position of a particle is given by $|\Psi|^2$, while its time evolution is governed by the time-dependent Schrödinger equation

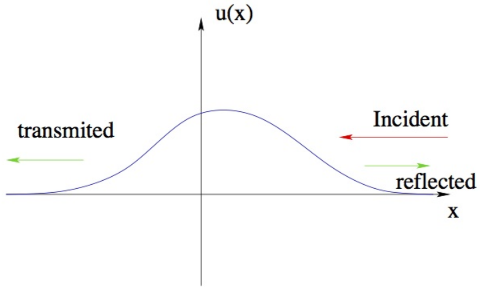
$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + u\Psi.$$

[1]Please email all corrections and suggestions to these notes to `A.Iserles@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html`.

It is easy to prove that, once $\Psi(\,\cdot\,,0)$ is a bound state, quantum-mechanical probability (a.k.a. Euclidean energy) is conserved, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\Psi,\Psi\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}|\Psi|^2\,\mathrm{d}x = 0.$$

The basic goal of particle physics is the solution of the following *inverse problem:* given scattering data produced by a Schrödinger equation with an unknown potential u, determine the potential. It has been solved by Gelfand, Levitan & Marchenko and then used by Kruskal *et al.* to solve the Cauchy problem for KdV. Changing variables in (3.1), we may assume that $\hbar^2/(2m) = 1$. We allow u to depend on time t but regard it as a parameter. Consider a beam of free particles incident from $+\infty$: some will be reflected by the potential (which, for the time being, is assumed to decay sufficiently fast at infinity), others will be transmitted in addition, there might also be a number of bound states with discrete energy levels.



Given *energy levels* E, *transmission probability* T and *reflection probability* R (all of which can be measured in a particle accelerator) we find the potential u: We commence from $u_0(x)$, the potential at $t = 0$ and find the scattering data $(E(0), T(0), R(0))$ at $t = 0$. Let $u(x,t)$ be the solution to KdV with $u(x,0) = u_0(x)$, then $(E(t), T(t), R(t))$ obeys a simple linear ODE and, in particular, $E(t) = \text{const}$. Once we have the scattering data, we recover $u(x,t)$ by solving a linear integral equation.

This leads to a scheme, due to Gardner, Green, Kruskal & Miura, to solve KdV, which is represented by the commutative diagram

$$\begin{tikzcd}
\begin{array}{l} u_t - 6uu_x + u_{xxx} = 0 \\ u(x,0) = u_0(x) \end{array} \arrow[rr, "\text{Schrödinger's eqn}"'] \arrow[d, dotted] & & \begin{array}{l}\text{Scattering}\\ \text{at } t=0\end{array} \arrow[d, "\text{KdV, Lax pair}"] \\
u(x,t) & & \begin{array}{l}\text{Scattering}\\ \text{at } t>0\end{array} \arrow[ll, "\text{The Gelfand–Levitan–Marchenko eqn}"']
\end{tikzcd}$$

Direct scattering Before we deal with inverse scattering, we need to understand *direct* scattering, i.e. the basic 1D quantum mechanics of particle scattering on a given potential. Letting $E = k^2$, we rewrite linear Schrödinger in the form $\mathcal{L}f = k^2 f$, where $\mathcal{L} = -\,\mathrm{d}^2/\,\mathrm{d}x^2 + u(x)$ is the *Schrödinger operator.* We assume that $u \in \mathcal{H}$, the linear space of all functions v s.t. $\int_{-\infty}^{\infty}(1+|x|)|v(x)|\,\mathrm{d}x < \infty$. This implies that $u(x) \overset{x\to\pm\infty}{\longrightarrow} 0$ and guarantees that the spectrum of $\mathcal{L}$ is made out of eigenvalues – in the language of physics, there is finite number of discrete energy levels. (This excludes many important cases, e.g. the harmonic oscillator and the hydrogen atom.)

At $x \to \pm\infty$ the problem reduces to $f_{xx} + k^2 f = 0$, thus $f(x) = c_1\mathrm{e}^{\mathrm{i}kx} + c_2\mathrm{e}^{-\mathrm{i}kx}$, where c_1, c_2 need not be the same at $\pm\infty$.