Integrable Systems – Lecture 6

The Liouville equation Bäcklund transformation is valid in a much broader context and they can link solutions of different PDEs. For example, suppose that
\[ \partial_\rho (\psi - \phi) = 2a \exp \frac{\phi + \psi}{2}, \quad \partial_\tau (\psi + \phi) = -a^{-1} \exp \frac{\phi - \psi}{2}, \quad a \neq 0. \]
Then (proof left as an exercise) \( \phi \) is the solution of the Liouville equation \( \phi_{\rho\tau} = e^\phi \) iff \( \psi \) is the solution of \( \psi_{\rho\tau} = 0 \) – the latter equation can, of course, be solved trivially.

The Camassa–Holm equation An interesting example of an integrable PDE is the Camassa–Holm equation which (in a simplified form) reads
\[ u_t - u_{xxxx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}. \]
It is used to model waves in shallow water. We will return to the CH equation once we deal with Hamiltonian formulation of soliton equations (this greedy equation has two such formulations!), now we just mention that solutions consist of a very strange version of non-smooth, crested solitons: peakons, of the form
\[ u(x,t) = \sum_j m_j(t)e^{-|x-x_j(t)|}. \]

3 The inverse scattering transform

The Schrödinger equation and scattering Consider an infinite-dimensional complex linear space \( \mathcal{H} \) of functions, referred to as wave functions or state vectors. We equip it with an inner product
\[ \langle \Psi, \Phi \rangle = \int_{-\infty}^{\infty} \bar{\Psi}(x)\Phi(x) \, dx. \]
Functions \( \Psi \) s.t. \( \langle \Psi, \Psi \rangle < \infty \) (think e\(^{-x^2}\)) are called bound states, while if \( \langle \Psi, \Psi \rangle = \infty \) (e.g. e\(^{ix}\)) then \( \psi \) is a scattering state. (The terminology originates in quantum mechanics: physicists are more careless than mathematicians when it comes to infinities...) The time-independent (univariate) Schrödinger equation is the spectral problem
\[ -\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} + u \Psi = E \Psi, \tag{3.1} \]
where \( u = u(x) \) is the interaction potential. \( \hbar \) (the Planck constant) and \( m \) (the mass) are constants, which need not concern us, while \( E \), the spectral point, is the energy of the underlying quantum system. The set of all \( E \) is in general discrete (point spectrum, a.k.a. eigenvalues) for bound states, continuous (continuous spectrum) for scattering states \( \Psi \), and this depends on the potential \( u \).

The quantum-mechanical interpretation is that the probability density of a position of a particle is given by \( |\Psi|^2 \), while its time evolution is governed by the time-dependent Schrödinger equation
\[ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + u \Psi. \]

1 Please email all corrections and suggestions to these notes to A.Iserles@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html.
It is easy to prove that, once $\Psi(\cdot, 0)$ is a bound state, quantum-mechanical probability (a.k.a. Euclidean energy) is conserved, since
\[
\frac{d}{dt} \langle \Psi, \Psi \rangle = \frac{d}{dt} \int_{-\infty}^{\infty} |\Psi|^2 \, dx = 0.
\]

The basic goal of particle physics is the solution of the following inverse problem: given scattering data produced by a Schrödinger equation with an unknown potential $u$, determine the potential. It has been solved by Gelfand, Levitan & Marchenko and then used by Kruskal et al. to solve the Cauchy problem for KdV. Changing variables in (3.1), we may assume that $\hbar^2/(2m) = 1$.

We allow $u$ to depend on time $t$ but regard it as a parameter. Consider a beam of free particles incident from $+\infty$: some will be reflected by the potential (which, for the time being, is assumed to decay sufficiently fast at infinity), others will be transmitted; in addition, there might also be a number of bound states with discrete energy levels.

Given energy levels $E$, transmission probability $T$ and reflection probability $R$ (all of which can be measured in a particle accelerator) we find the potential $u$: We commence from $u_0(x)$, the potential at $t = 0$ and find the scattering data $(E(0), T(0), R(0))$ at $t = 0$. Let $u(x, t)$ be the solution to KdV with $u(x, 0) = u_0(x)$, then $(E(t), T(t), R(t))$ obeys a simple linear ODE and, in particular, $E(t) = \text{const}$. Once we have the scattering data, we recover $u(x, t)$ by solving a linear integral equation.

This leads to a scheme, due to Gardner, Green, Kruskal & Miura, to solve KdV, which is represented by the commutative diagram

**Direct scattering** Before we deal with inverse scattering, we need to understand direct scattering, i.e. the basic 1D quantum mechanics of particle scattering on a given potential. Letting $E = k^2$, we rewrite linear Schrödinger in the form $\mathcal{L}f = k^2 f$, where $\mathcal{L} = -d^2/dx^2 + u(x)$ is the Schrödinger operator. We assume that $u \in \mathcal{H}$, the linear space of all functions $v$ s.t. $\int_{-\infty}^{\infty} (1 + |x|) |v(x)| \, dx < \infty$.

This implies that $u(x) \xrightarrow{x \to \pm \infty} 0$ and guarantees that the spectrum of $\mathcal{L}$ is made out of eigenvalues – in the language of physics, there is finite number of discrete energy levels. (This excludes many important cases, e.g. the harmonic oscillator and the hydrogen atom.)

At $x \to \pm \infty$ the problem reduces to $f_{xx} + k^2 f = 0$, thus $f(x) = c_1 e^{ikx} + c_2 e^{-ikx}$, where $c_1, c_2$ need not be the same at $\pm \infty$. 

\[
\int_{-\infty}^{\infty} (1 + |x|) |\phi(x)| \, dx < \infty.
\]

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\]