Mathematical Tripos Part II
Michaelmas 2012
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Integrable Systems – Lecture 8[1]

The Lax formulation and KdV Let $u(x,t)$ be a given function, then (in $(-\infty,\infty)$) $\mathcal{L} = -\,\mathrm{d}^2/\,\mathrm{d}x^2 + u(x,t)$ is a self adjoint linear operator, because, integrating by parts,

$$\langle \mathcal{L}f, g\rangle = \int_{-\infty}^{\infty} [-f''(x) + u(x,t)f(x)]g(x)\,\mathrm{d}x = \int_{\infty}^{\infty} [f'(x)g'(x) + u(x,t)f(x)g(x)]\,\mathrm{d}x = \langle f, \mathcal{L}g\rangle.$$

Let

$$\mathcal{A} = 4\frac{\mathrm{d}^3}{\mathrm{d}x^3} - 3\left(u\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}u\right) \qquad \Longrightarrow \qquad [\mathcal{A},\mathcal{L}] = 6uu_x - u_{xxx}, \qquad \mathcal{L}_t = u_t.$$

To prove, pick an arbitrary smooth function f and show that $(\mathcal{LA} - \mathcal{AL})f = (6uu_x - u_{xxx})f$. Therefore, the KdV is represented as an isospectral flow: a *Lax representation.*

Evolution of the scattering data Assume that the potential $u(x,t)$ in a Schrödinger equation obeys KdV and let f be an eigenfunction of the Schrödinger operator, $\mathcal{L}f = k^2 f$, such that $f(x) = \phi(x,k) \stackrel{x\to-\infty}{\longrightarrow} \mathrm{e}^{-\mathrm{i}kx}$. Recall that we have demonstrated that $(\mathcal{L}-k^2)(f_t + \mathcal{A}f) = (k^2)_t f = 0$ in the proof of the lemma. Therefore, if f is an eigenfunction then so is $f_t + \mathcal{A}f$, with the same eigenvalue. Since $u(x) \stackrel{x\to\pm\infty}{\longrightarrow} 0$ (solution of KdV!), we have

$$f_t + \mathcal{A}f = \phi_t + \mathcal{A}\phi \stackrel{x\to-\infty}{\longrightarrow} 4\frac{\mathrm{d}^3}{\mathrm{d}x^3}\mathrm{e}^{-\mathrm{i}kx} = 4\mathrm{i}k^3\mathrm{e}^{-\mathrm{i}kx} = 4\mathrm{i}k^3 \lim_{x\to-\infty}\phi(x,k).$$

We deduce that $4\mathrm{i}k^3\phi$ and $\phi_t + \mathcal{A}\phi$ are Schrödinger eigenfunctions with the same eigenvalue *and the same asymptotics:* it follows that they must be equal. The reason is that their difference must be in the kernel of $\mathcal{L} - k^2$, hence must be a linear combination of ψ and $\bar{\psi}$. Since this linear combination vanishes at $+\infty$ and given the linear independence of ψ and $\bar{\psi}$, it must vanish everywhere and we have equality. Summing up, we have the ODE

$$\phi_t + \mathcal{A}\phi = 4\mathrm{i}k^3\phi, \qquad x \in \mathbb{R}. \tag{3.3}$$

Recall, though, that $\phi(x,k) = a(k,t)\mathrm{e}^{-\mathrm{i}kx} + b(k,t)\mathrm{e}^{\mathrm{i}kx}$, $x \to +\infty$. Substituting into (3.3) and recalling that $k_t \equiv 0$, we have for $x \to -\infty$

$$a_t\mathrm{e}^{-\mathrm{i}kx} + b_t\mathrm{e}^{\mathrm{i}kx} = \left(-4\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 4\mathrm{i}k^3\right)(a\mathrm{e}^{-\mathrm{i}kx} + b\mathrm{e}^{\mathrm{i}kx}) = 8\mathrm{i}k^3 b\mathrm{e}^{\mathrm{i}kx}$$

– the reason is that, because $u, u_x \stackrel{x\to-\infty}{\longrightarrow} 0$, $\mathcal{A}g \stackrel{x\to-\infty}{\longrightarrow} -4g'''$. Comparing exponentials, we deduce that $a_t = 0$, $b_t = 8\mathrm{i}k^3 b$. Therefore

$$a(k,t) \equiv a(k,0), \qquad b(x,t) = b(k,0)\mathrm{e}^{8\mathrm{i}k^3 t}.$$

Likewise, because $\phi(x,\mathrm{i}\chi_n) \stackrel{x\to-\infty}{\longrightarrow} b_n\mathrm{e}^{-\chi_n x}$, using the same method we can prove that $b_n(t) = b_n(0)\mathrm{e}^{8\chi_n^3 t}$. This allows us to describe completely the evolution of scattering data:

$$a(k,t) \equiv a(k,0), \qquad b(k,t) = b(k,0)\mathrm{e}^{8\mathrm{i}k^3 t}, \qquad r(k,t) = r(k,0)\mathrm{e}^{8\mathrm{i}k^3 t},$$
$$\chi_n(t) \equiv \chi_n(0), \qquad b_n(t) = b_n(0)\mathrm{e}^{8\chi_n^8 t}, \qquad a_n(t) \equiv 0, \qquad \beta_n(t) = \frac{b_n(t)}{\mathrm{i}a'(\mathrm{i}\chi_n)} = \beta_n(0)\mathrm{e}^{8\chi_n^3 t}.$$

[1]Please email all corrections and suggestions to these notes to `A.Iserles@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html`.

Reflectionless potentials Suppose that the reflection coefficient is initially zero, $r(k,0)=0$. Then $r(k,t)\equiv 0$ and the inverse scattering algorithm can be carried out explicitly.

The 1-soliton solution Recall the function

$$F(x,t)=\sum_{n=1}^{N}\frac{b_n \mathrm{e}^{-\chi_n x}}{\mathrm{i}a_n'(\mathrm{i}\chi_n)}+\frac{1}{2\pi}\int_{-\infty}^{\infty} r(k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k.$$

(a' means a derivative w.r.t. k – don't be mislead by the fact that a is a constant *as a function of* t.) Letting $N=1$ and recalling that $r\equiv 0$, we have

$$F(x,t)=\frac{b_1(t)}{\mathrm{i}a'(\mathrm{i}\chi_n)}\mathrm{e}^{-\chi_1 x}=\beta_1(t)\mathrm{e}^{-\chi_1 x},\qquad \text{where}\qquad \beta(t)=\beta(0)\mathrm{e}^{8\chi_1 t}.$$

Hence, treating t as a parameter, the GLM equation becomes

$$K(x,y)+\beta_1\mathrm{e}^{-\chi_1(x+y)}+\int_x^{\infty}K(x,z)\beta_1\mathrm{e}^{-\chi_1(x+z)}\,\mathrm{d}z=0.$$

We look for separable solutions of the form $K(x,y)=K(x)\mathrm{e}^{-\chi_1 y}$, then

$$K(x)+\beta_1\mathrm{e}^{-\chi_1 x}+\beta_1 K(x)\int_x^{\infty}\mathrm{e}^{-2\chi_1 z}\,\mathrm{d}z=0 \qquad\Rightarrow\qquad K(x)=-\frac{\beta_1\mathrm{e}^{-\chi_1 x}}{1+\beta_1\mathrm{e}^{-2\chi_1 x}/(2\chi_1)}$$

and we deduce that

$$K(x,y)=-\frac{\beta_1\mathrm{e}^{-\chi_1(x+y)}}{1+\beta_1\mathrm{e}^{-2\chi_1 x}/(2\chi_1)}.$$

Therefore

$$\begin{aligned}u(x,t)&=-2\frac{\mathrm{d}K(x,x)}{\mathrm{d}x}=-\frac{4\beta_1(t)\chi_1\mathrm{e}^{-2\chi_1 x}}{\left[1+\beta_1(t)\mathrm{e}^{-2\chi_1 x}/(2\chi_1)\right]^2}\\&=-\frac{16\chi_1^2}{\left(\sqrt{\frac{2\chi_1}{\beta_1}}\mathrm{e}^{\chi_1 x}+\sqrt{\frac{\beta_1}{2\chi_1}}\mathrm{e}^{-\chi_1 x}\right)^2}=\frac{-16\chi_1^2}{(\hat\beta_1^{-1}\mathrm{e}^{-\chi_1 x}+\hat\beta_1\mathrm{e}^{\chi_1 x})^2}=\frac{4\chi_1^2}{\cosh^2(\chi_1 x-\rho_1)},\end{aligned}$$

where $\hat\beta_1=\sqrt{\beta_1/(2\chi_1)}$ and $\rho=\log\hat\beta_1$. Since $\beta_1(t)=\beta_1(0)=\mathrm{e}^{8\chi_1^3 t}$, we have $\hat\beta_1=\sqrt{\beta_0(0)/(2\chi_1)}\mathrm{e}^{4\chi_1^3 t}$, $\rho_1=\chi_1\phi_0+4\chi_1^3 t$, where $\phi_0=(2\chi_1)^{-1}\log(\beta_1(0)/(2\chi_1))$ and, finally, we derive the 1-soliton solution

$$u(x,t)=-\frac{2\chi_1^2}{\cosh^2(\chi_1(x-4\chi_1^2 t-\phi_0))}.$$

Note that $u(x,t)=U(x-4\chi_1^2 t)$. Therefore this solution represents a wave moving rightwards with velocity $4\chi_1^2$ and phase ϕ_0.

The N-soliton solution

We now assume N positive energy levels $\chi_N<\chi_{N-1}<\cdots<\chi_1$, therefore ($r$ is still zero!)

$$F(x)=\sum_{n=1}^{N}\beta_n\mathrm{e}^{-\chi_n x},$$

the GLM equation is

$$K(x,y)+\sum_{n=1}^{N}\beta_n\mathrm{e}^{-\chi_n(x+y)}+\int_x^{\infty}K(x,z)\sum_{n=1}^{N}\beta_n\mathrm{e}^{-\chi_n(x+z)}\,\mathrm{d}z=0$$

(as before, we suppress the dependence on t for the time being) and we seek solutions of the form

$$K(x,y)=\sum_{n=1}^{N}K_n(x)\mathrm{e}^{-\chi_n y}.$$