Mathematical Tripos Part II
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Integrable Systems – Lecture 7[1]

Recall the spectral problem $\mathcal{L}f = k^2 f$, where $\mathcal{L} = -\,\mathrm{d}^2/\,\mathrm{d}x^2 + u(x)$ is the Schrödinger operator. If $k \neq 0$ then functions f obeying the problem (abusing terminology, we'll call them 'eigenfunctions') form a 2D complex vector space $\mathcal{G}(k)$. $u(x)$ real $\Rightarrow$ if f is an eigenfunction then so is $\bar{f}$.

Let $\{\psi, \bar{\psi}\}$ and $\{\phi, \bar{\phi}\}$ be two bases of $\mathcal{G}(k)$ such that

$$\begin{aligned} x \to \infty: &\qquad \psi(x,k) \approx \mathrm{e}^{-\mathrm{i}kx}, \qquad \bar{\psi}(x,k) \approx \mathrm{e}^{\mathrm{i}kx}, \\ x \to -\infty: &\qquad \phi(x,k) \approx \mathrm{e}^{-\mathrm{i}kx}, \qquad \bar{\phi}(x,k) \approx \mathrm{e}^{\mathrm{i}kx}. \end{aligned}$$

Any eigenfunction can be expanded in a basis, hence $\exists\ a(k), b(k)$ s.t. $\phi(x,k) = a(k)\psi(x,k) + b(k)\bar{\psi}(x,k)$. Thus, if $a \neq 0$,

$$\frac{\phi(x,k)}{a(k)} \approx \begin{cases} \dfrac{\mathrm{e}^{-\mathrm{i}kx}}{a(k)}, & x \to -\infty, \\ \mathrm{e}^{-\mathrm{i}kx} + \dfrac{b(k)}{a(k)}\mathrm{e}^{\mathrm{i}kx}, & x \to \infty. \end{cases} \tag{3.2}$$

A particle emanates from $+\infty$ leftwards with the wave function $\mathrm{e}^{-\mathrm{i}kx}$. We let the *transmission* and *reflection* coefficients be

$$t(k) = \frac{1}{a(k)}, \qquad r(k) = \frac{b(k)}{a(k)},$$

because $|a(k)|^2 - |b(k)|^2 = 1$: this implies that $|t(k)|^2 + |r(k)|^2 = 1$ – we can thus interpret $|t(k)|^2$ and $|r(k)|^2$ as the probabilities of transmission and reflection, respectively.

To prove $|a(k)|^2 - |b(k)|^2 = 1$, consider the Wronskian $W(f,g) = fg_x - f_x g$. Therefore $W_x(f,g) = fg_{xx} - f_{xx}g$ and $W = 0$ if both f and g are eigenfunctions. It follows that $W(\phi, \bar{\phi}) \equiv c$ and

$$c = \lim_{x \to -\infty} W(\phi, \bar{\phi}) = \mathrm{e}^{-\mathrm{i}kx}(\mathrm{e}^{\mathrm{i}kx})_x - (\mathrm{e}^{-\mathrm{i}kx})_x \mathrm{e}^{\mathrm{i}kx} = 2\mathrm{i}k.$$

Likewise (letting $x \to +\infty$) $W(\psi, \bar{\psi}) = 2\mathrm{i}k$. However, since $\phi = a\psi + b\bar{\psi}$ and W is linear in both its arguments and skew-symmetric upon swapping the arguments,

$$\begin{aligned} W(\phi, \bar{\phi}) &= W(a\psi + b\bar{\psi}, \bar{a}\bar{\psi} + \bar{b}\psi) = |a|^2 W(\psi, \bar{\psi}) + a\bar{b}W(\psi, \psi) + \bar{a}bW(\bar{\psi}, \bar{\psi}) - |b|^2 W(\psi, \bar{\psi}) \\ &= 2\mathrm{i}k(|a|^2 - |b|^2), \end{aligned}$$

because $W(f,f) \equiv 0$ for any f. This proves that $|a|^2 - |b|^2 = 1$.

Properties of (forward) scattering Let $k \in \mathbb{C}$. Then it is possible to prove that

1. $a(k)$ is holomorphic in $\mathbb{C}^u = \{k \in \mathbb{C} : \operatorname{Im} k > 0\}$;
2. $\lim_{|k| \to \infty} |a(k)| = 1$ for $k \in \mathbb{C}^u$;
3. The zeros of a in C^u all live on $\mathrm{i}\mathbb{R}$ and, provided $\int_{-\infty}^{\infty}(1 + |x|)|u(x)|\,\mathrm{d}x < \infty$, their number is finite. We assume these are $\{\mathrm{i}\chi_k\}_{k=1}^N$, where $\chi_1 > \chi_2 > \cdots > \chi_N > 0$.

[1]Please email all corrections and suggestions to these notes to `A.Iserles@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html`.

4. Because of (3.2), we have

$$\phi(x, \mathrm{i}\chi_n) \approx \begin{cases} \mathrm{e}^{-\mathrm{i}(\mathrm{i}\chi_n)x} = \mathrm{e}^{\chi_n x}, & x \to -\infty, \\ a(\mathrm{i}\chi_n)\mathrm{e}^{-\mathrm{i}(\mathrm{i}\chi_n)x} + b(\mathrm{i}\chi_n)\mathrm{e}^{\mathrm{i}(\mathrm{i}\chi_n)x} = b_n \mathrm{e}^{-\chi_n x}, & x \to \infty, \end{cases} \qquad \text{where } b_n = b(\mathrm{i}\chi_n).$$

It is possible to show that $b_n \in \mathbb{R}$, the b_ns change sign, $b_n = (-1)^n |b_n|$, $a'(\mathrm{i}\chi_n) \in \mathrm{i}\mathbb{R}$ and $\operatorname{sgn} \mathrm{i}a'(\mathrm{i}\chi_n) = \operatorname{sgn} b_n$. Finally, $\phi(\,\cdot\,, \mathrm{i}\chi_n)$ is an eigenfunction,

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x)\right] \phi(x, \mathrm{i}\chi_n) = -\chi_n^2 \phi(x, \mathrm{i}\chi_n), \qquad x \in \mathbb{R}.$$

In particular, this indicates that $\phi(\,\cdot\,, \mathrm{i}\chi_n)$ is square integrable.

Inverse scattering We wish to recover $u(x)$ from the *scattering data:* reflection coefficients $r(k)$ and energy levels $\{\chi_1, \ldots, \chi_N\}$ – hence $E_n = (\mathrm{i}\chi_n)^2 = -\chi_n^2$ and

$$\phi(x, \mathrm{i}\chi_n) = \begin{cases} \mathrm{e}^{\chi_n x}, & x \to -\infty, \\ b_n \mathrm{e}^{-\chi_n x}, & x \to \infty. \end{cases}$$

To set the *inverse scattering transform* we need the following steps:

1. Set the function

$$F(x) = \sum_{n=1}^{N} \frac{b_n \mathrm{e}^{-\chi_n x}}{\mathrm{i}a'(\mathrm{i}\chi_n)} + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}k.$$

2. Solve for $K(x, y)$ the *GLM equation*

$$K(x, y) + F(x + y) + \int_x^{\infty} K(x, v) F(x + y) \, \mathrm{d}v = 0.$$

Then

$$u(x) = -2 \frac{\mathrm{d}K(x, x)}{\mathrm{d}x}.$$

We have done all this for the t-independent equation but we can introduce t as a parameter, provided time dependence of the scattering data is known. In that case $K = K(x, y, t)$ and $u = u(x, t)$.

The Lax formulation In general the eigenvalues of the t-dependent problem depend upon t, but inverse scattering is an exception: an example of an *isospectral problem,* whose evolution keeps eigenvalues intact.

Lemma *Let $\mathcal{L}$ be a time-dependent self-adjoint linear operator acting on the linear space closed w.r.t. the inner product $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \bar{g}(x) \, \mathrm{d}x$ and suppose that it obeys the ODE*

$$\mathcal{L}' = [\mathcal{A}, \mathcal{L}] = \mathcal{A}\mathcal{L} - \mathcal{L}\mathcal{A},$$

where $\mathcal{A}$ is itself a linear operator. Then the spectrum of $\mathcal{L}$ is independent of t.

Proof (We assume for simplicity that the spectrum consists of eigenvalues only.) Differentiate $\mathcal{L}f = Ef$. Therefore $\mathcal{L}_t f + \mathcal{L} f_t = E_t f + E f_t$. However, $\mathcal{L}_t = \mathcal{A}\mathcal{L} - \mathcal{L}\mathcal{A}$ and $\mathcal{A}\mathcal{L}f = \mathcal{A}(Ef) = E\mathcal{A}f$, therefore

$$(\mathcal{A}\mathcal{L} - \mathcal{L}\mathcal{A})f + \mathcal{L}f = E_t f + E f_t \qquad \Rightarrow \qquad -(\mathcal{L}\mathcal{A} - E\mathcal{A})f + (\mathcal{L} - E)f_t = E_t f$$

and we deduce that $E_t f = (\mathcal{L} - E)(\mathcal{A}f - f_t)$. Take an inner product with $f \neq 0$:

$$E_t \langle f, f \rangle = \langle f, (\mathcal{L} - E)(\mathcal{A}f - f_t) \rangle \overset{\mathcal{L} \text{ self adjoint}}{=} \langle (\mathcal{L} - E)f, \mathcal{A}f - f_t \rangle = 0$$

because $(\mathcal{L} - E)f = 0$, Since $f \neq 0$, it follows that $E_t = 0$. □