Recall the spectral problem \( \mathcal{L}f = k^2 f \), where \( \mathcal{L} = -d^2/dx^2 + u(x) \) is the Schrödinger operator. If \( k \neq 0 \) then functions \( f \) obeying the problem (abusing terminology, we’ll call them ‘eigenfunctions’) form a 2D complex vector space \( \mathcal{G}(k) \). \( u(x) \) real ⇒ if \( f \) is an eigenfunction then so is \( \bar{f} \).

Let \( \{ \psi, \bar{\psi} \} \) and \( \{ \phi, \bar{\phi} \} \) be two bases of \( \mathcal{G}(k) \) such that

\[
\begin{align*}
x \to +\infty: & \quad \psi(x,k) \approx e^{-ikx}, \quad \bar{\psi}(x,k) \approx e^{ikx}, \\
x \to -\infty: & \quad \phi(x,k) \approx e^{-ikx}, \quad \bar{\phi}(x,k) \approx e^{ikx}.
\end{align*}
\]

Any eigenfunction can be expanded in a basis, hence \( \exists a(k), b(k) \text{ s.t. } \phi(x,k) = a(k)\psi(x,k) + b(k)\bar{\psi}(x,k) \). Thus, if \( a \neq 0 \),

\[
\frac{\phi(x,k)}{a(k)} \approx \begin{cases} 
  e^{-ikx} \frac{a(k)}{a(k)}, & x \to -\infty, \\
  e^{-ikx} + b(k) \frac{a(k)}{a(k)} e^{ikx}, & x \to \infty.
\end{cases}
\]  

(3.2)

A particle emanates from \( +\infty \) leftwards with the wave function \( e^{-ikx} \). We let the transmission and reflection coefficients be

\[
t(k) = \frac{1}{a(k)}, \quad r(k) = \frac{b(k)}{a(k)},
\]

because \( |a(k)|^2 - |b(k)|^2 = 1 \): this implies that \( |t(k)|^2 + |r(k)|^2 = 1 \) - we can thus interpret \( |t(k)|^2 \) and \( |r(k)|^2 \) as the probabilities of transmission and reflection, respectively.

To prove \( |a(k)|^2 - |b(k)|^2 = 1 \), consider the Wronskian \( W(f,g) = fg_x - f_x g \). Therefore \( W_x(f,g) = fg_{xx} - f_{xx} g \) and \( W = 0 \) if both \( f \) and \( g \) are eigenfunctions. It follows that \( W(\phi, \bar{\phi}) \equiv c \) and

\[
c = \lim_{x \to -\infty} W(\phi, \bar{\phi}) = e^{-ikx}(e^{ikx})_x - (e^{-ikx})_x e^{ikx} = 2ik.
\]

Likewise (letting \( x \to +\infty \)) \( W(\psi, \bar{\psi}) = 2ik \). However, since \( \phi = a\psi + b\bar{\psi} \) and \( W \) is linear in both its arguments and skew-symmetric upon swapping the arguments,

\[
W(\phi, \bar{\phi}) = W(a\psi + b\bar{\psi}, a\bar{\psi} + b\psi) = |a|^2 W(\psi, \bar{\psi}) + a\bar{b}W(\psi, \psi) + \bar{a}bW(\bar{\psi}, \psi) - |b|^2 W(\psi, \bar{\psi}) = 2ik(|a|^2 - |b|^2),
\]

because \( W(f,f) \equiv 0 \) for any \( f \). This proves that \( |a|^2 - |b|^2 = 1 \).

**Properties of (forward) scattering**  Let \( k \in \mathbb{C} \). Then it is possible to prove that

1. \( a(k) \) is holomorphic in \( \mathbb{C}^u = \{ k \in \mathbb{C} : \text{Im} k > 0 \} \);
2. \( \lim_{|k| \to \infty} |a(k)| = 1 \) for \( k \in \mathbb{C}^u \);
3. The zeros of \( a \) in \( C^n \) all live on \( i\mathbb{R} \) and, provided \( \int_{-\infty}^{\infty} (1 + |x|)|u(x)| \, dx < \infty \), their number is finite. We assume these are \( \{i\chi_k\}_{k=1}^N \), where \( \chi_1 > \chi_2 > \cdots > \chi_N > 0 \).

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1 Please email all corrections and suggestions to these notes to A.Iserles@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html.
4. Because of (3.2), we have
\[
\phi(x, i\chi_n) \approx \begin{cases} 
 e^{-i(x_n x)} = e^{x_n x}, & x \to -\infty, \\
 b_n e^{-x_n x}, & x \to \infty,
\end{cases}
\]
where \( b_n = b(i\chi_n) \).

It is possible to show that \( b_n \in \mathbb{R} \), the \( b_n \)'s change sign, \( b_n = (-1)^n|b_n|, a'(i\chi_n) \in i\mathbb{R} \) and \( \text{sgn} \, a'(i\chi_n) = \text{sgn} \, b_n \). Finally, \( \phi(\cdot, i\chi_n) \) is an eigenfunction,
\[
\left[ -\frac{d^2}{dx^2} + u(x) \right] \phi(x, i\chi_n) = -\chi_n^2 \phi(x, i\chi_n), \quad x \in \mathbb{R}.
\]

In particular, this indicates that \( \phi(\cdot, i\chi_n) \) is square integrable.

**Inverse scattering** We wish to recover \( u(x) \) from the **scattering data**: reflection coefficients \( r(k) \) and energy levels \( \{\chi_1, \ldots, \chi_N\} \) - hence \( E_n = (i\chi_n)^2 = -\chi_n^2 \) and
\[
\phi(x, i\chi_n) = \begin{cases} 
 e^{x_n x}, & x \to -\infty, \\
b_n e^{-x_n x}, & x \to \infty.
\end{cases}
\]

To set the **inverse scattering transform** we need the following steps:

1. Set the function
\[
F(x) = \sum_{n=1}^{N} b_n e^{-x_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{ikx} dk.
\]

2. Solve for \( K(x, y) \) the **GLM equation**
\[
K(x, y) + F(x + y) + \int_{x}^{\infty} K(x, v) F(x + y) dv = 0.
\]

Then
\[
u(x) = -2 \frac{dK(x, x)}{dx}
\]

We have done all this for the \( t \)-independent equation but we can introduce \( t \) as a parameter, provided time dependence of the scattering data is known. In that case \( K = K(x, y, t) \) and \( u = u(x, t) \).

**The Lax formulation** In general the eigenvalues of the \( t \)-dependent problem depend upon \( t \), but inverse scattering is an exception: an example of an **isospectral problem**, whose evolution keeps eigenvalues intact.

**Lemma** Let \( \mathcal{L} \) be a time-dependent self-adjoint linear operator acting on the linear space closed w.r.t. the inner product \( (f, g) = \int_{-\infty}^{\infty} f(x) \bar{g}(x) dx \) and suppose that it obeys the ODE
\[
\mathcal{L}' = [A, \mathcal{L}] = A\mathcal{L} - \mathcal{L}A,
\]
where \( A \) is itself a linear operator. Then the spectrum of \( \mathcal{L} \) is independent of \( t \).

**Proof** (We assume for simplicity that the spectrum consists of eigenvalues only.) Differentiate \( \mathcal{L} f = Ef \). Therefore \( \mathcal{L} f + \mathcal{L} f_t = E_t f + Ef_t \). However, \( \mathcal{L}_t = A\mathcal{L} - \mathcal{L}A \) and \( A\mathcal{L} f = A(E f) = EAf \), therefore
\[
(\mathcal{L} f + \mathcal{L} f_t) + (\mathcal{L} f + \mathcal{L} f_t) = (E_t f + Ef_t + E_t f + Ef_t)
\]
and we deduce that \( E_t f = (\mathcal{L} - E)(Af - f_t) \). Take an inner product with \( f \neq 0 \):
\[
E_t(f, f) = \langle f, (\mathcal{L} - E)(Af - f_t) \rangle \mathcal{L} \text{ self adjoint} = \langle (\mathcal{L} - E)f, Af - f_t \rangle = 0
\]
because \( (\mathcal{L} - E)f = 0 \), Since \( f \neq 0 \), it follows that \( E_t = 0 \). \( \square \)